ASIL SHAAR (PROBABILITY THEORY(STAT3321)

CHAPTER 2

Chapter 2

multivariate Distributions

2.1 Distributions of Two Random variables

Example: Given the following:

(x_1, x_2)	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$f(x_1, x_2)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

$f : A \rightarrow [0,1]$ $\mathbb{R}$

$f(x_1, x_2) > 0 \quad \forall (x_1, x_2) \in A$

$f(x_1, x_2) = 0 \quad \forall (x_1, x_2) \in A^*$

$A \subseteq \mathbb{R}^2$

$\sum_{x_2} \sum_{x_1} f(x_1, x_2) = 1$

f is called the joint p.d.f of the discrete random variables X_1, X_2

$A = \{(0,0), (0,1), (1,0), (1,1)\}$

$f(x_1, x_2) = pr(X_1 = x_1, X_2 = x_2)$

$= pr((X_1, X_2) = (x_1, x_2))$

Note X_1: no. of heads in 1^{st} Toss

X_2: no. of heads in 2^{nd} Toss

Assuming Toss are independent

$\Rightarrow$

Rewrite the p.d.f in rectangular form (cross tabulation)

$X_2 \backslash X_1$	0	1	$f_2(x_2)$
0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
$f_1(x_1)$	$\frac{1}{2}$	$\frac{1}{2}$	1

بالنص الاسم joint p.d.f

بالأطراف الاسم marginal.

Define f_1 the marginal p.d.f of the discrete r.v X_1

$$f_1(x_1) = \sum_{X_2} f(x_1, x_2)$$

Define the marginal p.d.f of the discrete r.v X_2

$$f_2(x_2) = \sum_{x_1} f(x_1, x_2)$$

(x_1, x_2)	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$f(x_1,x_2)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

$$f_1(x_1) = \sum_{x_2} f(x_1, x_2)$$

$$= \begin{cases} 1/2 & , x_1 = 0 \\ 1/2 & , x_1 = 1 \\ 0 & , \text{else} \end{cases}$$

$$f_2(x_2) = \sum_{x_1} f(x_1, x_2)$$

$$= \begin{cases} 1/2 & , x_2 = 0 \\ 1/2 & , x_2 = 1 \\ 0 & , \text{else} \end{cases}$$

$\rightarrow$

Note: $f_1(x_1) = P_r(X_1 = x_1)$, X_1 discrete r.v

$f_2(x_2) = P_r(X_2 = x_2)$, X_2 discrete r.v

Example:

$$f(x,y) = \begin{cases} 6x^2y , & 0<x<1 , \ 0<y<1 \\ 0 , & \text{else} \end{cases}$$

$f : A \to \mathbb{R}$

$A \subseteq \mathbb{R}^2$

$f(x,y) > 0 \quad \forall (x,y) \in A$

$f(x,y) = 0 \quad \forall (x,y) \in A^*$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = \iint_{\mathbb{R}^2} f(x,y)\, dx\, dy = 1$$

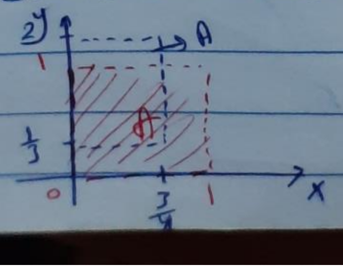
we call f a joint p.d.f of the continous random variables X, Y

check: $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = \int_0^1\int_0^1 6x^2y\, dx\, dy$

$$= \int_0^1 \frac{6x^3}{3}\Big|_0^1 y\, dy = \int_0^1 6y\,\frac{1}{3}(1-0)\, dy$$

$$= \int_0^1 2y\, dy = \frac{2y^2}{2}\Big|_0^1 = 1$$

$A = \{(x,y) \in \mathbb{R}^2 : 0<x<1 , 0<y<1\}$



Find $Pr\left(0 < X < \frac{3}{4}, \frac{1}{3} < Y < 2\right)$

Note: $P(A) = \iint_A f(x,y)\,dx\,dy$

Cont. r.v

$A = \left\{(x,y) \in \mathbb{R}^2,\ 0 < x < \frac{3}{4},\ \frac{1}{3} < y < 2\right\}$

$P(A) = \int_{\frac{1}{3}}^{2}\int_{0}^{\frac{3}{4}} f(x,y)\,dx\,dy = \int_{\frac{1}{3}}^{1}\int_{0}^{\frac{3}{4}} 6x^2y\,dx\,dy = \frac{3}{8}$ (check)

Define the marginal p.d.f of X

$f_1(x) = f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

Define the marginal p.d.f of Y

$f_2(y) = f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

Find $f_1(x)$

$f_1(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \begin{cases} \int_0^1 6x^2y\,dy, & 0<x<1 \\ 0, & \text{else} \end{cases}$

$f_1(x) = \begin{cases} \frac{1}{2}6x^2 = 3x^2, & 0<x<1 \\ 0, & \text{else} \end{cases}$

