ASIL SHAAR (PROBABILITY THEORY(STAT3321))

CHAPTER 3

Ch 3 Some special Distributions

3.1 Binomial and Related Distribution

Def A Binomial experiment Satisfie

① It has n identical trials

② Each trial has two outcomes:

sucess(s) & Failure (F)

③ $\left.\begin{aligned} p(S) &= p \\ p(F) &= 1-p \end{aligned}\right\}$ do not change in all trial

④ The trials are independent

Def A Binomial random variable X is defined as:

X = number of successes is a binomial experement

Notation $\boxed{X \sim b(n,p)}$

In words X has a binomial dist. with n trials and prob. of success equal to p

Theorm:

$$X \sim b(n,p) \Rightarrow f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & , x = 0,1,2,\dots n \\ 0 & , \text{else} \end{cases}$$

p	p	p		$1-p$
S	S	F	----	F
1	2	3		n

x succes

$n-\cancel{x}$ failures

Idea of proof

Theorm

$$f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & , x = 0, 1, \dots, n \\ 0 & , \text{else} \end{cases}$$

is a p.d.f

Proof $\quad 0 < p < 1$

$\boxed{1}$ $p^x > 0$

$(1-p)^{n-x} > 0$

$\binom{n}{x} > 0 \quad \Rightarrow \binom{n}{x} p^x (1-p)^{n-x} > 0 \quad , x = 0, 1, \dots, n$

$\Rightarrow f(x) \geq 0 \quad \forall x \in \mathbb{R}$

$\boxed{2}$ $\sum_{x \in \mathbb{R}} f(x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$

$$(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k}$$

Binomial Formula

$= (p + 1 - p)^n = 1$

Question $\quad X \sim b(n, p)$

Find μ

$\mu = E(x) = \sum_{x \in \mathbb{R}} x . f(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$

$= \dots\dots$

$\Rightarrow$

Question $X \sim b(n,p)$

Find the m.g.f

$$E(e^{tx}) = \sum_{x \in R} e^{tx} f(x) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x} = (1-p+pe^t)^n$$

$$\boxed{X \sim b(n,p) \rightarrow M(t) = (1-p+pe^t)^n, \ t \in R}$$

Question $X \sim b(n,p)$

① Find μ

② Find σ

① $M(t) = (1-p+pe^t)^n$

$M'(t) = n(1-p+pe^t)^{n-1}(pe^t)$

$= M'(0) = n(1-p+p)^{n-1}(p)$

$= np$

$$\boxed{E(x) = \mu = np}$$

② $M'(t) = npe^t(1-p+pe^t)^{n-1}$

$M''(t) = (npe^t)\left[(n-1)(1-p+pe^t)^{n-2}(pe^t)\right] + (1-p+pe^t)^{n-1}(npe^t)$

$M''(0) = (np)(n-1)(1-p+p)^{n-2}(p) + (1-p+p)^{n-1}(np)$

$E(x^2) = (np)(n-1)(p) + np$

$= (np)[(n-1)p+1]$

$\rightarrow$

$\sigma^2 = var(x)$

$= E(x^2) - (E(x))^2$

$\mu''(0) - (\mu'(0))^2$

$= np(n-1)p + np - (np)^2$

$= np^2(n-1) + np - (n^2p^2)$

$\cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2} = np - np^2 = n(p - p^2)$

$\boxed{np(1-p)}$

$\boxed{var(x) = \sigma^2 = np(1-p)}$ $\boxed{\sigma = \sqrt{np(1-p)}}$

Multinomial Distribution

$f(x_1, x_2, \ldots, x_{k-1})$

$$= \begin{cases} \dfrac{n!}{x_1!\,x_2!\cdots x_k!}\, p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k} & , \sum_{i=1}^{k} x_i = 0,1,2,\ldots,n \\ 0 & , \text{else} \end{cases}$$

We have $X_1, \ldots, X_{k-1}$ random variables

$X_k = n - (X_1 + \cdots + X_{k-1})$

It's better to write

$X_k = n - (X_1 + X_2 + \cdots + X_{k-1})$

$P_k = 1 - \sum_{i=1}^{k-1} P_i$ $\qquad$ P_i: prob. of Type i success.

