Chapter 28 Fluid Dynamics

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28.1 Ideal Fluids

An ideal fluid is a fluid that is incompressible and no internal resistance to flow (zero viscosity). In addition ideal fluid particles undergo no rotation about their center of mass (irrotational). An ideal fluid can flow in a circular pattern, but the individual fluid particles are irrotational. Real fluids exhibit all of these properties to some degree, but we shall often model fluids as ideal in order to approximate the behavior of real fluids. When we do so, one must be extremely cautious in applying results associated with ideal fluids to non-ideal fluids.

28.2 Velocity Vector Field

When we describe the flow of a fluid like water, we may think of the movement of individual particles. These particles interact with each other through forces. We could then apply our laws of motion to each individual particle in the fluid but because the number of particles is very large, this would be an extremely difficult computation problem. Instead we shall begin by mathematically describing the state of moving fluid by specifying the velocity of the fluid at each point in space and at each instant in time. For the moment we will choose Cartesian coordinates and refer to the coordinates of a point in space by the ordered triple (x, y, z) and the variable t to describe the instant in time, but in principle we may chose any appropriate coordinate system appropriate for describing the motion. The distribution of fluid velocities is described by the vector values function $\bar{v}(x, y, z, t)$. This represents the velocity of the fluid at the point (x, y, z) at the instant t. The quantity $\bar{v}(x, y, z, t)$ is called the **velocity vector** field. It can be thought of at each instant in time as a collection of vectors, one for each point in space whose direction and magnitude describes the direction and magnitude of the velocity of the fluid at that point (Figure 28.1). This description of the velocity vector field of the fluid refers to fixed points in space and not to fixed moving particles in the fluid.
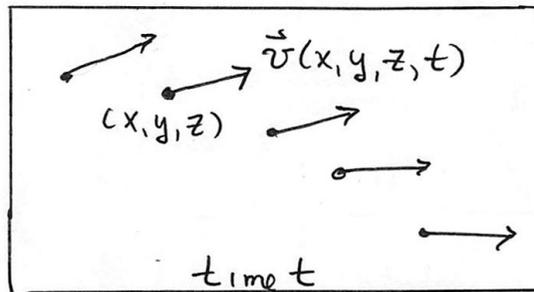
Figure 28.1: Velocity vector field for fluid flow at time t

We shall introduce functions for the pressure $P(x, y, z, t)$ and the density $\rho(x, y, z, t)$ of the fluid that describe the pressure and density of the fluid at each point in space and at

each instant in time. These functions are called **scalar fields** because there is only one number with appropriate units associated with each point in space at each instant in time.

In order to describe the velocity vector field completely we need three functions $v_x(x,y,z,t)$, $v_y(x,y,z,t)$, and $v_z(x,y,z,t)$. For a non-ideal fluid, the differential equations satisfied by these velocity component functions are quite complicated and beyond the scope of this discussion. Instead, we shall primarily consider the special case of **steady flow** of a fluid in which the velocity at each point in the fluid does not change in time. The velocities may still vary in space (non-uniform steady flow).

Let's trace the motion of particles in an ideal fluid undergoing steady flow during a succession of intervals of duration dt .
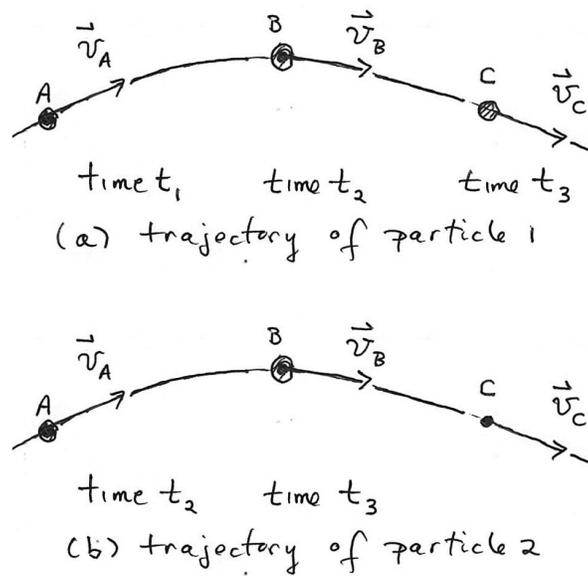
Figure 28.2: (a) trajectory of particle 1, (b) trajectory of particle 2