$\Rightarrow$

Find $f_2(y)$

$$f_2(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \begin{cases} \int_0^1 6x^2y\,dx, & 0<y<1 \\ 0, & \text{else.} \end{cases}$$

$$f_2(y) = \begin{cases} 2y, & 0<y<1 \\ 0, & \text{else} \end{cases}$$

* $f(x_1, x_2)$ joint p.d.f for x_1, x_2
* $f_1(x_1) = \int_{\mathbb{R}} f(x_1, x_2)\,dx_2$ marginal p.d.f of x_1
* $f_2(x_2) = \sum_{x_1 \in \mathbb{R}} f(x_1, x_2)$ marginal p.d.f of x_2
* $p(a<x_1<b,\ c<x_2<d) = \int_c^d \int_a^b f(x_1, x_2)\,dx_1\,dx_2$
* $p(a<x_1<b) = \int_a^b f_1(x_1)\,dx_1$
* $p(c \leq x_2 \leq d) = \sum_{x_2=c}^{d} f_2(x_2)$

Def The joint CDF

$$F(x_1, x_2) = pr(X_1 \leq x_1,\ X_2 \leq x_2)$$

$= \Rightarrow$

$$\begin{cases} \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t_1, t_2)\, dt_1\, dt_2 & , \quad X_1, X_2 \text{ Cont} \\[2ex] \displaystyle\sum_{t_2 \le x_2}\sum_{t_1 \le x_1} f(t_1, t_2) & , \quad X_1, X_2 \text{ dis} \end{cases}$$

2.2 Conditional Distributions and expectations

Def

The Conditional p.d.f of X_2 given $X_1 = x_1$ is defined as:

$$f_{2|1}(x_2 / x_1) = \frac{f(x_1, x_2)}{f_1(x_1)} \quad , f_1(x_1) \neq 0$$

Function
بدلالة x_2

Def

The Conditional p.d.f of X_1 given $X_2 = x_2$ is defined as:

$$f_{1|2}(x_1 / x_2) = \frac{f(x_1, x_2)}{f_2(x_2)} \quad , f_2(x_2) \neq 0$$

Def

$$P(a < X_1 < b \mid X_2 = c) = \int_a^b f_{1|2}(x_1 / c)\, dx_1 \quad , X_1 X_2 \text{ cont.}$$

$$= \sum_{a < x_1 < b} f_{1|2}(x_1 / c) \quad , X_1, X_2 \text{ dis.}$$

$\rightarrow$

$$P(c < X_2 < d \mid X_1 = a) = \int_c^d f_{2/1}(x_2/a)\, dx_2 \quad , X_1, X_2 \text{ cont}$$

$$= \sum_{c<x_2<d} f_{2/1}(x_2/a) \quad , X_1, X_2 \text{ disc.}$$

Def

$$E(X_1 \mid X_2 = x_2) = \mu_{1/2} = \int_{-\infty}^{\infty} x_1 f_{1/2}(x_1/x_2)\, dx_1 \quad , X_1, X_2 \text{ cont}$$

$$E(X_2 / X_1 = x_1) = \mu_{2/1} = \sum_{x_2 \in R} x_2 f_{2/1}(x_2/x_1) \quad , X_1, X_2 \text{ dis}$$

$$E(X_1 . X_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2)\, dx_1\, dx_2 \quad , X_1, X_2 \text{ cont}$$

Example :

$$f(x_1, x_2) = \begin{cases} 2 , & 0 < x_1 < x_2 < 1 \\ 0 , & \text{else} \end{cases}$$

1. verify that f is a p.d.f.

$f(x_1, x_2) > 0 \quad \forall (x_1, x_2) \in A$

$f(x_1, x_2) = 0 \quad \forall (x_1, x_2) \in A^*$

where $A = \{ (x_1, x_2) : 0 < x_1 < x_2 < 1 \}$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1\, dx_2 = \int_0^1 \int_0^{x_2} 2\, dx_1\, dx_2$$

→

$$= \int_0^1 2x_1 \Big|_0^{x_2} dx_2 = \int_0^1 2x_2 \, dx_2 = \frac{2x_2^2}{2} \Big|_0^1$$

$$= x_2^2 \Big|_0^1 = \boxed{1}$$

② find $f_1(x_1)$

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2) \, dx_2 = \begin{cases} \int_{x_1}^{1} 2 \, dx_2 & , 0 < x_1 < 1 \\ 0 & , \text{else} \end{cases}$$

$$f_1(x_1) = \begin{cases} 2(1 - x_1) & , 0 < x_1 < 1 \\ 0 & , \text{else.} \end{cases}$$

③ Find $F_{2/1}(x_2/x_1)$

$$f_{2/1}(x_2/x_1) = \frac{f(x_1, x_2)}{f_1(x_1)} \quad , f_1(x_1) \neq 0$$

$$= \frac{2}{2(1 - x_1)} \quad , x_1 < x_2 < 1 \quad , 0 < x_1 < 1$$

$$f_{2/1}(x_2/x_1) = \frac{1}{1 - x_1} \quad , x_1 < x_2 < 1 \quad , 0 < x_1 < 1$$

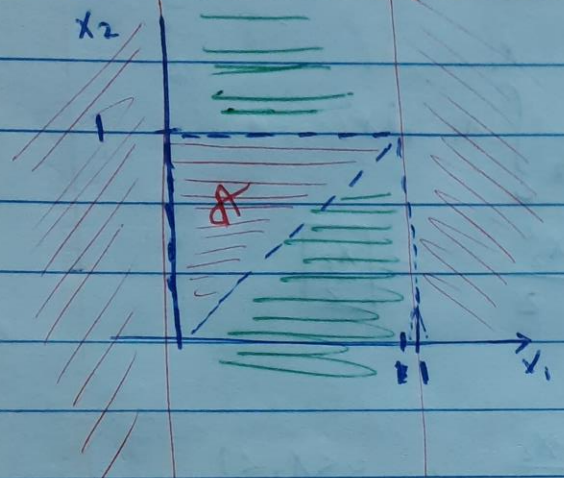
④ Find $F_2(x_2)$

$$f_2(x_2) = \begin{cases} 2x_2 & , 0 < x_2 < 1 \\ 0 & , \text{else} \end{cases}$$