$M(t_1, t_2, \ldots, t_{k-1}) = (P_1 e^{t_1} + P_2 e^{t_2} + \cdots + P_{k-1} e^{t_{k-1}} + 1 - P_1 - P_2 - \cdots - P_{k-1})^n$

$= \left(\sum_{i=1}^{k-1} P_i e^{t_i} + P_k \right)^n \qquad (t_1, t_2, \ldots, t_{k-1}) \in \mathbb{R}^{k-1}$

Consider

Trinomial Distribution:

$$f(x,y) = \begin{cases} \dfrac{n!}{x!\,y!\,(n-x-y)!}\, p_1^x p_2^y (1-p_1-p_2)^{n-x-y} & , x+y = 0,1,2,\ldots,n \\ 0 & , \text{else} \end{cases}$$

We have 2 random variables X and Y

P_1 : prob of succes of Type I

P_2 : prob of succes of Type II

you can write:

$Z = n - x - y$

$P_3 = 1 - P_1 - P_2$

$$M(t_1, t_2) = (P_1 e^{t_1} + P_2 e^{t_2} + 1 - P_1 - P_2)^n \quad , (t_1, t_2) \in \mathbb{R}^2$$

Find the marginal and Conditionals densities, expected value, variance

check!

3.2 The poission Distribution

occurance / event　　　occurance / event

$(x) \longleftrightarrow (x)$

time or distance between 2 occurances.

Def

$X \sim \text{Poisson}(\mu)$

In words, X has a poisson distribution with parameter μ, if

$$f(x) = \begin{cases} \dfrac{\mu^x e^{-\mu}}{x!} & , x = 0, 1, 2, \ldots \\ 0 & , \text{else} \end{cases}$$

$\rightarrow$

Theorem

$X \sim \text{possion}(\mu) \Rightarrow M(t) = e^{\mu(e^t - 1)}$, $t \in R$

Q: Find $E(x)$, $E(x^2)$, σ^2, σ

$X \sim \text{poisson}(\mu)$

$E(e^{tX})$ + Taylor formula for exponential function

$M(t) = e^{\mu(e^t - 1)}$

$M'(t) = M(t) \cdot [\mu e^t]$

$M'(0) = M(0) \cdot [\mu e^0] = \mu$

$M''(t) = M(t) \cdot [\mu e^t] + [\mu e^t] M'(t)$

$M''(0) = M(0) \cdot [\mu e^0] + [\mu e^0] M'(0)$

$\mu + \mu \cdot \mu = \mu + \mu^2$

$E(x) = M'(0) = \mu$

$E(x^2) = M''(0) = \mu + \mu^2$

$Var(x) = (\mu + \mu^2) - (\mu)^2 = \mu$

standard deviation of $X = \sigma = \sqrt{\mu}$

check Ex 1, Ex 2, Ex 3

3.3 The Gamma and the chi-squared Distributions

Def :

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} \, dy \qquad \text{Gamma function}$$

Note: $\Gamma(\alpha)$ exisits and positive for all $\alpha > 0$

Note 1.

$$\Gamma(1) = \int_0^\infty y^{1-1} e^{-y} dy = \int_0^\infty e^{-y} dy = \frac{e^{-y}}{-1}\Big|_0^\infty = e^{-y}\Big|_\infty^0$$

$$= 1 - 0 = 1$$

$$\boxed{\Gamma(1) = 1}$$

Note:

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy \qquad (\alpha > 1)$$

$f = y^{\alpha-1}$ $\quad dg = e^{-y} dy$

$df = (\alpha-1) y^{\alpha-2} dy \quad g = -e^{-y}$

$$\Gamma(\alpha) = f.g\Big|_0^\infty - \int_0^\infty g\, df$$

$$\Gamma(\alpha) = \left[(y^{\alpha-1})(-e^{-y})\right]\Big|_0^\infty - \int_0^\infty (-e^{-y})(\alpha-1) y^{\alpha-2} dy$$

$$= [0 - 0(-1)] + \int_0^\infty (\alpha-1) y^{\alpha-2} e^{-y} dy$$

$$= (\alpha-1)\int_0^\infty y^{(\alpha-1)-1} e^{-y} dy = \boxed{\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)} \qquad \alpha > 1$$