Consider particle 1 located at point A with coordinates (x_A, y_A, z_A) . At the instant t_1 , particle 1 will have velocity $\vec{v}(x_A, y_A, z_A)$ and move to a point B with coordinates (x_B, y_B, z_B) , arriving there at the instant $t_2 = t_1 + dt$. During the next interval, particle 1 will move to point C arriving there at instant $t_3 = t_2 + dt$, where it has velocity $\vec{v}(x_B, y_B, z_B)$ (Figure 28.2(a)). Because the flow has been assumed to be steady, at instant t_2 , a different particle, particle 2, is now located at point A but it has the same velocity $\vec{v}(x_A, y_A, z_A)$ as particle 1 had at point A and hence will arrive at point B at the end of the next interval, at the instant $t_3 = t_2 + dt$ (Figure 28.2(b)). In the third interval, particle 2, which began the interval at point B will end the interval at point C . In this way every particle that lies on the trajectory that our first particle traces out in time will follow the same trajectory. This trajectory is called a *streamline*. The particles in the fluid will not

have the same velocities at points along a streamline because we have not assumed that the velocity field is uniform.

28.3 Mass Continuity Equation

A set of streamlines for an ideal fluid undergoing steady flow in which there are no sources or sinks for the fluid is shown in Figure 28.3.
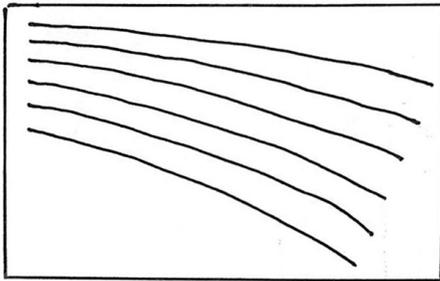
Figure 28.3: Set of streamlines for an ideal fluid flow
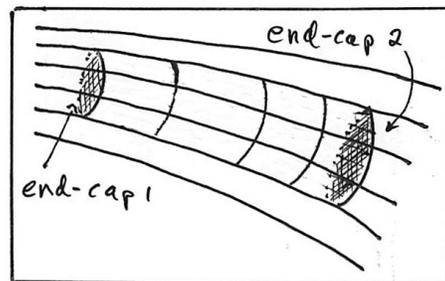
Figure 28.4: Flux Tube associated with set of streamlines

We also show a set of closely separated streamlines that form a *flow tube* in Figure 28.4 We add to the flow tube two open surface (end-caps 1 and 2) that are perpendicular to velocity of the fluid, of areas A_1 and A_2, respectively. Because all fluid particles that enter end-cap 1 must follow their respective streamlines, they must all leave end-cap 2. If our streamlines that form the tube are sufficiently close together, we can assume that the velocity of the fluid in the vicinity of each end-cap surfaces is uniform.
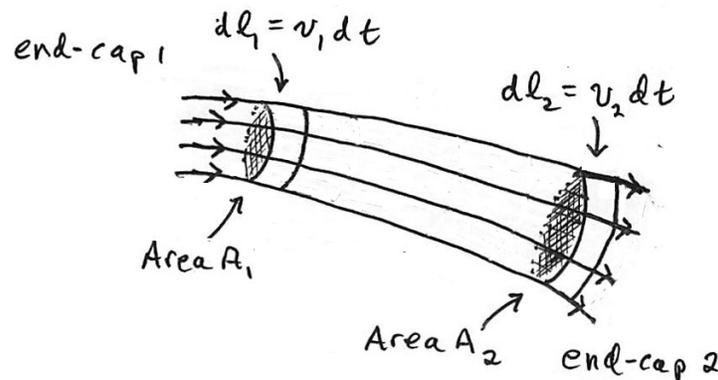
Figure 28.5: Mass flow through flux tube