$\Rightarrow$

(5) find $f_{1/2}(x_1/x_2)$

$$f_{1/2}(x_1/x_2) = \frac{1}{x_2}, \quad 0<x_1<x_2, \quad 0<x_2<1$$



$$f_{2/1}(x_2/x_1) = \begin{cases} \frac{1}{1-x_1}, & x_1<x_2<1, \quad 0<x_1<1 \\ 0, & \text{else}, \quad 0<x_1<1 \end{cases}$$

$$f_{2/1}(x_2/x_1) = \begin{cases} \frac{1}{1-x_1}, & x_1<x_2<1 \\ 0, & \text{else} \end{cases}, \quad 0<x_1<1$$

$$f_{1/2}(x_1/x_2) = \begin{cases} \frac{1}{x_2}, & 0<x_1<x_2 \\ 0, & \text{else} \end{cases}, \quad 0<x_2<1$$

(6) $E(x_1) = \mu_1 = \int_{-\infty}^{\infty} f_1(x_1)\,dx_1 = \int_0^1 x_1 2(1-x_1)\,dx_1 = \frac{1}{3}$ check

(7) $\mu_2 = \int_{-\infty}^{\infty} f_2(x_2)\,dx_2 = \int_0^1 2x_2\,dx_2 = \frac{2}{3}$ check

$\Rightarrow$

(8) $\mu_{1/2} = E(X_1 / X_2 = x_2)$

$$= \int_{-\infty}^{\infty} x_1 f_{1/2}(x_1 / x_2)\, dx_1$$

$$= \int_{0}^{x_2} x_1 \cdot \left(\frac{1}{x_2}\right) dx_1 \quad (0 < x_2 < 1) \qquad = \frac{1}{x_2} \int_{0}^{x_2} x_1\, dx_1$$

$$= \frac{1}{x_2}\left(\frac{x_1^2}{2}\Big|_0^{x_2}\right) = \frac{1}{x^2} \cdot \frac{x_2^2}{2} = \boxed{\frac{x_2}{2}}$$

$$\mu_{1/2} = E(X_1 / X_2 = x_2) = \frac{x_2}{2}, \quad 0 < x_2 < 1$$

(9) $\mu_{2/1} = E(X_2 / X_1 = x_1)$ = check

$$\int_{-\infty}^{\infty} x_2 f_{2/1}(x_2 / x_1)\, dx_2$$

$$0 < x_1 < 1 \quad = \int_{x_1}^{1} x_2 \left(\frac{1}{1 - x_1}\right) dx_2 =$$

$$= \frac{1}{1 - x_1} \int_{x_1}^{1} x_2\, dx_2 = \left(\frac{x_2^2}{2}\Big|_{x_1}^{1}\right) \frac{1}{1 - x_1}$$

$$= \frac{1 + x_1}{2}, \quad 0 < x_1 < 1$$

$$\left(\frac{1}{2} - \frac{x_1^2}{2}\right) \frac{1}{1 - x_1}$$

$$= \frac{1 - x_1^2}{2} \cdot \frac{1}{(1 - x_1)} = \frac{\cancel{(1 - x_1)}(1 + x_1)}{2\cancel{(1 - x_1)}}$$

$$= \boxed{\frac{(1 + x_1)}{2}}$$

Define

$$\sigma^2_{1/2} = Var(X_1 / X_2 = x_2)$$
$$= E\left[(X_1 - \mu_{1/2})^2 / X_2 = x_2\right]$$

$$\sigma^2_{2/1} = Var(X_2 / X_1 = x_1)$$
$$= E\left[(X_2 - \mu_{2/1})^2 / X_1 = x_1\right]$$

Proposition

① $E(u_1(X_1) / X_2 = x_2)$

$$= \int_{-\infty}^{\infty} u_1(x_1) f_{1/2}(x_1/x_2)\, dx_1$$

Continous X_1, X_2. Use sum for discrete X_1, X_2

② $E(u_2(X_2) / X_1 = x_1)$

$$= \sum_{x_2} u_2(x_2) f_{2/1}(x_2/x_1)$$

discrete X_1, X_2. Use integral for cont. X_1, X_2.

③ $E(u(X_1, X_2)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x_1, x_2) f(x_1, x_2)\, dx_1\, dx_2$

Cont. X_1, X_2. Use sum for discrete X_1, X_2.

$\rightarrow$

Proposition

① $\sigma^2_{1/2} = var(X_1 / X_2 = x_2)$

$= E(X_1^2 / X_2 = x_2) - (\mu_{1/2})^2$

② $\sigma^2_{2/1} = E(X_2^2 / X_1 = x_1) - (E(X_2 / X_1 = x_1))^2$

Back to example

⑩ σ_1^2 = check = $\frac{1}{18}$ $\quad \sigma_1^2 = E(X_1^2) - (\mu_1)^2$

$= \int_0^1 x_1^2 f_1(x_1) - \left(\frac{1}{3}\right)^2 = \int_0^1 2(1-x_1)x_1^2 \, dx_1 - \frac{1}{9}$