Note 3

$\alpha = 2, 3, 4, \ldots$

$\Gamma(2) = (1)\,\Gamma(1) = 1 = 1!$

$\Gamma(3) = (2)\,\Gamma(2) = (2)(1) = 2!$

$\Gamma(4) = (3)\,\Gamma(3) = (3)(2)(1) = 3!$

$$\boxed{\Gamma(\alpha) = (\alpha - 1)! \quad \alpha = 2, 3, 4, \ldots}$$

Def $0! = 1$

Def $X \sim \text{Gamma}(\alpha, \beta)$

In words, X has a gamma distribution with parameters α & β, if:

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\beta}} & , x > 0 \\ 0 & , \text{else} \end{cases}$$

for $\alpha > 0$, $\beta > 0$

Note: $f(x) \geq 0 \quad \forall x \in \mathbb{R}$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\beta}}\,dx$$

$y = \frac{x}{\beta} \qquad dy = \frac{1}{\beta}\,dx$

$x = 0 \Rightarrow y = 0$

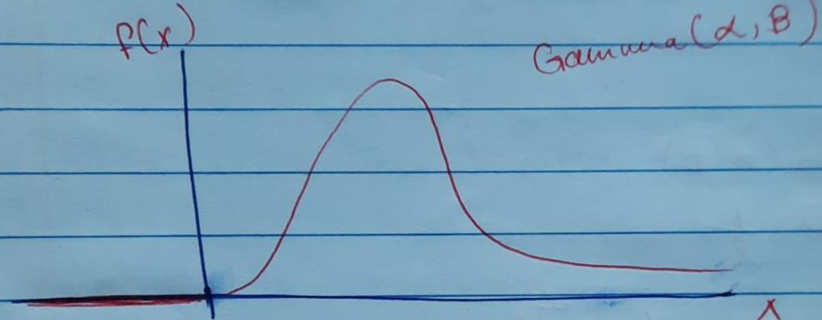
$x \to \infty \Rightarrow y \to \infty$

$\Rightarrow$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} (y\beta)^{\alpha-1} e^{-y} \beta \, dy$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \left(\int_0^{\infty} y^{\alpha-1} e^{-y} dy \right)$$

$$= \frac{\Gamma(\alpha)\beta^{\alpha}}{\Gamma(\alpha)\beta^{\alpha}}$$



<u>Theorem</u> $X \sim \text{Gamma}(\alpha, \beta)$

$$M(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} dx$$

$$M(t) = \frac{1}{(1-\beta t)^{\alpha}}, \quad t < \frac{1}{\beta}$$

<u>Note:</u>

$$M(t) = (1-\beta t)^{-\alpha}, \quad t < \frac{1}{\beta}$$
$$M'(t) = -\alpha(1-\beta t)^{-\alpha-1}(-\beta)$$
$$= \alpha\beta(1-\beta t)^{-\alpha-1}$$
$$M''(t) = \alpha\beta(-\alpha-1)(1-\beta t)^{-\alpha-2}.(-\beta)$$
$$= \alpha\beta(\alpha+1)\beta(1-\beta)^{-\alpha-2}$$

$\rightarrow$

$E(x) = \mu = \mu'(0) = \alpha\beta(1-0)^{-\alpha-1} = \alpha\beta$

$E(x^2) = \mu''(0) = \alpha\beta(\alpha+1)\beta(1-0)^{-\alpha-2}$

$= \alpha\beta(\alpha+1)\beta = \alpha(\alpha+1)\beta^2$

$Var(X) = \sigma^2 = E(x^2) - (E(x))^2$

$\alpha(\alpha+1)\beta^2 - (\alpha\beta)^2$

$= \cancel{\alpha^2\beta^2} + \alpha\beta^2 - \cancel{\alpha^2\beta^2} = \boxed{\alpha\beta^2}$

Special Case

$X \sim Gamma(\alpha, \beta)$

$\alpha = 1 \quad \beta = \frac{1}{\lambda}$

$$f(x) = \begin{cases} \frac{1}{\Gamma(1)(\frac{1}{\lambda})^1} \cdot x^{1-1} e^{\frac{-x}{\frac{1}{\lambda}}} & , x > 0 \\ 0 & , else \end{cases}$$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x > 0 \\ 0 & , else \end{cases}$$

$X \sim$ Exponential (λ)

