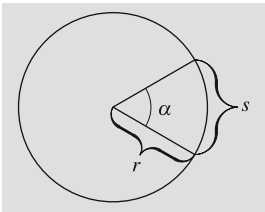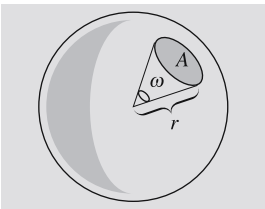# A. Mathematics

A.1 Geometry

Units of Angle and Solid Angle. *Radian* is the angular unit most suitable for theoretical studies. One radian is the angle subtended by a circular arc whose length equals the radius. If r is the radius of a circle and s the length of an arc, the arc subtends an angle

$$\alpha = s/r \ .$$

Since the circumference of the circle is $2\pi r$, we have

$$2\pi \ \text{rad} = 360° \quad \text{or} \quad 1 \ \text{rad} = 180°/\pi \ .$$

In an analogous way we can define a *steradian*, a unit of solid angle, as the solid angle subtended by a unit area on the surface of a unit sphere as seen from the centre. An area A on the surface of a sphere with radius r subtends a solid angle
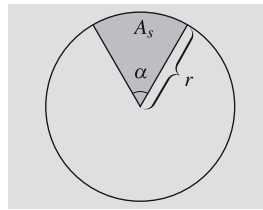
$$\omega = A/r^2 \ .$$

Since the area of the sphere is $4\pi r^2$, a full solid angle equals 4π steradians.

Circle

Area $A = \pi r^2$.

Area of a sector $A_{\text{s}} = \dfrac{1}{2}\alpha r^2$.

Sphere

Area $A = 4\pi r^2$.

Volume $V = \dfrac{4}{3}\pi r^3$.

Volume of a sector $V_{\text{s}} = \dfrac{2}{3}\pi r^2 h = \dfrac{2}{3}\pi r^3(1 - \cos\alpha)$

$$= V_{\text{sphere}} \ \text{hav} \ \alpha \ .$$

Area of a segment $A_{\text{s}} = 2\pi r h = 2\pi r^2(1 - \cos\alpha)$

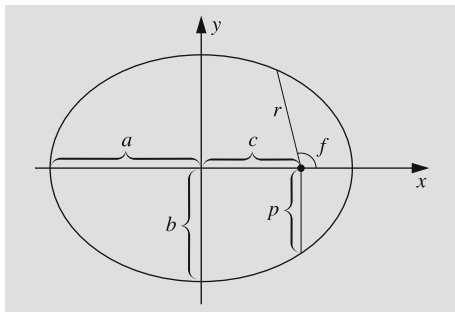$$= A_{\text{sphere}} \ \text{hav} \ \alpha \ .$$
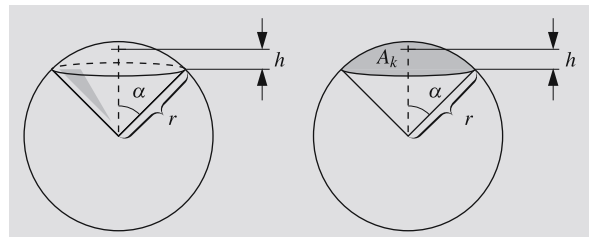
A.2 Conic Sections

As the name already says, conic sections are curves obtained by intersecting circular cones with planes.

Hyperbola. Equations in rectangular and polar coordinates

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 , \quad r = \frac{p}{1 + e \cos f} .$$
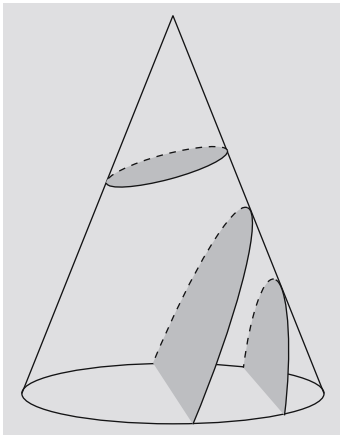
Ellipse. Equation in rectangular coordinates

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 .$$

Eccentricity $e > 1$.