$= \int_0^1 2x_1^2 - 2x_1^3 \, dx_1 - \frac{1}{9} = \left(\frac{2x_1^3}{3} - \frac{2x_1^4}{4}\Big|_0^1\right) - \frac{1}{9}$

$= \left(\frac{2}{3} - \frac{2}{4}\right) - \frac{1}{9} = \boxed{\frac{1}{18}}$

⑪ σ_2^2 = check = $\frac{1}{18}$ $\quad \sigma_2^2 = E(x_2^2) - (\mu_2)^2$

$= \int_0^1 x_2^2 \cdot \underbrace{2x_2}_{f_2(x_2)} \, dx_2 - \left(\frac{2}{3}\right)^2 = \frac{2x_2^4}{4}\Big|_0^1 - \frac{4}{9}$

$= \frac{1}{2} - \frac{4}{9} = \boxed{\frac{1}{18}}$

⑫ $\sigma^2_{1/2}$ = check = $\frac{x_2^2}{12}$, $0 < x_2 < 1$

$\sigma^2_{1/2} = E((X_1 - \mu_{1/2})^2 / X_2)$

$= E(X_1^2 / X_2) - (\mu_{1/2})^2$

$= \int_0^{x_2} x_1^2 \cdot f_{1/2}(x_1/x_2) \, dx_1 - \left(\frac{x_2}{2}\right)^2 = \int_0^{x_2} \frac{x_1^2}{x_2} dx_1 - \frac{x_2^2}{4}$

$= \frac{1}{x_2} \cdot \frac{x_1^3}{3}\Big|_0^{x_2} - \frac{x_2^2}{4} = \frac{x_2^2}{3} - \frac{x_2^2}{4}$

$= \boxed{\frac{x_2^2}{12}}$ #

(13) $\sigma^2_{2/1}$ = Check = $\dfrac{(1-x_1)^2}{12}$, $0<x_1<1$

$$\sigma^2_{2/1} = E(X_2^2/X_1) - (\mu_{2/1})^2$$

$$= \int_{x_1}^{1} x_2^2 \cdot f_{2/1}(x_2/x_1)\, dx_2 - \left(\frac{1+x_1}{2}\right)^2$$

$$\int_{x_1}^{1} x_2^2 \cdot \frac{1}{1-x_1}\, dx_2 - \frac{(1+x_1)^2}{4} = \frac{(1-x_1)^2}{12}$$

Ex:

(x_1, x_2)	(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)	Sum
$f(x_1, x_2)$	$\frac{1}{k}$	$\frac{3}{k}$	$\frac{4}{k}$	$\frac{3}{k}$	$\frac{6}{k}$	$\frac{1}{k}$	1

$f(x_1, x_2)$ is a p.d.f

(1) Find K

$$\sum_{x_2}\sum_{x_1} f(x_1, x_2) = \frac{1+3+4+3+6+1}{k} = 1 \Rightarrow \frac{18}{k} = 1 \Rightarrow \boxed{k = 18}$$

(2) $E(x_1 x_2) = \sum_{x_2}\sum_{x_1} x_1 x_2 f(x_1, x_2)$

$$0+0+0+\frac{3}{18}+0+\frac{2}{18} = \boxed{\frac{5}{18}}$$

(3) $f_1(x_1) = \begin{cases} \frac{4}{18} & , x_1 = 0 \\ \frac{7}{18} & , x_1 = 1 \\ \frac{7}{18} & , x_1 = 2 \\ 0 & , \text{else} \end{cases}$

يعني بدي اخلص من x_2

$\Rightarrow$

(4) $f_2(x_2) = \begin{cases} \frac{11}{18} & , x_2 = 0 \\ \frac{7}{18} & , x_2 = 1 \\ 0 & , \text{else} \end{cases}$

(5) $\mu_1 = 0 + \frac{7}{18} + \frac{14}{18} = \frac{21}{18} = \frac{7}{6}$ $\quad \mu_1 = \sum x_1 f(x_1)$

(6) $\mu_2 = 0 + \frac{7}{18} = \frac{7}{18}$

(7) $\mu_{1/2} = ?$

$f_{1/2}(x_1/x_2) = ?$

$$f_{1/2}(x_1/x_2) = \frac{f(x_1, x_2)}{f_2(x_2)}$$

$$f_{1/2}(x_1/0) = \frac{f(x_1, 0)}{f_2(0)} = \frac{f(x_1, 0)}{\frac{11}{18}}$$

$$= \begin{cases} \frac{\frac{1}{18}}{\frac{11}{18}} = \frac{1}{11} & , x_1 = 0 \\ \frac{\frac{4}{18}}{\frac{11}{18}} = \frac{4}{11} & , x_1 = 1 \\ \frac{\frac{6}{18}}{\frac{11}{18}} \quad \frac{6}{11} & , x_1 = 2 \\ 0 & , \text{else.} \end{cases}$$

$$f_{1/2}(x_1/1) = \begin{cases} \frac{3}{7} & , x_1 = 0 \\ \frac{3}{7} & , x_1 = 1 \\ \frac{1}{7} & , x_1 = 2 \\ 0 & , \text{else} \end{cases}$$

$$\mu_{1/2} = E(X_1 / X_2 = x_2)$$

$$= \sum_{x_1} x_1 \, f_{1/2}(x_1/x_2)$$

$$= \begin{cases} 0 + \frac{4}{11} + \frac{12}{11} = \frac{16}{11} & , x_2 = 0 \\ 0 + \frac{3}{7} + \frac{2}{7} = \frac{5}{7} & , x_2 = 1 \end{cases}$$