$E(X) = \frac{1}{\lambda} \qquad Var(x) = \frac{1}{\lambda^2}$

$M(t) = \frac{1}{(1-\frac{t}{\lambda})}, \quad t < \lambda$

$\Rightarrow$

<u>Def</u> $X \sim \chi^2(r)$

In words, X has chi-squared distribution with degrees of freedom $df = r$ if:

$X \sim \text{Gamma}(\alpha, \beta)$

$\alpha = \frac{r}{2}$, $\beta = 2$

Note:-

$M(t) = \frac{1}{(1-2t)^{\frac{r}{2}}}$, $t < \frac{1}{2}$

$E(X) = r$, $var(X) = 2r$

check Q3.36

χ^2 table

<u>Example 6</u>

$X \sim \text{Gamma}(\alpha, \beta)$

$pr(a < X < b) = ?$

$Y = \frac{2X}{\beta} \sim \chi^2(r)$

$r = 2\alpha$

$X \sim \text{Gamma}(\alpha, \beta)$

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{\frac{-x}{\beta}} & , x > 0 \\ 0 & , \text{else} \end{cases}$$

$G(y) = pr(Y \leq y) = pr\left(\frac{2X}{\beta} \leq y\right)$

$= pr\left(X \leq \frac{\beta y}{2}\right) = \int_0^{\beta y/2} f(x)\,dx$, $y > 0$

$\Rightarrow$

$$g(y) = \frac{d}{dy}\int_0^{\beta y/2} f(x)\,dx = f\left(\frac{\beta y}{2}\right)\cdot\frac{\beta}{2} - f(0)\cdot 0 \quad , y>0$$

$$0 \quad , \text{else}$$

$$g(y) = \frac{\beta}{2} f\left(\frac{\beta}{2} y\right) \quad , y>0$$

$$= \frac{\beta}{2}\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\left(\frac{\beta y}{2}\right)^{\alpha-1} e^{-\frac{\left(\frac{\beta y}{2}\right)}{\beta}} \quad , y>0$$

$$= \left(\frac{\beta}{2}\right)^{\alpha}\cdot\frac{1}{\Gamma(\alpha)}\cdot\frac{1}{\beta^{\alpha}}\, y^{\alpha-1} e^{\frac{-y}{2}} \quad , y>0$$

$$= \frac{1}{\Gamma(\alpha)2^{\alpha}}\, y^{\alpha-1} e^{\frac{-y}{2}} \quad , y>0$$

$Y \sim Gamma(\alpha, 2)$

$\boxed{Y \sim \chi^2(r) ,\ r = 2\alpha}$

$X \sim Gamma(\alpha, \beta)$

$Y = \frac{2X}{\beta}$ $\quad Y \sim \chi^2(r) \quad r = 2\alpha$

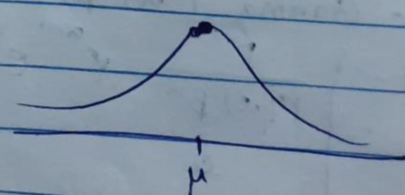
3.4 The normal distribution

$X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \quad , x \in \mathbb{R}$$

$E(X) = \mu$

$Var(X) = \sigma^2$



$$Z = \frac{X-\mu}{\sigma}$$

$\Rightarrow Z \sim N(0,1)$

$E(Z) = 0$

$Var(Z) = 1$

Question

$$f(x) = \frac{1}{\sqrt{2\pi}\,6} e^{-\frac{(x-\mu)^2}{26^2}}, \quad x \in R$$

$\mu \in R, \ 6 > 0$

why $f(x)$ is a p.d.f?

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,6} e^{-\frac{(x-\mu)^2}{26^2}} dx$$

$\frac{x-\mu}{6} = z$

$dx = 6\,dz$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,6} e^{-\frac{1}{2}z^2} 6\,dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = I$$

$$I^2 = \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} d\cancel{z}\right)^2$$

$$= \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw\right)\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}(w^2+u^2)} dw\,du$$