Semi-minor axis $b = a\sqrt{e^2 - 1}$.

Parameter $p = a(e^2 - 1)$.

Asymptotes $y = \pm\frac{b}{a}x$.

Parabola. Parabola is a limiting case between the previous ones; its eccentricity is $e = 1$.

Equations

$$x = -ay^2 , \quad r = \frac{p}{1 + \cos f} .$$

$a = $ the semimajor axis,

$b = $ the semiminor axis $b = a\sqrt{1 - e^2}$,

$e = $ eccentricity $0 \le e < 1$.

Distance of the foci from the centre $c = ea$.

Parameter (semilatus rectum) $p = a(1 - e^2)$.

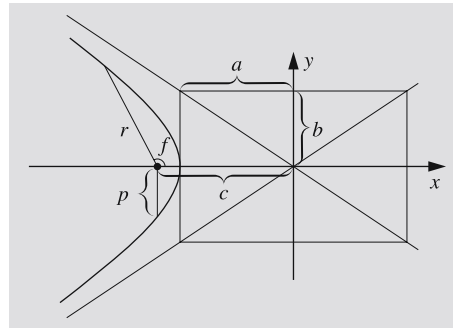Area $A = \pi ab$.

Equation in polar coordinates

$$r = \frac{p}{1 + e \cos f} ,$$

where the distance r is measured from one focus, not from the centre.

When $e = 0$, the curve becomes a circle.
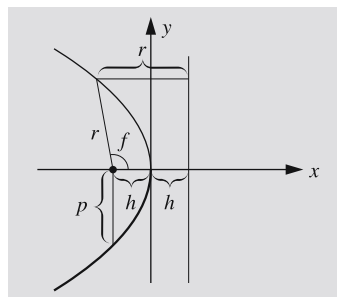
Distance of the focus from the apex $h = 1/4\,a$.

Parameter $p = 1/2\,a$.

A.3 Taylor Series

Let us consider a differentiable real-valued function of one variable $f : \mathbf{R} \to \mathbf{R}$. The tangent to the graph of the function at x_0 is

$$y = f(x_0) + f'(x_0)(x - x_0) \,,$$

where $f'(x_0)$ is the derivative of f at x_0. Now, if x is close to x_0, the graph of the tangent at x will not be very far from the graph of the function itself. Thus, we can approximate the function by

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \,.$$

The approximation becomes worse, the more the derivative f' varies in the interval $[x_0, x]$. The rate of change of f' is described by the second derivative f'', and so on. To improve accuracy, we have to also include higher derivatives. It can be shown that the value of the function f at x is (assuming that the derivatives exist)

$$f(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$+ \frac{1}{2} f''(x_0)(x - x_0)^2 + \dots$$

$$+ \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n + \dots \,,$$

where $f^{(n)}(x_0)$ is the nth derivative at x_0 and $n!$ is the n-factorial, $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$. This expansion is called the *Taylor series* of the function at x_0.
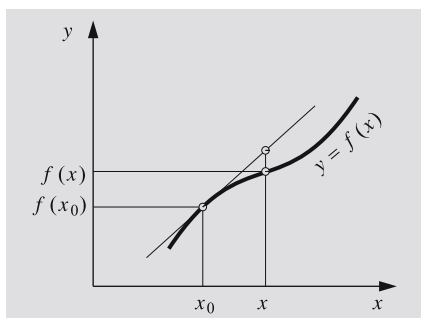
The following list gives some useful Taylor series (in all these cases we have $x_0 = 0$):

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$
$$\text{converges if } |x| < 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots$$

$$\sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \dots$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 - \dots$$