(8) $\mu_{2/1}$ = check $= \begin{cases} \frac{3}{4} & , x_1 = 0 \\ \frac{3}{7} & , x_1 = 1 \\ \frac{1}{7} & , x_1 = 2 \end{cases}$

* $f_{2/1}(x_2/0) = \dfrac{f(0, x_2)}{\frac{4}{18}}$

$$= \begin{cases} \frac{1}{4} & , x_2 = 0 \\ \frac{3}{4} & , x_2 = 1 \\ 0 & , \text{else} \end{cases}$$

* $f_{2/1}(x_2/1) = \dfrac{f(1, x_2)}{\frac{7}{18}}$

$$= \begin{cases} \frac{4}{7} & , x_2 = 0 \\ \frac{3}{7} & , x_2 = 1 \\ 0 & , \text{else} \end{cases}$$

* $f_{2/1}(x_2/2) = \dfrac{f(2, x_2)}{\frac{7}{18}}$

$$= \begin{cases} \frac{6}{7} & , x_2 = 0 \\ \frac{1}{7} & , x_2 = 1 \\ 0 & , \text{else} \end{cases}$$

2.3 The Correlation Coefficient

Def

let X, Y be r.v with joint p.d.f $f(x, y)$. The Covariance of X and Y is defined as:

$$\mathrm{Cov}(x, y) = E\left[(X - \mu_x)(Y - \mu_y)\right]$$

Proposition

$$\mathrm{Cov}(x, y) = E(XY) - E(x)E(y)$$
$$= E(xy) - \mu_x \mu_y$$

Def let X, Y be r.v the correlation coefficient ρ_{xy} is defined as:

$$\rho_{xy} = \frac{\mathrm{Cov}(x, y)}{\sigma_x \sigma_y}$$

proposition

① $Cov(x,y) = \rho_{\sigma_1 \sigma_2}$

② $E(xy) = \rho_{\sigma_1 \sigma_2} + \mu_1 \mu_2$

Theorem :

let x, y be random variables &

If $E(y/x) = a + bx$

then :

① $E(y/x) = \mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x - \mu_1)$

② $E(var(y/x)) = \sigma_2^2 (1 - \rho^2)$

Def

let x_1, x_2 be r.v

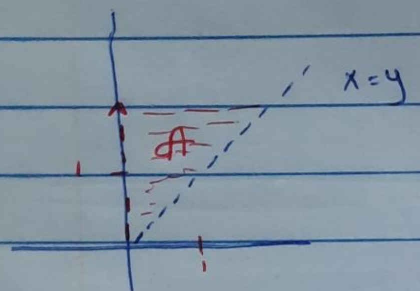
The m.g.f of x_1, x_2 is defined as :

$$M(t_1, t_2) = E(e^{t_1 x_1 + t_2 x_2})$$

The m.g.f exists if the expected value exists for $-h_1 < t_1 < h_1$, $-h_2 < t_2 < h_2$ where $h_1 > 0$, $h_2 > 0$

Ex:

$$f(x,y) = \begin{cases} e^{-y}, & 0 < x < y < \infty \\ 0, & \text{else} \end{cases}$$



① Find the m.g.f (if it exists)

$$M(t_1, t_2) = E\left(e^{t_1 X_1 + t_2 X_2}\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x + t_2 y} f(x,y)\, dx\, dy$$

$$= \int_0^{\infty}\int_0^{y} e^{t_1 x + t_2 y} \cdot e^{-y}\, dx\, dy = \int_0^{\infty}\int_x^{\infty} e^{t_1 x + t_2 y} e^{-y}\, dy\, dx$$

$$\downarrow$$

$$= \int_0^{\infty} e^{t_2 y} \cdot e^{-y} \left(\int_0^{y} e^{t_1 x}\, dx \right) dy$$

$$= \int_0^{\infty} e^{(t_2 - 1)y} \left(\frac{e^{t_1 x}}{t_1} \Big|_0^y \right) dy = \int_0^{\infty} e^{(t_2-1)y} \left(\frac{e^{t_1 y}}{t_1} - \frac{1}{t_1} \right) dy \qquad t_1 \neq 0$$

$$= \int_0^{\infty} \frac{e^{(t_1 + t_2 - 1)y}}{t_1}\, dy - \int_0^{\infty} \frac{e^{(t_2 - 1)y}}{t_1}\, dy$$

$$\rightarrow$$

$$= \frac{1}{t_1}\left\{ \frac{e^{(t_1+t_2-1)y}}{(t_1+t_2-1)} \Bigg|_0^\infty \right\} - \frac{1}{t_1}\left\{ \frac{e^{(t_2-1)y}}{(t_2-1)} \Bigg|_0^\infty \right\}$$