$r^2 = w^2 + u^2$

$r\,dr\,d\theta = dw\,du$

$$I^2 = \int_0^{2\pi}\int_0^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}r^2} r\,dr\,d\theta = \left(\int_0^{2\pi} \frac{1}{2\pi} d\theta\right)\left(\int_0^{\infty} r e^{-\frac{r^2}{2}} dr\right)$$

$$= (1) \int_0^{\infty} \frac{1}{2} e^{-\frac{y}{2}} dy$$

$$\boxed{\begin{matrix} r^2 = y \\ 2r\,dr = dy \end{matrix}}$$

$$= \frac{1}{2} \frac{e^{-\frac{y}{2}}}{-\frac{1}{2}} \Big|_0^{\infty} = e^{-\frac{y}{2}} \Big|_{\infty}^{0} = \boxed{1}$$

$$\left(\int_{-\infty}^{\infty} f(x)\,dx\right)^2 = 1$$

But $f(x) > 0 \quad \forall x \in \mathbb{R}$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}$$

Theorem

$X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}$$

$$\Rightarrow M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}, \quad t \in \mathbb{R}$$

Question

$X \sim N(\mu, \sigma^2)$

$M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}, \quad t \in \mathbb{R}$

$M'(t) = M(t)\left[\mu + \sigma^2 t\right]$

$M''(t) = M(t)\left[\sigma^2\right] + \left[\mu + \sigma^2 t\right] M'(t)$

$E(x) = M'(0) = M(0).\left[\mu + 0\right]$

$1.\left[\mu\right] = \mu$

$E(x^2) = M''(0) = M(0)\left[\sigma^2\right] + \left[\mu + 0\right] M'(0)$

$\sigma^2 + \mu . \mu$

$= \sigma^2 + \mu^2$

$Var(x) = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$

Theorem 1

$X \sim N(\mu, \sigma^2)$

$Z = \frac{X - \mu}{\sigma}$

$\Rightarrow Z \sim N(0, 1)$

Proof

$$G(z) = pr(Z \leq z) = \int_{-\infty}^{z} g(t)\, dt$$

او بعد التفاضل
ابسط

$$= pr\left(\frac{X - \mu}{\sigma} \leq z\right)$$

$$= pr(X - \mu \leq \sigma z) = pr(X \leq \mu + \sigma z)$$

$$= \int_{-\infty}^{\mu + \sigma z} f(x)\, dx$$

الحد الادنى

$$g(z) = f(\mu + \sigma z) \cdot \sigma - \lim_{c \to -\infty} f(c) \cdot c'$$

$$= \sigma \cdot \frac{1}{\sqrt{2\pi}\, \sigma} e^{-\frac{(\mu + \sigma z - \mu)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad z \in R$$

Theorem 2

$X \sim N(\mu, \sigma^2)$

$W = \left(\frac{X - \mu}{\sigma}\right)^2$

$\Rightarrow W \sim \chi^2(1)$

$X \sim N(\mu, \sigma^2)$

$W \sim \text{Gamma}\left(\frac{1}{2}, 2\right)$

$W \sim \chi^2(1)$

Note: $X \sim N(\mu, \sigma^2)$, $Z = \frac{X-\mu}{\sigma}$, $Z \sim N(0,1)$

$$p(a \le Z \le b) = p(Z \le b) - p(Z \le a)$$

$$\Phi(b) - \Phi(a)$$

$$\Phi(Z) = \int_{-\infty}^{Z} \frac{1}{\sqrt{2\pi}} e^{\frac{-w^2}{2}} \, dw$$

$\boxed{\text{Page 548 Table III}} \longrightarrow$ section 3.4

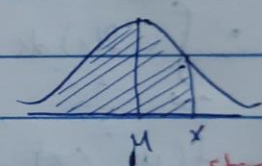
Note: $X \sim \text{Gamma}(\alpha, \beta)$, $Y = \frac{2X}{\beta}$, $Y \sim \chi^2(2\alpha)$

$$pr(a \le x \le b) = pr(c \le y \le d)$$

$\boxed{\text{page 547 Table II}} \longrightarrow$ section 3.3

Recall $X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} , \quad x \in \mathbb{R}$$



$$F(x) = \int_{-\infty}^{x} f(t) \, dt$$

$$M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$E(x) = \mu \qquad var(x) = \sigma^2$

standard normal

$\mu = 0$, $\sigma = 1$ $\quad Z \sim N(0, 1)$

$$g(Z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-Z^2}{2}} , \quad Z \in \mathbb{R}$$

$$\Phi(Z) = G(Z) = \int_{-\infty}^{Z} \frac{1}{\sqrt{2\pi}} e^{\frac{-w^2}{2}} \, dw$$

