Hamza AlHasan   Mathmatical 330
1181636   HW5

$\boxed{1}$ $\nabla^2\Psi(r,\theta,\phi)=0$

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

$$\nabla^2\Psi = \underbrace{\frac{\partial^2\Psi}{\partial r^2} + \frac{2}{r}\frac{\partial\Psi}{\partial r}}_{r\text{ terms}} + \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}}_{\phi\text{ term}} + \underbrace{\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\Psi}{\partial\theta}}_{\theta\text{ terms}} = 0$$

Using separation of variables :

Let $\Psi(r,\theta,\phi) = R(r)\,S(\theta,\phi)$

$$\Rightarrow \left(\nabla^2\Psi = R''S + \frac{2}{r}R'S + \frac{1}{r^2\sin^2\theta}R\frac{\partial^2 S}{\partial\phi^2} + \frac{1}{r^2}R\frac{\partial^2 S}{\partial\theta^2} + \frac{\cot\theta}{r^2}R\frac{\partial S}{\partial\theta} = 0\right)/\Psi$$

$$\Rightarrow \left(\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2\sin^2\theta}R\frac{\partial^2 S}{\partial\phi^2} + \frac{R}{r^2}\frac{\partial^2 S}{\partial\theta^2} + \frac{R\cot\theta}{r^2}\frac{\partial S}{\partial\theta} = 0\right) * r^2$$

$$\Rightarrow \underbrace{\left(\frac{r^2R''+2rR'}{R}\right)}_{\lambda} + \underbrace{\left(\frac{1}{\sin^2\theta}\frac{\partial^2 S}{\partial\phi^2}\cdot\frac{1}{S} + \frac{\partial^2 S}{\partial\theta^2}\frac{1}{S} + \cot\theta\frac{\partial S}{\partial\theta}\frac{1}{S}\right)}_{-\lambda} = 0$$

$r^2R''+2rR'-\lambda R=0$ —(I) , Let $S(\theta,\phi)=\Theta(\theta)\,\phi(\phi)$

$$\frac{1}{\sin^2\theta}\frac{\phi''}{\phi} + \frac{\Theta''}{\Theta} + \cot\theta\frac{\Theta'}{\Theta} = -\lambda$$

$$\underbrace{\left(\frac{\phi''}{\phi}\right)}_{-m^2} + \underbrace{\left(\frac{\sin^2\theta\,\Theta'' + \sin\theta\cos\theta\,\Theta'}{\Theta} + \lambda\sin^2\theta\right)}_{m^2} = 0$$

let $-m^2$ where m^2 is a scalar

$\phi''+m^2\phi=0$ —(II)

$$\sin\theta\,\Theta'' + \cos\theta\,\Theta' + \lambda\sin\theta\,\Theta - \frac{m^2}{\sin\theta}\Theta = 0 \quad \text{(III)}$$

We have 3 Eigen value problems (Sturm-Liouville Theory)

$$[p(x)y']' + (q(x)+\lambda W(x))y = 0$$

First, Let start with problem (I) related to $R(r)$:

$r^2R''+2rR'-\lambda R=0$ → This is a Cauchy-Euler equation with the auxiliary equation as following:

$r^2R''+2rR'-n(n+1)=0 \Rightarrow \lambda \equiv n(n+1)$

λ must be this value to ~~have~~ get a physical solution.

$\Rightarrow m^2+(2-1)m-n(n+1)=0$

$m^2+m-n(n+1)=0$

$(m-n)(m+(n+1))=0$

$m_1=n$, $m_2=-(n+1)$

$$\boxed{R_n(r) = C_1r^n + C_2r^{-(n+1)}}$$

Secondly, the problem (II) :

$\phi'' + m^2 \phi = 0$ (It look like the spring - mass equation : $\ddot{x} + w_0^2 x = 0$)

~~the auxiliary equation:~~

Hamza

$$\Rightarrow \boxed{\phi = C_3 \sin(m\phi) + C_4 \cos(m\phi)}$$

Finally, we solve the problem (III) :

$$\sin\theta\, \Theta'' + \cos\theta\, \Theta' + \lambda \sin\theta\, \Theta - \frac{m^2}{\sin\theta}\Theta = 0$$