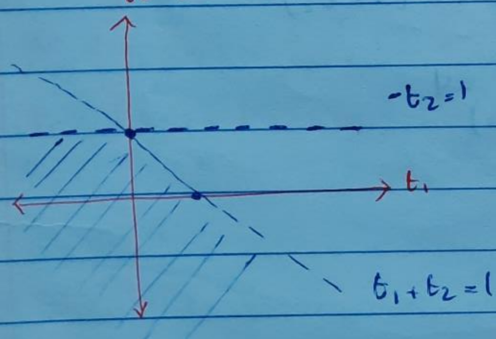
$t_1+t_2<1$ = $t_2<1)$ بشروط

$$\frac{1}{t_1}\left(0 - \frac{1}{t_1+t_2-1}\right) - \frac{1}{t_1}\left(0 - \frac{1}{t_2-1}\right)$$

$$= \frac{1}{t_1}\left[\frac{1}{1-t_1-t_2} - \frac{1}{1-t_2}\right]$$

$$= \frac{1}{t_1}\left[\frac{(1-t_2)-(1-t_1-t_2)}{(1-t_1-t_2)(1-t_2)}\right] = \frac{1}{(1-t_2)(1-t_1-t_2)}$$

$$M(t_1, t_2) = \frac{1}{(1-t_2)(1-t_1-t_2)} \quad , t_2<1 \quad , t_1+t_2<1$$



$M(t_1, t_2) = E(e^{t_1X_1 + t_2X_2})$

$M(0,0) = 1$ for all

r.v X_1, X_2

Properties of m.g.f

1. $M(0,0) = 1$
2. $M(t_1, 0) = M_1(t_1)$ marginal m.g.f of X_1
3. $M(0, t_2) = M_2(t_2)$ marginal m.g.f of X_2
4. $$\frac{\partial^{k+l} M(t_1, t_2)}{\partial t_1^k \, \partial t_2^k}\Bigg|_{(t_1,t_2)=(0,0)} = E(X_1^k X_2^k)$$

$\rightarrow$

Back to example :

② Find ρ

$$\rho = \frac{Cov(X,Y)}{\sigma_x \sigma_y} = \frac{E(X-\mu_X)(Y-\mu_Y)}{\sigma_x \sigma_y}$$

$$\rho = \frac{E(XY) - \mu_X \mu_Y}{\sigma_x \sigma_y}$$

$$\rho = \frac{\frac{\partial^2}{\partial t_1 \partial t_2} M(0,0) - (M_1'(0))(M_2'(0))}{\sqrt{M_1''(0) - (M_1'(0))^2} \sqrt{M_2''(0) - (M_2'(0))^2}}$$

$M(t_1, t_2) = (1-t_2)^{-1}(1-t_1-t_2)^{-1}$, $t_2 < 1$, $t_1+t_2 < 1$

$\frac{\partial}{\partial t_1} M(t_1, t_2) = (1-t_2)^{-1} (+1(1-t_1-t_2)^{-2})$

$\frac{\partial}{\partial t_1} M \Big|_{t_1=0} = (1-t_2)^{-1}(1-t_2)^{-2} = (1-t_2)^{-3}$

$\frac{\partial}{\partial t_2} \left(\frac{\partial}{\partial t_1} M \Big|_{t_1=0} \right) = +3(1-t_2)^{-4}$

$\frac{\partial}{\partial t_2} \left(\frac{\partial}{\partial t_1} M \Big|_{t_1=0} \right) \Big|_{t_2=0} = \boxed{3}$

$E(XY) = \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} M \Big|_{t_1=0, t_2=0} = 3$

$M_1(t_1) = M(t_1, 0) = (1-t_1)^{-1}$, $t_1 < 1$

$M_1'(t_1) = +(1-t)^{-2}$

$M_1'(0) = 1 = \mu_X = E(X)$

$M_1''(t_1) = +2(1-t_1)^{-3}$ $\quad M_1''(0) = E(X^2) = 2 \Rightarrow$

$\boxed{\sigma_x = 1}$

$M_2(t_2) = (1-t_2)^{-2}$, $t_2 < 1$

$M_2'(t_2) = +2(1-t_2)^{-3}$

$E(y) = \mu_y = M_2'(0) = 2$

$M_2''(t_2) = +6(1-t_2)^{-4}$

$E(y^2) = M_2''(0) = 6 \Rightarrow \boxed{\sigma_y = \sqrt{2}}$

$$\rho = \frac{3-(1)(2)}{(1)(\sqrt{2})} = \frac{1}{\sqrt{2}}$$

2.4 Independent random variables

Def X_1, X_2 are called independent random variables if:

$f_{1/2}(x_1/x_2) = f_1(x_1) \quad \forall x_1 \; (\forall x_2 \; f_2(x) \neq 0)$

$f_{2/1}(x_2/x_1) = f_2(x_2) \quad \forall x_2 \; (\forall x_1 \; f_1(x_1) \neq 0)$

Th 1

let X_1, X_2 be random variables with joint p.d.f $f(x_1, x_2)$.

X_1, X_2 are independent $\Leftrightarrow f(x_1, x_2) = h(x_1) \cdot g(x_2)$, $\forall (x_1, x_2) \in \mathbb{R}^2$

where $h(x_1) \geq 0$

$g(x_2) \geq 0$

$\Rightarrow$

Ex:

$$f(x,y) = \begin{cases} e^{-y} & , \ 0<x<y<\infty \\ 0 & , \ \text{else} \end{cases}$$

Are X and Y independent? why?

No. X and Y are not independent

$$f(x,y) = \begin{cases} e^{-y} & , \ \underbrace{x<y<\infty} \\ 0 & , \ \text{else} \end{cases} \cdot \begin{cases} 1 & , \ \underbrace{0<x<y} \\ 0 & , \ \text{else} \end{cases}$$

$$= j(x,y) \cdot k(x,y)$$

Because, $f(x,y) \neq h(x) \cdot g(y) \ \forall (x,y) \in \mathbb{R}^2$

Ex:

$$f(x_1, x_2) = \begin{cases} x_1 + x_2 & , \ 0<x_1<1 \ , \ 0<x_2<1 \\ 0 & , \ \text{else} \end{cases}$$

Are X_1, X_2 independent? why?