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots$$

$$\mathrm{e}^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n + \dots$$
$$\text{converges for all } x$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$
$$x \in (-1, 1]$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots \quad \text{for all } x$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots \quad \text{for all } x$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots \quad |x| < \frac{\pi}{2} \,.$$

Many problems involve small perturbations, in which case it is usually possible to find expressions having very rapidly converging Taylor expansions. The great advantage of this is the reduction of complicated functions to simple polynomials. Particularly useful are linear approximations, such as

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x \,, \quad \frac{1}{\sqrt{1+x}} \approx 1 - \frac{1}{2}x \,, \text{ etc.}$$

A.4 Vector Calculus

A *vector* is an entity with two essential properties: *magnitude* and *direction*. Vectors are usually denoted by boldface letters $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{A}$, $\boldsymbol{B}$ etc. The *sum* of the vectors

A and *B* can be determined graphically by moving the origin of *B* to the tip of *A* and connecting the origin of *A* to the tip of *B*. The vector $-A$ has the same magnitude as *A*, is parallel to *A*, but points in the opposite direction. The *difference* $A - B$ is defined as $A + (-B)$.
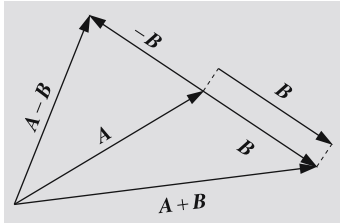
Addition of vectors satisfies the ordinary rules of commutativity and associativity,

$$A + B = B + A ,$$
$$A + (B + C) = (A + B) + C .$$
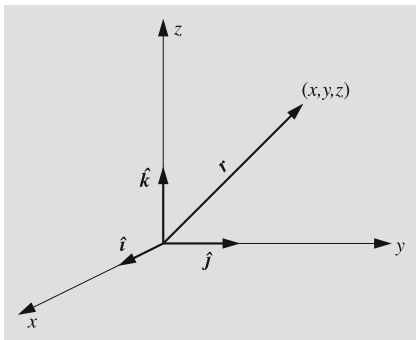
A point in a coordinate frame can be specified by giving its *position* or *radius vector*, which extends from the origin of the frame to the point. The position vector *r* can be expressed in terms of *basis vectors*, which are usually *unit vectors*, i.e. have a length of one distance unit. In a rectangular xyz-frame, we denote the basis vectors parallel to the coordinate axes by $\hat{i}$, $\hat{j}$ and $\hat{k}$. The position vector corresponding to the point (x, y, z) is then

$$r = x\hat{i} + y\hat{j} + z\hat{k} .$$

The numbers x, y and z are the *components* of *r*. Vectors can be added by adding their components. For example, the sum of

$$A = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} ,$$
$$B = b_x\hat{i} + b_y\hat{j} + b_z\hat{k} ,$$

is

$$A + B = (a_x + b_x)\hat{i} + (a_y + b_y)\hat{j} + (a_z + b_z)\hat{k} .$$

The magnitude of a vector *r* in terms of its components is

$$r = |r| = \sqrt{x^2 + y^2 + z^2} .$$

The *scalar product* of two vectors *A* and *B* is a real number (scalar)

$$A \cdot B = a_xb_x + a_yb_y + a_zb_z = |A||B| \cos(A, B) ,$$

where (A, B) is the angle between the vectors *A* and *B*. We can also think of the scalar product as the projection of, say, *A* in the direction of *B* multiplied by the length of *B*. If *A* and *B* are perpendicular, their scalar product vanishes. The magnitude of a vector expressed as a scalar product is $A = |A| = \sqrt{A \cdot A}$.

