Mathematical Method

HW3

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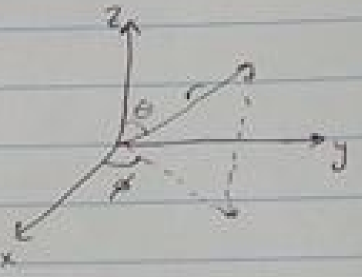
1 a $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ } where $q_1 = r$, $q_2 = \theta$, $q_3 = z$ in cylindrical coordinates

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \quad \begin{aligned} q_1 &= r \\ q_2 &= \theta \\ q_3 &= \phi \end{aligned}$$

(In spherical)



b In spherical:

$$dx = \frac{\partial x}{\partial q_1} dq_1 + \frac{\partial x}{\partial q_2} dq_2 + \frac{\partial x}{\partial q_3} dq_3$$

$$dx = \sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi$$

$$dy = \sin\theta\sin\phi\, dr + r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d\phi$$

$$dz = \cos\theta\, dr - r\sin\theta\, d\theta$$

In Cylindrical:

$$dx = -r\sin\theta\, d\theta + \cos\theta\, dr$$

$$dy = \sin\theta\, dr + r\cos\theta\, d\theta$$

$$dz = dz$$

c $d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

In Spherical:
$$\begin{aligned} d\vec{r} = &(\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k})\, dr \\ &+ (r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k})\, d\theta \\ &+ (-r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j})\, d\phi \end{aligned}$$

In Cylindrical:
$$\begin{aligned} d\vec{r} &= (-r\sin\theta\, d\theta + \cos\theta\, dr)\,\hat{i} + (\sin\theta\, dr + r\cos\theta\, d\theta)\,\hat{j} + dz\,\hat{k} \\ &= (\cos\theta\,\hat{i} + \sin\theta\,\hat{j})\, dr + (-r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j})\, d\theta + dz\,\hat{k} \end{aligned}$$

$\boxed{d}$ $ds^2 = d\vec{r} \cdot d\vec{r}$

I In Cylindrical: $ds^2 = dx^2 + dy^2 + dz^2$

$= dr^2 + r^2 d\theta^2 + dz^2$

I In spherical: $ds^2 = dx^2 + dy^2 + dz^2$

and $= dr^2 + \frac{1}{2}(2d\theta^2 + d\phi^2) r^2 - \frac{1}{2} d\phi^2 r^2 \cos(2\theta)$

$\boxed{e}$ $ds^2 = \sum_i (h_i dq_i)^2$; $h_i^2 = \frac{\partial \vec{r}}{\partial q_i} \cdot \frac{\partial \vec{r}}{\partial q_i}$ #

In Cylindrical:

$h_1^2 = h_r^2 = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 = \cos^2\phi + \sin^2\phi + 0 = 1$

$h_2^2 = h_\theta^2 = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = (-r\sin\phi)^2 + (r\cos\phi)^2 + 0 = r^2$

$h_3^2 = h_z^2 = 0 + 0 + 1 = 1$

$\Rightarrow ds^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2$

$= 1 \cdot dr^2 + r^2 d\theta^2 + dz^2$ (same as above)

In spherical:

$h_r^2 = (\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + (\cos\theta)^2 = 1$

$h_\theta^2 = (r\cos\theta\cos\phi)^2 + (r\cos\theta\sin\phi)^2 + (-r\sin\theta)^2 = r^2$

$h_\phi^2 = (-r\sin\theta\sin\phi)^2 + (r\sin\theta\cos\phi)^2 + 0 = r^2\sin^2\theta$

$\Rightarrow ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$

OR: $ds^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2 = d\vec{r} \cdot d\vec{r}$

let $\vec{r} = r_1\hat{q}_1 + r_2\hat{q}_2 + r_3\hat{q}_3$

$d\vec{r} = \frac{\partial \vec{r}}{\partial q_1} dq_1 + \frac{\partial \vec{r}}{\partial q_2} dq_2 + \frac{\partial \vec{r}}{\partial q_3} dq_3$; let $\vec{h}_i = \frac{\partial \vec{r}}{\partial q_i}$

$\Rightarrow d\vec{r} = \vec{h}_1 dq_1 + \vec{h}_2 dq_2 + \vec{h}_3 dq_3$ and $h_i^2 = \vec{h}_i \cdot \vec{h}_i = \frac{\partial \vec{r}}{\partial q_i} \cdot \frac{\partial \vec{r}}{\partial q_i}$

$ds^2 = d\vec{r} \cdot d\vec{r} = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2$

$\Rightarrow ds^2 = \sum_i (h_i dq_i)^2$; #

$\boxed{2}$ In polar coordinate: $q_1 = r$, $q_2 = \theta$

where $x = r\cos\theta$

$y = r\sin\theta$

$$dx\,dy = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr\,d\theta$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr\,d\theta$$

$$= (r\cos^2\theta + r\sin^2\theta)\,dr\,d\theta = \underline{r\,dr\,d\theta}$$

In spherical: $q_1 = r$, $q_2 = \theta$, $q_3 = \phi$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$$dx\,dy\,dz = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} dr\,d\theta\,d\phi$$

$$\Rightarrow dx\,dy\,dz = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} dr\,d\theta\,d\phi$$

$$= (r^2\sin\theta)\,dr\,d\theta\,d\phi$$

(This is the Jacobians ~~matrix~~ determinants)

3 From the previous question:

Cylindrical:

$h_r = h_1 = 1$

$h_\theta = h_2 = r$

$h_z = h_3 = 1$

Spherical:

$h_r = h_1 = 1$

$h_\theta = h_2 = r$

$h_\phi = h_3 = r\sin\theta$

Cylindrical: (let $\vec{V} = V_r\hat{r} + V_\theta\hat{\theta} + V_z\hat{z}$)

$$\vec{\nabla}\Psi = \frac{\partial\Psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\Psi}{\partial\theta}\hat{\theta} + \frac{\partial\Psi}{\partial z}\hat{z}$$

$$\vec{\nabla}\cdot\vec{V} = \frac{1}{r}\left[\frac{\partial(rV_r)}{\partial r} + \frac{\partial V_\theta}{\partial\theta} + \frac{\partial V_z}{\partial z}\right]$$

$$\vec{\nabla}\times\vec{V} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \partial r & \partial\theta & \partial z \\ V_r & rV_\theta & V_z \end{vmatrix}$$

Spherical: (let $\vec{V} = V_r\hat{r} + V_\theta\hat{\theta} + V_\phi\hat{\phi}$)

$$\vec{\nabla}\Psi = \frac{\partial\Psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\Psi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial\phi}\hat{\phi}$$

$$\vec{\nabla}\cdot\vec{V} = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}(r^2V_r) + r\frac{\partial}{\partial\theta}(\sin\theta V_\theta) + r\frac{\partial V_\phi}{\partial\phi}\right]$$

$$\vec{\nabla}\times\vec{V} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \partial r & \partial\theta & \partial\phi \\ V_r & rV_\theta & r\sin\theta V_\phi \end{vmatrix}$$

Find the laplacian:

$$\nabla^2\Psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} + \frac{\partial^2\Psi}{\partial z^2} \quad \text{(Cylindrical)}$$

$$\nabla^2\Psi = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2\Psi}{\partial\phi^2}\right] \quad \text{(Spherical)}$$