No. X_1 and X_2 are not independent.

Because, $f(x_1, x_2) \neq h(x_1) \cdot g(x_2) \ \forall (x_1, x_2) \in \mathbb{R}^2$

Theorem 2

X_1, X_2 independent random variables

$\Rightarrow pr(a < X_1 < b, \; c < X_2 < d) = pr(a < X_1 < b) \cdot pr(c < X_2 < d)$

Theorem 3

X_1, X_2 independent random variables

$\Rightarrow E(u_1(X_1) \cdot u_2(X_2)) = E(u_1(X_1)) \cdot E(u_2(X_2))$

Theorem 4

X_1, X_2 independent random variabels

$\Leftrightarrow M(t_1, t_2) = M_1(t_1) \cdot M_2(t_2)$

Recall

Def X_1, X_2 indep $\Leftrightarrow f_{1/2}(x_1 / x_2) = f_1(x_1) \Leftrightarrow f(x_1, x_2) = f_1(x_1) \cdot f_2(x_2)$

Th 1 X_1, X_2 indep $\Leftrightarrow f(x_1, x_2) = h(x_1) \cdot g(x_2)$, $h(x_1) \geq 0$, $g(x_2) \geq 0$

Note: we say X_1, X_2 are uncorrelated random variables if and only if $\rho = 0$

Question

which statment is True (T)

& which is False (F) ? why

① $E(xy) = E(x) \cdot E(y) \Rightarrow \rho = 0$ (T)

$$\rho = \frac{Cov(x, y)}{\sigma_x \sigma_y} = \frac{E(xy) - E(x)E(y)}{\sigma_x \sigma_y} = \frac{0}{\sigma_x \sigma_y} = 0$$

$\Rightarrow$

② $\rho = 0 \Rightarrow E(xy) = E(x) \cdot E(y)$ (T)

$$\rho = \frac{Cov(x,y)}{\sigma_x \sigma_y} = \frac{E(xy) - E(x)E(y)}{\sigma_x \sigma_y} = 0 \Rightarrow$$

$$E(xy) - E(x)E(y) = 0 \Rightarrow E(xy) = E(x)E(y)$$

③ X, Y independent $\Rightarrow$ X, Y uncorrelated (T)

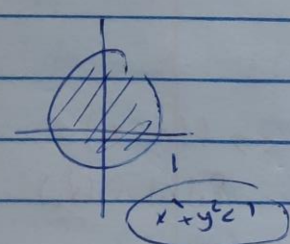
X, Y indep $\Rightarrow E(xy) = E(x)E(y)$ (Th.3)

$\Rightarrow \rho = 0$ (part (1))

$\Rightarrow X, Y$ uncorrelated.

④ X, Y uncorrelated $\Rightarrow$ X, Y independent (F)

Find a Counter example



uniform density on a Disk

2.5 Extension to several Random variables

$X_1, X_2, \ldots, X_n$ random variables

Joint p.d.f $f(x_1, x_2, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$

- $f(x_1, \ldots, x_n) \geq 0 \quad \forall (x_1, \ldots, x_n) \in \mathbb{R}^n$
- * $\sum_{x_n \in R} \cdots \sum_{x_1 \in R} f(x_1, \ldots, x_n) = 1$

  * $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n = 1$

marginals densities

$$* \; f_j(x_j) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{(n-1) \text{ integrals}} f(x_1, \ldots, x_n)\, dx_1 \ldots dx_{j-1}\, dx_{j+1} \ldots dx_n$$

$$* \; f_j(x_j) = \underbrace{\sum_{x_1 \in R_1} \cdots \sum_{x_{j-1} \in R} \; \sum_{x_{j+1} \in R} \cdots \sum_{x_n \in R}}_{(n-1) \text{ sum}} f(x_1, \ldots, x_n)$$

Joint marginal densities

نكامل كل variables ماعدا i و j

نفرض انهم Cont. r.v

$$f_{ij}(x_i, x_j) = \int_{\mathbb{R}^{n-2}} \cdots \int f(x_1, \ldots, x_n) \prod_{\substack{k=1 \\ k \neq i \\ k \neq j}}^{n} dx_k$$

$$f_{i_1, \ldots, i_\ell}(x_{i_1}, \ldots, x_{i_\ell}) = \int_{\mathbb{R}^{n-2}} \cdots \int f(x_1, \ldots, x_n) \prod_{\substack{k=1 \\ k \notin \{i_1, \ldots, i_\ell\}}}^{n} dx_k$$

$n = 4$

f	f_{12}	f_{123}
f_1	f_{13}	f_{124}
f_2	f_{14}	f_{234}
f_3	f_{23}	
f_4	f_{24}	
	f_{34}	

Joint conditional densites

هي افرض انهم !

$$f_{1/2}(x_1/x_2) = \frac{f_{12}(x_1, x_2)}{f_2(x_2)}$$

$$f_{1,2/3}(x_1, x_2 / x_3) = \frac{f_{123}(x_1, x_2, x_3)}{f_3(x_3)}$$

$$f_{1/2,3}(x_1 / x_2, x_3) = \frac{f_{123}(x_1, x_2, x_3)}{f_{2,3}(x_2, x_3)}$$

$$f_{1,2,3/4}(x_1, x_2 x_3 / x_4) = \frac{f(x_1, x_2, x_3, x_4)}{f_4(x_4)}$$

Def

① $X_1, ----, X_n$ pairwise independent

$\rightarrow f_{i,j}(x_i, x_j) = f_i(x_i) \cdot f_j(x_j) \quad \forall (x_i, x_j) \in \mathbb{R}^2$

② $X_1, -----, X_n$ (mutually) independent

$\Rightarrow f_{i_1, \ldots, i_L}(x_{i_1}, \ldots, x_{i_L}) = f_{i_1}(x_{i_1}) \ldots f_{i_L}(x_{i_L})$

for all $(x_{i_1}, \ldots x_{i_L}) \in \mathbb{R}^2$

$\{i_1, \ldots, i_L\} \subseteq \{1, \ldots, n\}$

In words, $X_1, \ldots, X_n$ are indep. if the joint density of any Collection of random variables can be written as the product of their marginals.