The *vector product* of the vectors *A* and *B* is a vector

$$A \times B = (a_yb_z - a_zb_y)\hat{i} + (a_zb_x - a_xb_z)\hat{j}$$
$$+ (a_xb_y - a_yb_x)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} .$$

This is perpendicular to both *A* and *B*. Its length gives the area of the parallelogram spanned by *A* and *B*. The vector product of parallel vectors is a null vector. The vector product is anti-commutative:

$$A \times B = -B \times A .$$
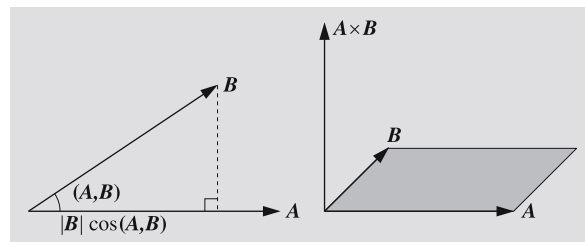
Scalar and vector products satisfy the laws of distributivity:

$$A \cdot (B + C) = A \cdot B + A \cdot C \ ,$$

$$A \times (B + C) = A \times B + A \times C \ ,$$

$$(A + B) \cdot C = A \cdot C + B \cdot C \ ,$$

$$(A + B) \times C = A \times C + B \times C \ .$$

A *scalar triple product* is a scalar

$$A \times B \cdot C = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \ .$$

Here the cross and dot can be interchanged and the factors permuted cyclically without affecting the value of the product. For example $A \times B \cdot C = B \times C \cdot A = B \cdot C \times A$, but $A \times B \cdot C = -B \times A \cdot C$.

A *vector triple product* is a vector, which can be evaluated using one of the expansions

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B) \ ,$$

$$(A \times B) \times C = B(A \cdot C) - A(B \cdot C) \ .$$

In all these products, scalar factors can be moved around without affecting the product:

$$A \cdot kB = k(A \cdot B) \ ,$$

$$A \times (B \times kC) = k(A \times (B \times C)) \ .$$

The position vector of a particle is usually a function of time $r = r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$. The velocity of the particle is a vector, tangent to the trajectory, obtained by taking the derivative of r with respect to time:
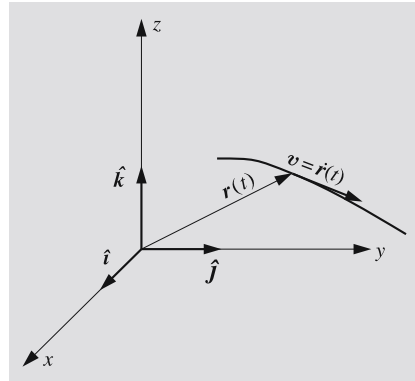
$$v = \frac{\mathrm{d}}{\mathrm{d}t} r(t) = \dot{r} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} \ .$$

The acceleration is the second derivative, $\ddot{r}$.

Derivatives of the various products obey the same rules as derivatives of products of real-valued function:

$$\frac{\mathrm{d}}{\mathrm{d}t}(A \cdot B) = \dot{A} \cdot B + A \cdot \dot{B} \ ,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(A \times B) = \dot{A} \times B + A \times \dot{B} \ .$$

When computing a derivative of a vector product, one must be careful to retain the order of the factors, since the sign of the vector product changes if the factors are interchanged.

A.5 Matrices

Assume we have a vector x with components (x, y, z). We can calculate another vector $x' = (x', y', z')$, the components of which are linear combinations of the original components:

$$x' = a_{11}x + a_{12}y + a_{13}z \ ,$$

$$y' = a_{21}x + a_{22}y + a_{23}z \ ,$$

$$z' = a_{31}x + a_{32}y + a_{33}z \ .$$

This is a linear transform that maps the vector x to a vector x'.

We can collect the coefficients to an array, called a *matrix A*:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \ .$$

A general matrix can consist of an arbitrary number of rows and columns. In this book we need only matrices operating on vectors of a three-dimensional space, and they always have three rows and columns. Two subscripts refer to the different elements of the matrix, the first one giving the row and the second one the column.