Let v_1 denote the speed of the fluid near end-cap 1 and v_2 denote the speed of the fluid near end-cap 2. Let ρ_1 denote the density of the fluid near end-cap 1 and ρ_2 denote the density of the fluid near end-cap 2. The amount of mass that enters and leaves the tube in

a time interval dt can be calculated as follows (Figure 28.5): suppose we consider a small volume of space of cross-sectional area A_1 and length $dl_1 = v_1 dt$ near end-cap 1. The mass that enters the tube in time interval dt is

$$dm_1 = \rho_1 dV_1 = \rho_1 A_1 dl_1 = \rho_1 A_1 v_1 dt \ . \tag{28.3.1}$$

In a similar fashion, consider a small volume of space of cross-sectional area A_2 and length $dl_2 = v_2 dt$ near end-cap 2. The mass that leaves the tube in the time interval dt is then

$$dm_2 = \rho_2 dV_2 = \rho_2 A_2 dl_2 = \rho_2 A_2 v_2 dt \ . \tag{28.3.2}$$

An equal amount of mass that enters end-cap 1 in the time interval dt must leave end-cap 2 in the same time interval, thus $dm_1 = dm_2$. Therefore using Eqs. (28.3.1) and (28.3.2), we have that $\rho_1 A_1 v_1 dt = \rho_2 A_2 v_2 dt$. Dividing through by dt implies that

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2 \quad \text{(steady flow)} \ . \tag{28.3.3}$$

Eq. (28.3.3) generalizes to any cross sectional area A of the thin tube, where the density is ρ, and the speed is v,

$$\rho A v = \text{constant} \quad \text{(steady flow)} \ . \tag{28.3.4}$$

Eq. (28.3.3) is referred to as the *mass continuity equation for steady flow*. If we assume the fluid is incompressible, then Eq. (28.3.3) becomes

$$A_1 v_1 = A_2 v_2 \quad \text{(incompressable fluid, steady flow)} \ . \tag{28.3.5}$$

Consider the steady flow of an incompressible with streamlines and closed surface formed by a streamline tube shown in Figure 28.5. According to Eq. (28.3.5), when the spacing of the streamlines increases, the speed of the fluid must decrease. Therefore the speed of the fluid is greater entering end-cap 1 then when it is leaving end-cap 2. When we represent fluid flow by streamlines, regions in which the streamlines are widely spaced have lower speeds than regions in which the streamlines are closely spaced.

28.4 Bernoulli's Principle

Let's again consider the case of an ideal fluid that undergoes steady flow and apply energy methods to find an equation of state that relates pressure, density, and speed of the flow at different points in the fluid. Let's examine the case of a steady horizontal flow in as seen in the overhead view shown in Figure 28.6. We represent this flow by streamlines and a flow tube associated with the streamlines. Let's consider the motion of a fluid particle along one streamline passing through points A and B in Figure 28.6. The cross-

sectional area of the flow tube at point A is less than the cross-sectional area of the flow tube at point B.
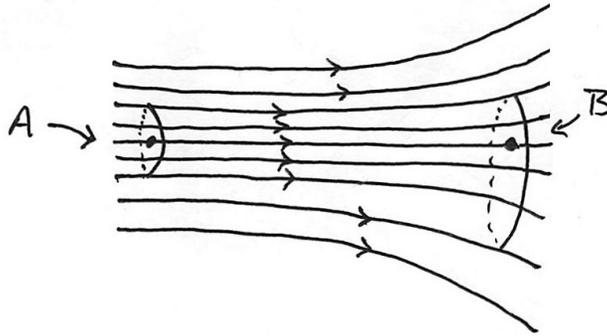
Figure 28.6 Overhead view of steady horizontal flow: in regions where spacing of the streamlines increases, the speed of the fluid must decrease

According to Eq. (28.3.5), the particle located at point A has a greater speed than a fluid particle located at point B. Therefore a particle traveling along the streamline from point A to point B must decelerate. Because the streamline is horizontal, the force responsible is due to pressure differences in the fluid. Thus, for this steady horizontal flow in regions of lower speed there must be greater pressure than in regions of higher speed.