$X \sim N(\mu, \sigma^2)$	$X \sim N(\mu, \sigma^2)$
$Z = \frac{X-\mu}{\sigma}$	$W = \left(\frac{X-\mu}{\sigma}\right)^2$
$\Rightarrow Z \sim N(0,1)$	$\Rightarrow W \sim \chi^2(1)$

3.5 Bivariate Normal Distribution

Def:

X and Y are called bivariate normal random variables with parameters $\mu_x, \mu_y, \sigma_x > 0, \sigma_y > 0, \rho_{xy}$ if we the following density:

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho_{xy}^2}} e^{-\frac{q}{2}}, \quad (x,y) \in \mathbb{R}^2$$

$$q = \frac{1}{1-\rho_{xy}^2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho_{xy}\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]$$

assuming: $\mu_x \in \mathbb{R}, \mu_y \in \mathbb{R}, \sigma_x > 0, \sigma_y > 0$ فش داعي نحفظها

$-1 < \rho_{xy} < 1$

Theorem:

let X, Y be bivariate normal random variables with parameters $\mu_x, \mu_y, \sigma_x, \sigma_y, \rho_{xy}$

1. $f(x,y)$ is a p.d.f
2. $X \sim N(\mu_x, \sigma_x^2)$
   $Y \sim N(\mu_y, \sigma_y^2)$
3. $\rho_{xy} = \frac{Cov(X,Y)}{\sigma_x\sigma_y} = \frac{E(XY) - \mu_x\mu_y}{\sigma_x\sigma_y}$
4. $Y/X=x \sim N\left(\mu_y + \rho_{xy}\frac{\sigma_y}{\sigma_x}(x-\mu_x), \sigma_y^2(1-\rho_{xy}^2)\right)$
   $X/Y=y \sim N\left(\mu_x + \rho_{xy}\frac{\sigma_x}{\sigma_y}(y-\mu_y), \sigma_x^2(1-\rho_{xy}^2)\right)$

→

⑤ $M(t_1, t_2) = e^{\mu_x t_1 + \mu_y t_2 + \frac{1}{2}\sigma_x^2 t_1^2 + \frac{1}{2}\sigma_y^2 t_2^2 + \rho_{xy}\sigma_x \sigma_y t_1 t_2}$ $\quad (t_1, t_2) \in \mathbb{R}^2$

فش داعي تحفظها

Proof: Not required

Theorem:

X, Y bivariate normal random variables

X, Y independent $\Leftrightarrow$ X, Y uncorrelated

proof: Not required.

Ex 1 page 149

X_1: height of husband.

X_2: " " wife.

$$\left[\begin{array}{lll} X_1, X_2 \text{ bivariate normal.} & & \\ \mu_1 = 5.8 & \sigma_1 = 0.2 & \\ \mu_2 = 5.3 & \sigma_2 = 0.2 & \rho = 0.6 \end{array}\right]$$

① $pr(5.28 < X_2 < 5.92)$

$= pr\left(\frac{5.28 - 5.3}{0.2} < \frac{X_2 - \mu_2}{\sigma_2} < \frac{5.92 - 5.3}{0.2}\right)$ $\quad X_2 \sim N(\mu_2, \sigma_2^2)$

$= pr(-0.1 < Z < 3.1)$ $\quad \frac{X_2 - \mu_2}{\sigma_2} \sim N(0,1)$

$= \Phi(3.1) - \Phi(-0.1)$

$= \Phi(3.1) - (1 - \Phi(0.1))$

$0.999 - (1 - 0.54) = 0.539$

$\rightarrow$

② $pr(5.28 < X_2 < 5.92 / X_1 = 6.3)$

$X_2 / X_1 = x \sim N\left(\mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x - \mu_1),\ \sigma_2^2(1 - \rho^2)\right)$

$X_2 / X_1 = 6.3 \sim N\left(5.3 + 0.6 \frac{0.2}{0.2}(6.3 - 5.8),\ (0.2)^2(1 - 0.6^2)\right)$

$X_2 / X_1 = 6.3 \sim N(5.6,\ (0.16)^2)$

$= pr\left(\frac{5.28 - 5.6}{0.16} < Z < \frac{5.92 - 5.6}{0.16}\right)$

$= pr(-2 < Z < 2)$

$= \Phi(2) - \Phi(-2) = \Phi(2) - (1 - \Phi(2)) = 0.954.$