Let $x = \cos\theta \Rightarrow dx = -\sin\theta\, d\theta$

$$\frac{d\Theta}{dx} = \frac{d\Theta}{d\theta}\frac{d\theta}{dx} = \frac{-1}{\sin\theta}\frac{d\Theta}{d\theta} = \frac{-\Theta'}{\sin\theta} \Rightarrow \Theta' = -\sin\theta \frac{d\Theta}{dx}$$

$$\frac{d}{dx}\left(\frac{d\Theta}{dx}\right) = \frac{d}{x}\left(\frac{-\Theta'}{\sin\theta}\right) = \frac{d}{d\theta}\left(\frac{-\Theta'}{\sin\theta}\right)\frac{d\theta}{dx} = \frac{-\sin\theta\,\Theta'' + \Theta'\cos\theta}{\sin^2\theta} * \frac{-1}{\sin\theta}$$

$$\frac{d^2\Theta}{dx^2} = \frac{\sin\theta\,\Theta'' - \cos\theta\,\Theta'}{\sin^3\theta}$$

$$\Rightarrow \Theta''(\theta) = \frac{\sin^3\theta \frac{d^2\Theta}{dx^2} + \cos\theta\,\Theta'}{\sin\theta} = \sin^2\theta \frac{d^2\Theta}{dx^2} - \cos\theta \frac{d\Theta}{dx}$$

The eq (III) becomes :

$$\left(\sin^3\theta \frac{d^2\Theta}{dx^2} - \sin\theta\cos\theta \frac{d\Theta}{dx} - \sin\theta\cos\theta\frac{d\Theta}{dx} + \lambda\sin\theta\,\Theta - \frac{m^2}{\sin\theta}\Theta = 0\right)/\sin\theta$$

$$\Rightarrow \sin^2\theta \frac{d^2\Theta}{dx^2} - 2\cos\theta\frac{d\Theta}{dx} + \lambda\Theta - \frac{m^2}{\sin^2\theta}\Theta = 0 \;;$$

$$\Rightarrow (1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \lambda\Theta - \frac{m^2}{1-x^2}\Theta = 0$$

But $\lambda \equiv n(n+1)$, hence,

$$(1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \left[n(n+1) - \frac{m^2}{1-x^2}\right]\Theta = 0 \rightarrow$$

This is Associated Legendre polynomials equation

where $m, n \in \mathbb{Z}$

The solution for this equation is :

$$\Theta(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m}(P_n(x)) = P_n^m(x)$$

$$\Rightarrow \boxed{\Theta(\theta) = P_n^m(\cos\theta)}$$

$$\Longrightarrow \Psi(r,\theta,\phi) = R(r)\,\Theta(\theta)\,\phi(\phi)$$

$$= \left(C_1 r^n + C_2 r^{-(n+1)}\right)\left(C_3\sin(m\phi) + C_4\cos(m\phi)\right)\left(P_n^m(\cos\theta)\right)$$

2 Generating function for Legendre's Polynomial $P_n(x)$:

$$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x) \quad , \quad |h|<1$$

I need to ~~prove~~ find $P_n(0)$:

I start with Laplace's First Integral for $P_n(x)$:

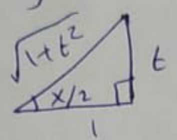
$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} [\, x \pm \sqrt{x^2-1}\cos\phi \,]^n \, d\phi \quad — (*)$$

proof:

First, I need to use this integral: $\int_0^{\pi} \frac{dx}{a+b\cos x} = I$

It can be solved by using Weierstrass Substitution:

Let $t = \tan\left(\frac{x}{2}\right)$



$\frac{x}{2} = \tan^{-1} t$

$dx = \frac{2}{1+t^2}\, dt$

$\cos(x) = \cos\left(2\cdot\frac{x}{2}\right)$

$= \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$

$= \frac{1}{1+t^2} - \frac{t^2}{1+t^2}$

$= \frac{1-t^2}{1+t^2}$

$$I = \int \frac{2}{1+t^2} \frac{dt}{a+b\left(\frac{1-t^2}{1+t^2}\right)} = \int \frac{2\,dt}{(1+t^2)a + (1-t^2)b}$$

$$= \int \frac{2\,dt}{(a-b)t^2 + (a+b)}$$

(Now, we know that $\int \frac{1}{A^2+t^2} dt = \frac{1}{A}\tan^{-1}\left(\frac{t}{A}\right)$)

$$\Rightarrow I = \frac{2}{a-b}\int \frac{dt}{t^2 + \frac{(a+b)}{a-b}}$$

$$= \frac{2}{a-b}\left[\sqrt{\frac{a-b}{a+b}} * \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\, t\right)\right]$$

$$= \frac{2}{\sqrt{a^2-b^2}} \cdot \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)\Big|_0^{\pi}$$