Now suppose the steady flow of the ideal fluid is not horizontal, with the y-representing the vertical directi. The streamlines and flow tube for this steady flow are shown in Figure 28.7.
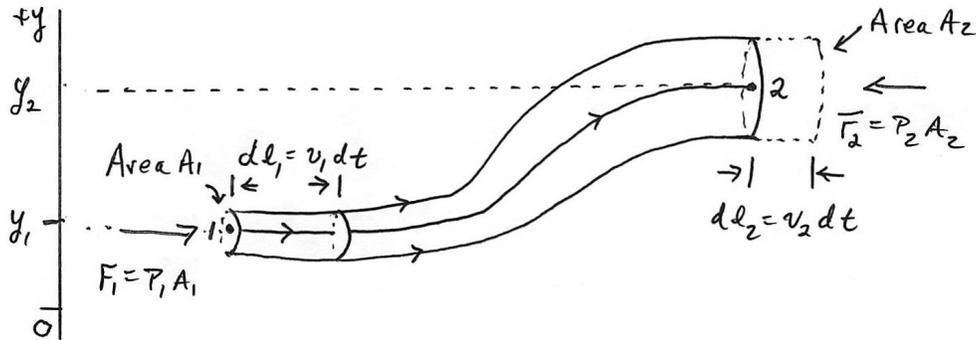
Figure 28.7: Non-horizontal steady flow

In order to determine the equation relating the pressure, speed and height difference of the tube, we shall use the work-energy theorem. We take as a system the mass contained in the flow tube shown in Figure 28.7. The external forces acting on our system are due to the pressure acting at the two ends of the flow tube and the gravitational force. Consider a streamline passing through points 1 and 2 at opposite ends of the flow tube. Let's assume that the flow tube is narrow enough such that the velocity of the fluid is

uniform on the cross-sectional areas of the tube at points 1 and 2. At point 1, denote the speed of a fluid particle by v_1, the cross-sectional area by A_1, the fluid pressure by P_1, and the height of the center of the cross-sectional area by y_1. At point 2, denote the speed of a fluid particle by v_2, the cross-sectional area by A_2, the fluid pressure by P_2, and the height of the center of the cross-sectional area by y_2.

Consider the flow tube at time t as illustrated in Figure 28.7. At the left end of the flow, in a time interval dt, a particle at point 1 travels a distance $dl_1 = v_1 dt$. Therefore a small volume $dV_1 = A_1 dl_1 = A_1 v_1 dt$ of fluid is displaced at the right end of the flow tube. In a similar fashion, at particle at point 2, travels a distance $dl_2 = v_2 dt$. Therefore a small volume of fluid $dV_2 = A_2 dl_2 = A_2 v_2 dt$ is also displaced to the right in the flow tube during the time interval dt. Because we are assuming the fluid is incompressible, by Eq.(28.3.5), these volume elements are equal, $dV \equiv dV_1 = dV_2$.

There is a force of magnitude $F_1 = P_1 A_1$ in the direction of the flow arising from the fluid pressure at the left end of the tube acting on the mass element that enters the tube. The work done displacing the mass element is then

$$dW_1 = F_1 dl_1 = P_1 A_1 dl_1 = P_1 dV \ . \tag{28.4.1}$$

There is also a force of magnitude $F_2 = P_2 A_2$ in the direction opposing the flow arising from the fluid pressure at the right end of the tube. The work done opposing the displacement of the mass element leaving the tube is then