When using matrix formalism it is convenient to write vectors in the form of column vectors:

$$A = \begin{pmatrix} x \\ y \\ z \end{pmatrix} .$$

We now define that the product of a matrix and a column vector

$$x' = Ax$$

or

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

means just

$$x' = a_{11}x + a_{12}y + a_{13}z ,$$
$$y' = a_{21}x + a_{22}y + a_{23}z ,$$
$$z' = a_{31}x + a_{32}y + a_{33}z .$$

Comparing these equations we see, for example, that the first component of x' is obtained by taking the first row of the matrix, multiplying the components of the vector x by the corresponding components of that row, and finally adding the products.

This definition can easily be generalised to the product of two matrices. The elements of the matrix

$$C = AB$$

are

$$c_{ij} = \sum_k a_{ik}b_{kj} .$$

This is easy to remember by noting that we take the row i of the first factor A and the column j of the second factor B and evaluate the scalar product of the two vectors. For example

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1+2+1 & 2+1+3 & 0+1+2 \\ 0+2+2 & 0+1+6 & 0+1+4 \\ 1+4+3 & 2+2+9 & 0+2+6 \end{pmatrix}$$
$$= \begin{pmatrix} 4 & 6 & 3 \\ 4 & 7 & 5 \\ 8 & 13 & 8 \end{pmatrix} .$$

When multiplying matrices, we have to be careful with the order of the factors, because usually $AB \neq BA$. If we multiply the matrices of the previous example in the reverse order, we get quite a different result:

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 3 & 8 & 13 \end{pmatrix} .$$

A *unit matrix I* is a matrix, which has ones on its diagonal and zeros elsewhere:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

If a vector or a matrix is multiplied by a unit matrix, it will remain unchanged.

If the product of two matrices is a unit matrix, the two matrices are *inverse matrices* of each others. The inverse matrix of A is denoted by A^{-1}. It satisfies the equations

$$A^{-1}A = AA^{-1} = I .$$

In spherical astronomy we need mainly *rotation matrices*, describing the rotation of a coordinate frame. The following matrices correspond to rotations around the x, y and z axes, respectively:

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} ,$$

$$R_y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} ,$$

$$R_z(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

If the angle is $\alpha = 0$, only a unit matrix remains.

The elements of a rotation matrix can easily be determined. For example, a rotation around the x axis will leave the x coordinate unaffected, and thus the first row and column must be zeroes, except for the diagonal element, which must be one. This will leave four elements. When the angle is zero, the matrix has to reduce to a unit

matrix; thus the diagonal elements must be cosines and the other ones sines. The only problem is to decide, which of the sines will get the minus sign. This is most easily done by testing the effect of the matrix on some basis vector.

The inverse matrix of a rotation matrix corresponds to a rotation in the opposite direction. Thus it is obtained from the original matrix by replacing the angle α by $-\alpha$. The only change in the matrix is that the signs of the sines are changed.

For example, the *precession matrix* is a product of three rotation matrices. Since the matrix product is not commutative, these rotations must be carried out in the correct order.

A.6 Multiple Integrals

An integral of a function f over a surface A

$$I = \int_A f \, dA$$

can be evaluated as a double integral by expressing the surface element dA in terms of coordinate differentials. In rectangular coordinates,
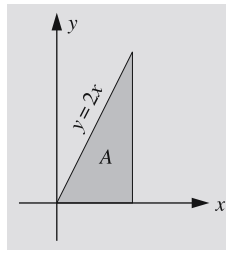
$$dA = dx \, dy$$

and in polar coordinates

$$dA = r \, dr \, d\varphi .$$

The integration limits of the innermost integral may depend on the other integration variable. For example, the function xe^y integrated over the shaded area is

$$I = \int_A xe^y \, dA = \int_{x=0}^{1} \int_{y=0}^{2x} xe^y \, dx \, dy$$

$$= \int_0^1 \left[\left.\vphantom{\int} xe^y \right|_0^{2x} \right] dx = \int_0^1 (xe^{2x} - x) \, dx$$

$$= \left. \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} - \frac{1}{2}x^2 \right|_0^1 = \frac{1}{4}(e^2 - 1) .$$