$0 = \tan^{-1}(\tan(0))$, $\tan\frac{\pi}{2} = \infty$

, $\tan^{-1}(\infty) = \frac{\pi}{2}$

$$\Rightarrow I = \frac{2}{\sqrt{a^2-b^2}}\left[\frac{\pi}{2} - 0\right]$$

$$= \boxed{\frac{\pi}{\sqrt{a^2-b^2}}}$$

First proof is done

Now, we have $\int_0^{\pi} \frac{dx}{a+b\cos x} = \frac{\pi}{\sqrt{a^2-b^2}}$

choose $a = 1-hx$

$b = h\sqrt{x^2-1}$

So, $a^2-b^2 = 1-2xh+h^2$

$$\Rightarrow \pi(1-2xh+h^2)^{-1/2} = \int_0^{\pi}[1-hx \pm h\sqrt{x^2-1}\cos\phi]^{-1}\, d\phi$$

$$= \int_0^{\pi}[1 - h(x \mp \sqrt{x^2-1}\cos\phi)]^{-1}\, d\phi$$

Let $t = x \mp \sqrt{x^2-1}\cos\phi$

$$\Rightarrow = \int_0^{\pi}[1-ht]^{-1}\, d\phi$$

Now we know that

$$\frac{1}{1-ht} = \sum_{n=0}^{\infty}(ht)^n \; ; |ht|<1$$

condition

$$\Rightarrow \text{So, } \pi(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$$

$$= \sum_{n=0}^{\infty}\left[h^n \int_0^{\pi} t^n\, d\phi\right]$$

Comparing:

$$\pi P_n(x) = \int_0^{\pi} t^n\, d\phi$$

$$\Rightarrow \boxed{P_n(x) = \frac{1}{\pi}\int_0^{\pi}[\, x \pm \sqrt{x^2-1}\cos\phi\,]^n\, d\phi}$$

Laplace's First Integral $P_n(x)$ is done

Now, $P_n(0) = \frac{1}{\pi}\int_0^{\pi}(\pm i\cos\phi)^n\, d\phi \; ; \; i = \sqrt{-1}$

$$\Rightarrow P_n(0) = \frac{(-1)^n\, i^n}{\pi}\int_0^{\pi}(\cos\phi)^n\, d\phi$$

Hamza

We have until now, $P_n(0) = \frac{(-1)^n i^n}{\pi} \int_0^{\pi} \cos^n\phi \, d\phi$

$\cos^n\phi = \frac{(e^{ix} + e^{-ix})^n}{2^n}$

Binomial expansion

$\Rightarrow 2^n \cos^n\phi = \sum_{k=0}^{n} \binom{n}{k} (e^{ix})^k (e^{-ix})^{n-k} = \sum_{k=0}^{n} \binom{n}{k} e^{(2k-n)ix}$

$2^n \int_0^{\pi} \cos^n\phi = \sum_{k=0}^{n} \binom{n}{k} \int_0^{\pi} e^{(2k-n)ix} \, dx$

If $2k-n \neq 0$:

$\left. \frac{e^{(2k-n)ix}}{i(2k-n)} \right|_0^{\pi} = \frac{1}{i(2k-n)} \left[\cos((2k-n)\pi) + i \cancel{\sin((2k-n)\pi)}^{\,0} - 1 \right]$

$\cos((2k-n)\pi) = (-1)^{2k-n}$

$= \frac{1}{i(2k-n)} \left[(-1)^{2k-n} - 1 \right]$

if $2k-n$ even $\Rightarrow$ zero

if $2k-n$ odd $\Rightarrow \frac{-2}{i(2k-n)} = \frac{2i}{2k-n}$

or If $2k-n=0$

$\Rightarrow \int_0^{\pi} e^{(2k-n)ix} dx = \int_0^{\pi} dx = \pi$

Now, if n is even, then $(2k-n)$ is even, so:

$2^n \int_0^{\pi} \cos^n\phi = \sum_{k=0}^{n} \binom{n}{k} \int_0^{\pi} e^{(2k-n)ix} dx$ is zero

all the terms are zeros except that when $2k - n = 0$ — since n is even it is possible to have $2k = n$ at some k

so $\int_0^{\pi} \cos^n\phi = \frac{\pi}{2^n} \sum_{k=0}^{n} \binom{n}{k}$ — at $2k = n$ —

$\neq 0$ (Let it $= \alpha$) $\Rightarrow \alpha = \frac{\pi}{2^n} \binom{n}{n/2}$

Therefore, $P_n(0) = \frac{(\pm i)^n}{\pi} \cdot \alpha$ — when n is even — (α is above)

But, if n is odd, then $2k-n \neq 0, \forall k$, and $2k-n$ is always odd

$\Rightarrow \sum_{k=0}^{n} \int_0^{\pi} e^{(2k-n)ix} dx = \sum_{k=0}^{n} \frac{2i}{2k-n}$

$= 2i \left[\cancel{\frac{1}{-n}} + \cancel{\frac{1}{2-n}} + \cancel{\frac{1}{4-n}} + \cdots + \cancel{\frac{1}{n-4}} + \cancel{\frac{1}{n-2}} + \cancel{\frac{1}{n}} \right] = \text{zero}$

[All the minus terms cancel with the positive terms]

$\boxed{\text{The result} : P_n(0)} = \begin{cases} 0 & , n \text{ is odd} \\ \frac{(\pm i)^n}{\pi} \alpha, & n \text{ is even} \end{cases}$

↳ where α is above $\left(\alpha = \frac{\pi}{2^n} * \binom{n}{n/2} \right)$

3 Prove: (a) $L P_L(x) = (2L-1)\, x\, P_{L-1}(x) - (L-1) P_{L-2}$

Hamza

$\Phi(x,h) = (1-2xh+h^2)^{-1/2}$

$\frac{\partial \Phi}{\partial h} = \frac{-1}{2}(1-2xh+h^2)^{-3/2}(-2x+2h)$

$(1-2xh+h^2)\frac{\partial \Phi}{\partial h} = (x-h)\Phi$

Substitute: $\Phi = \sum_{L=0}^{\infty} h^L P_L(x)$

$\frac{\partial \Phi}{\partial h} = \sum_{L=1}^{\infty} L h^{L-1} P_L(x)$

$\Rightarrow (1-2xh+h^2)\sum_{L=1}^{\infty} L h^{L-1} P_L(x) = (x-h)\sum_{L=0}^{\infty} h^L P_L(x)$

For simplicity, I write $P_L(x)$ as P_L, and $\sum_L^{\infty}$ will not be written.

$\Rightarrow L h^{L-1} P_L - 2xLh^L P_L + L h^{L+1} P_L = x h^L P_L - h^{L+1} P_L$

$(L h^{L-1} P_L - 2x(L-1) h^{L-1} P_{L-1} + (L-2) h^{L-1} P_{L-2} = x h^{L-1} P_{L-1} - h^{L-1} P_{L-2}) / h^{L-1}$

$L P_L - 2x(L-1) P_{L-1} + (L-2) P_{L-2} = x P_{L-1} - h^{L-1} P_{L-2}$

$L P_L = (2L-1)\, x\, P_{L-1} - (L-1) P_{L-2}$ # a ✓

(b) $x P_L'(x) - P_{L-1}'(x) = L P_L(x)$

$\frac{\partial \Phi}{\partial x} = \frac{-1}{2}(1-2xh+h^2)^{-3/2}(-2h)$

$\frac{\partial \Phi}{\partial x} \div \frac{\partial \Phi}{\partial h} = \frac{-2h}{-2x+2h} = \frac{h}{x-h}$

$\Rightarrow (x-h)\frac{\partial \Phi}{\partial x} = h\frac{\partial \Phi}{\partial h}$