$$dW_1 = -F_2 dl_2 = -P_2 A_2 dl_2 = -P_2 dV \ . \tag{28.4.2}$$

Therefore the external work done by the force associated with the fluid pressure is the sum of the work done at each end of the tube

$$dW^{ext} = dW_1 + dW_2 = (P_1 - P_2)dV \ . \tag{28.4.3}$$

In a time interval dt, the work done by the gravitational force is equal to

$$dW^g = -dm \, g(y_2 - y_1) = -\rho dV g(y_2 - y_1) \ . \tag{28.4.4}$$

Because we only chose the mass in the flow tube as our system, and we assumed that the fluid was ideal (no frictional losses due to viscosity) the change in the potential energy of the system is

$$dU = -W^g = \rho dV g(y_2 - y_1) \ . \tag{28.4.5}$$

At time t, the kinetic energy of the system is the sum of the kinetic energy of the small mass element of volume $dV = A_1 dl_1$ moving with speed v_1 and the rest of the mass in the flow tube. At time $t + dt$, the kinetic energy of the system is the sum of the kinetic energy of the small mass element of volume $dV = A_2 dl_2$ moving with speed v_2 and the rest of the mass in the flow tube. The change in the kinetic energy of the system is due to the mass elements at the two ends and therefore

$$dK = \frac{1}{2} dm_2 v_2^2 - \frac{1}{2} dm_1 v_1^2 = \frac{1}{2} \rho dV (v_2^2 - v_1^2) \ . \tag{28.4.6}$$

The work-energy theorem $dW^{ext} = dU + dK$ for system is then

$$(P_1 - P_2) dV = \frac{1}{2} \rho dV (v_2^2 - v_1^2) + \rho g (y_2 - y_1) dV \ . \tag{28.4.7}$$

We now divide Eq. (28.4.7) through by the volume dV and rearrange terms, yielding

$$P_1 + \rho g y_1 + \frac{1}{2} \rho v_1^2 = P_2 + \rho g y_2 + \frac{1}{2} \rho v_2^2 \ . \tag{28.4.8}$$

Because points 1 and 2 were arbitrarily chosen, we can drop the subscripts and write Eq. (28.4.8) as

$$P + \rho g y + \frac{1}{2} \rho v^2 = \text{constant} \quad \text{(ideal fluid, steady flow)} \ . \tag{28.4.9}$$

Eq. (28.4.9) is known as **Bernoulli's Equation**.

28.5 Worked Examples: Bernoulli's Equation

Example 28.1 Venturi Meter

Figure 28.8 shows a Venturi Meter, a device used to measure the speed of a fluid in a pipe. A fluid of density ρ_f is flowing through a pipe. A U-shaped tube partially filled with mercury of density ρ_{Hg} lies underneath the points 1 and 2.
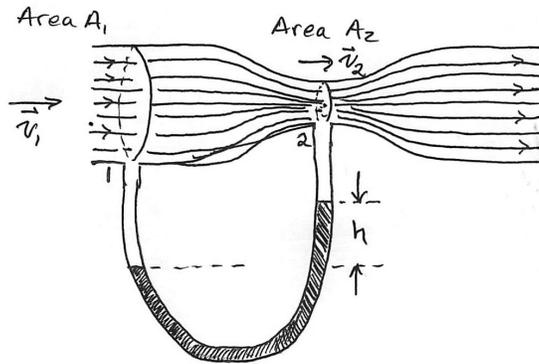
Figure 28.8: Venturi Meter

The cross-sectional areas of the pipe at points 1 and 2 are A_1 and A_2 respectively. Determine an expression for the flow speed at the point 1 in terms of the cross-sectional areas A_1 and A_2, and the difference in height h of the liquid levels of the two arms of the U-shaped tube.