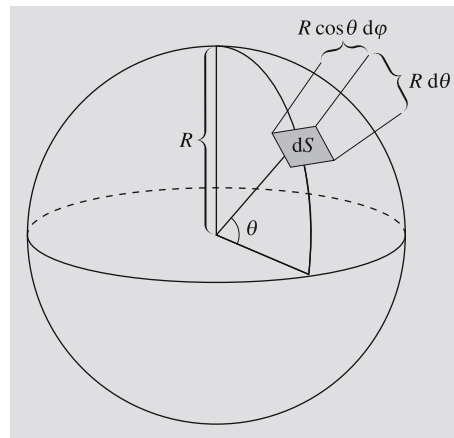The surface need not be confined to a plane. For example, the area of a sphere is
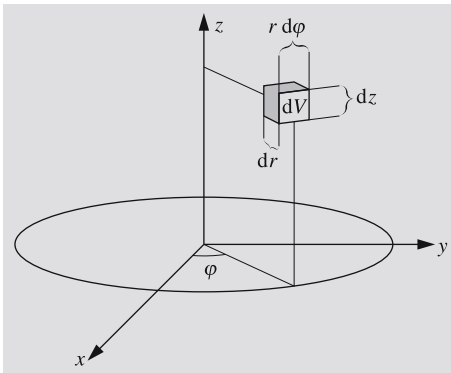
$$A = \int_S dS ,$$

where the integration is extended over the surface S of the sphere. In this case the surface element is

$$dS = R^2 \cos\theta \, d\varphi \, d\theta ,$$

and the area is

$$A = \int_{\varphi=0}^{2\pi} \int_{\theta=-\pi/2}^{\pi/2} R^2 \cos\theta \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} \left[\left. R^2 \sin\theta \right|_{-\pi/2}^{\pi/2} \right] d\varphi$$

$$= \int_0^{2\pi} 2R^2 \, d\varphi = 4\pi R^2 .$$

Similarly, a volume integral

$$I = \int_V f \, \mathrm{d}V$$

can be evaluated as a triple integral. In rectangular coordinates, the volume element $\mathrm{d}V$ is

$$\mathrm{d}V = \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \, ;$$

in cylindrical coordinates

$$\mathrm{d}V = r \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}z \, ,$$

and in spherical coordinates

$$\mathrm{d}V = r^2 \cos\theta \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}\theta$$

(θ measured from the xy plane)

or

$$\mathrm{d}V = r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}\theta$$

(θ measured from the z axis) .
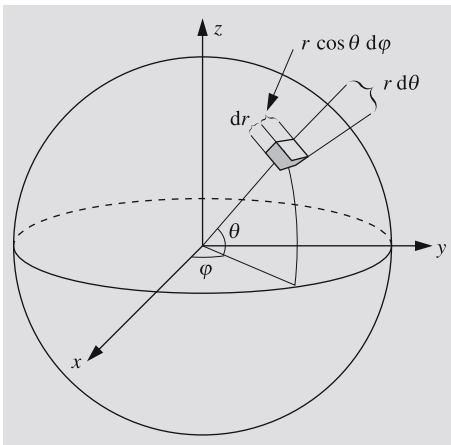
For example, the volume of a sphere with radius R is

$$V = \int_V \mathrm{d}V$$

$$= \int_{r=0}^{R} \int_{\varphi=0}^{2\pi} \int_{\theta=-\pi/2}^{\pi/2} r^2 \cos\theta \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}\theta$$

$$= \int_0^R \int_0^{2\pi} \left[\Big|_{-\pi/2}^{\pi/2} r^2 \sin\theta \right] \mathrm{d}r \, \mathrm{d}\varphi$$

$$= \int_0^R \int_0^{2\pi} 2r^2 \, \mathrm{d}r \, \mathrm{d}\varphi$$

$$= \int_0^R 4\pi r^2 \, \mathrm{d}r = \Big|_0^R \frac{4\pi r^3}{3} = \frac{4}{3}\pi R^3 \, .$$

A.7 Numerical Solution of an Equation

We frequently meet equations defying analytical solutions. Kepler's equation is a typical example. If we cannot do anything else, we can always apply some numerical method. Next we shall present two very simple methods, the first of which is particularly suitable for calculators.