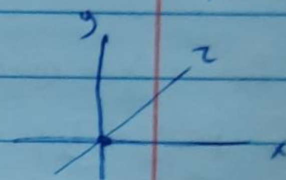
Ex:

X, Y, Z random variables with joint p.d.f

$$f(x,y,z) = \begin{cases} e^{-x-y-z} & , x>0 \ , y>0 \ , z>0 \\ 0 & , \text{else} \end{cases}$$

Find the joint CDF

1st octant



$$F(x,y,z) = pr(X \le x, Y \le y, Z \le z)$$

$$= \int_{-\infty}^{z}\int_{-\infty}^{y}\int_{-\infty}^{x} f(t_1,t_2,t_3)\, dt_1\, dt_2\, dt_3$$

$$= \int_{0}^{z}\int_{0}^{y}\int_{0}^{x} e^{-t_1-t_2-t_3}\, dt_1\, dt_2\, dt_2$$

$$= \left(\int_0^x e^{-t_1} dt_1\right)\left(\int_0^y e^{-t_2} dt_2\right)\left(\int_0^z e^{-t_3} dt_3\right)$$

$$= \left(\frac{e^{-t_1}}{-1}\Big|_0^x\right)\left(\frac{e^{-t_2}}{-1}\Big|_0^y\right)\left(\frac{e^{-t_3}}{-1}\Big|_0^z\right)$$

$$= (1-e^{-x})(1-e^{-y})(1-e^{-z})$$

$$F(x,y,z) = \begin{cases} (1-e^{-x})(1-e^{-y})(1-e^{-z}) & , x\ge 0 \ , y \ge 0 \ , z \ge 0 \\ 0 & , \text{else} \end{cases}$$

Ex:

X_1, X_2, X_3 mutually (indep) random variables, each with p.d.f $f(x) = \begin{cases} 2x & , 0<x<1 \\ 0 & , \text{else} \end{cases}$

Define $Y = \max(X_1, X_2, X_3)$

① $pr(Y \le \frac{1}{2}) = ?$

② Find the CDF of Y

③ Find the p.d.f of Y

Solution

① $pr(Y \le \frac{1}{2}) = pr(\max(X_1, X_2, X_3) \le \frac{1}{2})$

$= pr(X_1 \le \frac{1}{2}, X_2 \le \frac{1}{2}, X_3 \le \frac{1}{2})$

X_1, X_2, X_3 indep. $= pr(X_1 \le \frac{1}{2}) \cdot pr(X_2 \le \frac{1}{2}) \cdot pr(X_3 \le \frac{1}{2})$

$$= \left(\int_{-\infty}^{\frac{1}{2}} f(x_1)\,dx_1\right) \cdot \left(\int_{-\infty}^{\frac{1}{2}} f(x_2)\,dx_2\right) \cdot \left(\int_{-\infty}^{\frac{1}{2}} f(x_3)\,dx_3\right)$$

$$= \left(\int_0^{\frac{1}{2}} 2x_1\,dx_1\right) \cdot \left(\int_0^{\frac{1}{2}} 2x_2\,dx_2\right) \cdot \left(\int_0^{\frac{1}{2}} 2x_3\,dx_3\right)$$

$$= \left(\int_0^{\frac{1}{2}} 2t\,dt\right)^3 = \left(t^2 \Big|_0^{\frac{1}{2}}\right)^3 = \left(\frac{1}{4}\right)^3 = \frac{1}{64}$$

$\Rightarrow$

(2) $G(y) = pr(Y \le y) = pr(\max(X_1, X_2, X_3) \le y)$

$= pr(X_1 \le y, X_2 \le y, X_3 \le y)$

indep. X_1, X_2, X_3 $= pr(X_1 \le y) \cdot pr(X_2 \le y) \cdot pr(X_3 \le y)$

$= F(y)\, F(y)\, F(y)$

$= (F(y))^3$

$F(y) = \int_{-\infty}^{y} f(x)\, dx = \begin{cases} 0 & , y<0 \\ \int_0^y 2x\, dx = y^2 & , 0 \le y < 1 \\ \int_0^1 2x\, dx = 1 & , 1 \le y \end{cases}$

$F(y) = \begin{cases} 0 & , y<0 \\ y^2 & , 0 \le y < 1 \\ 1 & , 1 \le y \end{cases}$

$G(y) = \begin{cases} 0 & , y<0 \\ y^6 & , 0 \le y < 1 \\ 1 & , 1 \le y \end{cases}$

(3) $g(y) = \dfrac{d\, G(y)}{dy}$

$= \begin{cases} 6y^5 & , 0 < y < 1 \\ 0 & , \text{else}. \end{cases}$

$\Rightarrow$