Solution:
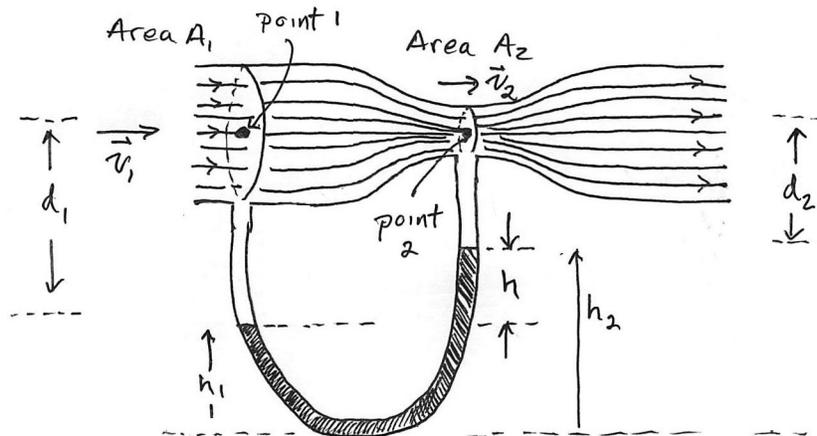
Figure 28.8: Coordinate system for Venturi tube

We shall assume that the pressure and speed are constant in the cross-sectional areas A_1 and A_2. We also assume the fluid is incompressible so the density ρ_f is constant throughout the tube. The two points 1 and 2 lie on the streamline passing through the midpoint of the tube so they are at the same height. Using $y_1 = y_2$ in Eq. (28.4.8), the pressure and flow speeds at the two points 1 and 2 are related by

$$P_1 + \frac{1}{2}\rho_f v_1^2 = P_2 + \frac{1}{2}\rho_f v_2^2 \; . \tag{28.4.10}$$

We can rewrite Eq. (28.4.10) as

$$P_1 - P_2 = \frac{1}{2}\rho_f(v_2^2 - v_1^2) \ . \tag{28.4.11}$$

Let h_1 and h_2 denote the heights of the liquid level in the arms of the U-shaped tube directly beneath points 1 and 2 respectively. Pascal's Law relates the pressure difference between the two arms of the U-shaped tube according to in the left arm of the U-shaped tube according to

$$P_{bottom} = P_1 + \rho_f g d_1 + \rho_{Hg} g h_1 \ . \tag{28.4.12}$$

In a similar fashion, the pressure at point 2 is given by

$$P_{bottom} = P_2 + \rho_f g d_2 + \rho_{Hg} g h_2 \ . \tag{28.4.13}$$

Therefore, setting Eq. (28.4.12) equal to Eq. (28.4.13), we determine that the pressure difference on the two sides of the U-shaped tube is

$$P_1 - P_2 = \rho_f g(d_2 - d_1) + \rho_{Hg} g(h_2 - h_1). \tag{28.4.14}$$

From Figure 28.8, $d_2 + h_2 = d_1 + h_1$, therefore $d_2 - d_1 = h_1 - h_2 = -h$. We can rewrite Eq. (28.4.14) as

$$P_1 - P_2 = (\rho_{Hg} - \rho_f)gh. \tag{28.4.15}$$

Substituting Eq. (28.4.11) into Eq. (28.4.15) yields

$$\frac{1}{2}\rho_f(v_2^2 - v_1^2) = (\rho_{Hg} - \rho_f)gh \ . \tag{28.4.16}$$

The mass continuity condition (Eq.(28.3.5)) implies that $v_2 = (A_1 / A_2)v_1$ and so we can rewrite Eq. (28.4.16) as

$$\frac{1}{2}\rho_f((A_1 / A_2)^2 - 1)v_1^2 = (\rho_{Hg} - \rho_f)gh \ . \tag{28.4.17}$$

We can now solve Eq. (28.4.17) for the speed of the flow at point 1;

$$v_1 = \sqrt{\frac{2(\rho_{Hg} - \rho_f)gh}{\rho_f((A_1 / A_2)^2 - 1)}} \ . \tag{28.4.18}$$