Method 1: Direct Iteration. We shall write the equation as $f(x) = x$. Next we have to find an initial value x_0 for the solution. This can be done, for example, graphically or by just guessing. Then we compute a succession of new iterates $x_1 = f(x_0)$, $x_2 = f(x_1)$, and so on, until the difference of successive solutions becomes smaller than some preset limit. The last iterate x_i is the solution. After computing a few x_i's, it is easy to see if they are going to converge. If not, we rewrite the equation as $f^{-1}(x) = x$ and try again. (f^{-1} is the inverse function of f.)

As an example, let us solve the equation $x = -\ln x$. We guess $x_0 = 0.5$ and find

$$x_1 = -\ln 0.5 = 0.69 \, , \quad x_2 = 0.37 \, , \quad x_3 = 1.00 \, .$$
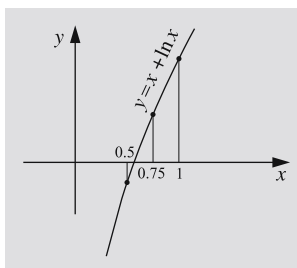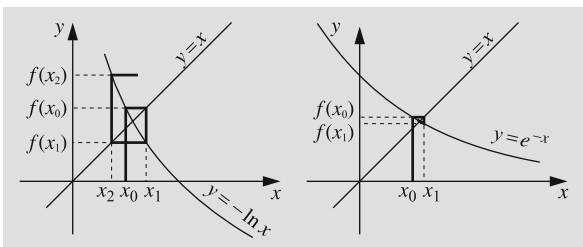
This already shows that something is wrong. Therefore we change our equation to $x = e^{-x}$ and start again:

$$x_0 = 0.5 \,,$$
$$x_1 = e^{-0.5} = 0.61 \,,$$
$$x_2 = 0.55 \,,$$
$$x_3 = 0.58 \,,$$
$$x_4 = 0.56 \,,$$
$$x_5 = 0.57 \,,$$
$$x_6 = 0.57 \,.$$

Thus the solution, accurate to two decimal places, is 0.57.

Method 2. Interval Halving. In some pathological cases the previous method may refuse to converge. In such situations we can use the foolproof method of interval halving. If the function is continuous (as most functions of classical physics are) and we manage to find two points x_1 and x_2 such that $f(x_1) > 0$ and $f(x_2) < 0$, we know that somewhere between x_1 and x_2 there must be a point x in which $f(x) = 0$. Now we find the sign of f in the midpoint of the interval, and select the half of the interval in which f changes sign. We repeat this procedure until the interval containing the solution is small enough.

We shall try also this method on our equation $x = -\ln x$, which is now written as $f(x) = 0$, where $f(x) = x + \ln x$. Because $f(x) \to -\infty$, when $x \to 0$ and $f(1) > 0$, the solution must be in the range $(0, 1)$. Since $f(0.5) < 0$, we know that $x \in (0.5, 1)$. We continue in this way:

$$f(0.75) > 0 \quad \Rightarrow \; x \in (0.5, 0.75) \,,$$

$$f(0.625) > 0 \Rightarrow \; x \in (0.5, 0.625) \,,$$

$$f(0.563) < 0 \Rightarrow \; x \in (0.563, 0.625) \,,$$

$$f(0.594) > 0 \Rightarrow \; x \in (0.563, 0.594) \,.$$

The convergence is slow but certain. Each iteration restricts the solution to an interval which is half as large as the previous one, thus improving the solution by one binary digit.