Example 28.2 Water Pressure

A cylindrical water tower of diameter $3.0\,\text{m}$ supplies water to a house. The level of water in the water tower is $35\,\text{m}$ above the point where the water enters the house through a pipe that has an inside diameter $5.1\,\text{cm}$. The intake pipe delivers water at a maximum rate of $2.0 \times 10^{-3}\,\text{m}^3 \cdot \text{s}^{-1}$. The pipe is connected to a narrower pipe leading to the second floor that has an inside diameter $2.5\,\text{cm}$. What is the pressure and speed of the water in the narrower pipe at a point that is a height $5.0\,\text{m}$ above the level where the pipe enters the house?
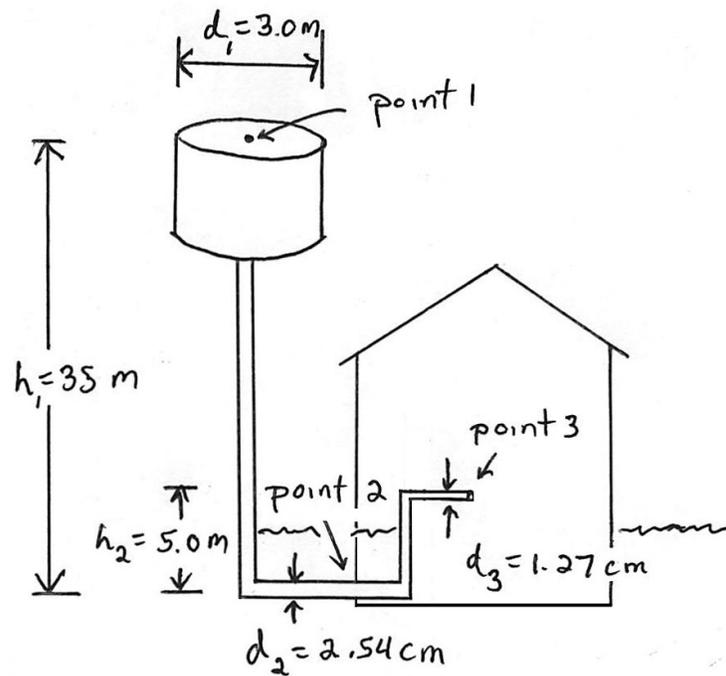
Figure 28.9: Example 28.2 (not to scale)

Solution: We shall assume that the water is an ideal fluid and that the flow is a steady flow and that the level of water in the water tower is constantly maintained. Let's choose three points, point 1 at the top of the water in the tower, point 2 where the water just enters the house, and point 3 in the narrow pipe at a height $h_2 = 5.0$ m above the level where the pipe enters the house.

We begin by applying Bernoulli's Equation to the flow from the water tower at point 1, to where the water just enters the house at point 2. Bernoulli's equation (Eq. (28.4.8)) tells us that

$$P_1 + \rho g y_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho g y_2 + \frac{1}{2}\rho v_2^2 \ . \tag{28.4.19}$$

We assume that the speed of the water at the top of the tower is negligibly small due to the fact that the water level in the tower is maintained at the same height and so we set $v_1 = 0$. The pressure at point 2 is then

$$P_2 = P_1 + \rho g(y_1 - y_2) - \frac{1}{2}\rho v_2{}^2 \ .$$

(28.4.20)

In Eq. (28.4.20) we use the value for the density of water $\rho = 1.0 \times 10^3 \text{ kg} \cdot \text{m}^{-3}$, the change in height is $(y_1 - y_2) = 35 \text{ m}$, and the pressure at the top of the water tower is $P_1 = 1 \text{ atm}$. The rate R that the water flows at point 1 satisfies $R = A_1 v_1 = \pi (d_1 / 2)^2 v_1$. Therefore, the speed of the water at point 1 is

$$v_1 = \frac{R}{\pi(d_1 / 2)^2} = \frac{2.0 \times 10^{-3} \text{ m}^3 \cdot \text{s}^{-1}}{\pi(1.5 \text{ m})^2} = 2.8 \times 10^{-4} \text{ m} \cdot \text{s}^{-1},$$

(28.4.21)

which is negligibly small and so we are justified in setting $v_1 = 0$. Similarly the speed of the water at point 2 is

$$v_2 = \frac{R}{\pi(d_2 / 2)^2} = \frac{2.0 \times 10^{-3} \text{ m}^3 \cdot \text{s}^{-1}}{\pi(2.5 \times 10^{-2} \text{ m})^2} = 1.0 \text{ m} \cdot \text{s}^{-1},$$

(28.4.22)

We can substitute Eq. (28.4.21) into Eq. (28.4.22), yielding

$$v_2 = (d_1{}^2 / d_2{}^2) v_1,$$

(28.4.23)

a result which we will shortly find useful. Therefore the pressure at point 2 is

$$P_2 = 1.01 \times 10^5 \text{ Pa} + (1.0 \times 10^3 \text{ kg} \cdot \text{m}^{-3})(9.8 \text{ m} \cdot \text{s}^{-2})(35 \text{ m}) - \frac{1}{2}(1.0 \times 10^3 \text{ kg} \cdot \text{m}^{-3})(1.0 \text{ m} \cdot \text{s}^{-1})^2$$

$$P_2 = 1.01 \times 10^5 \text{ Pa} + 3.43 \times 10^5 \text{ Pa} - 5.1 \times 10^2 \text{ Pa} = 4.4 \times 10^5 \text{ Pa}$$

.

(28.4.24)

The dominant contribution is due to the height difference between the top of the water tower and the pipe entering the house. The quantity $(1/2)\rho v_2{}^2$ is called the **dynamic pressure** due to the fact that the water is moving. The amount of reduction in pressure due to the fact that the water is moving at point 2 is given by

$$\frac{1}{2}\rho v_2^2 = \frac{1}{2}(1.0 \times 10^3 \text{ kg} \cdot \text{m}^{-3})(1.0 \text{ m} \cdot \text{s}^{-1})^2 = 5.1 \times 10^3 \text{ Pa},$$

(28.4.25)

which is much smaller than the contributions from the other two terms.

We now apply Bernoulli's Equation to the points 2 and 3,

$$P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2 = P_3 + \frac{1}{2}\rho v_3^2 + \rho g y_3 \ . \tag{28.4.26}$$

Therefore the pressure at point 3 is

$$P_3 = P_2 + \frac{1}{2}\rho(v_2^2 - v_3^2) + \rho g(y_2 - y_3) \ . \tag{28.4.27}$$

The change in height $y_2 - y_3 = -5.0 \ \text{m}$. The speed of the water at point 3 is

$$v_3 = \frac{R}{\pi(d_3 / 2)^2} = \frac{2.0 \times 10^{-3} \ \text{m}^3 \cdot \text{s}^{-1}}{\pi(1.27 \times 10^{-2} \ \text{m})^2} = 3.9 \ \text{m} \cdot \text{s}^{-1}, \tag{28.4.28}$$

Then the pressure at point 3 is

$$P_3 = (4.4 \times 10^5 \ \text{Pa}) + \frac{1}{2}(1.0 \times 10^3 \ \text{kg} \cdot \text{m}^{-3})((1.0 \ \text{m} \cdot \text{s}^{-1})^2 - (3.9 \ \text{m} \cdot \text{s}^{-1})^2)$$
$$-(1.0 \times 10^3 \ \text{kg} \cdot \text{m}^{-3})(9.8 \ \text{m} \cdot \text{s}^{-2})(5.0 \ \text{m})$$

$$\tag{28.4.29}$$

$$= (4.4 \times 10^5 \ \text{Pa}) - (7.1 \times 10^3 \ \text{Pa}) - 4.9 \times 10^4 \ \text{Pa}$$
$$= 3.8 \times 10^5 \ \text{Pa}$$

Because the speed of the water at point 3 is much greater than at point 2, the dynamic pressure contribution at point 3 is much larger than at point 2.