$\frac{\partial \Phi}{\partial x} = \sum_{L=0}^{\infty} h^L P_L'(x)$

$\Rightarrow (x-h)\sum_{L=0}^{\infty} h^L P_L'(x) = h\sum_{L=1}^{\infty} L h^{L-1} P_L(x)$

$(x h^L P_L' - h^{L+1} P_L' = L h^L P_L) / h^L$

$x P_L' - P_{L-1}' = L P_L$ # b ✓

(c) $P_L'(x) - x P_{L-1}'(x) = L P_{L-1}(x)$

Differentiate (a): $L P_L' = (2L-1)\, x\, P_{L-1}' + (2L-1) P_{L-1} - (L-1) P_{L-2}'$

Use (b) with replacing L by $L-1$: $x P_{L-1}' - P_{L-2}' = (L-1) P_{L-1}$

$\Rightarrow L P_L' = (2L-1)\, x P_{L-1}' + (2L-1) P_{L-1} - (L-1)(x P_{L-1}' - (L-1) P_{L-1})$

$= ((2L-1) + (L-1)^2) P_{L-1} + ((2L-1)x - (L-1)x) P_{L-1}'$

$= L^2 P_{L-1} + Lx P_{L-1}'$

$\Rightarrow P_L' - x P_{L-1}' = L P_{L-1}$ # c ✓

Hamza

(d) $(1-x^2)P_L'(x) = LP_{L-1}(x) - LxP_L(x)$

from (b) : $P_{L-1}' = xP_L' - LP_L$

Using (c) : $P_L' - xP_{L-1}' = LP_{L-1}$

$P_L' - x^2P_L' + LxP_L = LP_{L-1}$

$\Rightarrow (1-x^2)P_L' = LP_{L-1} - LxP_L$ # d ✓

(f) $(1-x^2)P_{L-1}' = LxP_{L-1} - LP_L$

· Substitute P_L' from (b) in (c) :

$x(L.P_{L-1} + xP_{L-1}') - P_{L-1}' = LP_L$

$xLP_{L-1} + x^2P_{L-1}' - P_{L-1}' = LP_L$

$(1-x^2)P_{L-1}' = LxP_{L-1} - LP_L$ # f ✓

(e) $(2L+1)P_L(x) = P_{L+1}'(x) - P_{L-1}'(x)$

From (c) and replacing L by L+1 :

$P_{L+1}' - xP_L' = (L+1)P_L$

Subsiture xP_L' from (c) in (b) :

$(P_{L+1}' - (L+1)P_L) - P_{L-1}' = LP_L$

$P_{L+1}' - P_L(2L+1) - P_{L-1}' = 0$

$\Rightarrow (2L+1)P_L = P_{L+1}' - P_{L-1}'$ # e ✓

$\boxed{4}$ $\int_{-1}^{1} x\, P_L(x)\, P_n(x)\, dx$ Hamza

Using this recursion relation:

(a) $L\, P_L(x) = (2L-1)\, x\, P_{L-1}(x) - (L-1)\, P_{L-2}(x)$

We want $x\, P_L(x)$:

$\Rightarrow$ (a) becomes:

$$x\, P_L(x) = \frac{1}{2L+1}\left[(L+1)\, P_{L+1}(x) + L\, P_{L-1}(x)\right]$$

Multiply both sides by $P_n(x)$ and take the integral from $x=-1$ to $x=1$:

$$\int_{-1}^{1} x\, P_L(x)\, P_n(x)\, dx = \frac{1}{2L+1}\left[(L+1)\int_{-1}^{1} P_{L+1}(x)\, P_n(x)\, dx + L\int_{-1}^{1} P_{L-1}(x)\, P_n(x)\, dx\right]$$

We know that $\int_{-1}^{1} P_L(x)\, P_m(x)\, dx = \begin{cases} 0, & L \neq m \\ \frac{2}{2m+1}, & L = m \end{cases}$

Hence, if $L+1 = n$, the second integral on RHS will be zero since $n = L+1 \neq L-1$ and the first integral on RHS is $\frac{2}{2n+1} = \frac{2}{2L+3}$.

So, $\int_{-1}^{1} x\, P_L(x)\, P_n(x)\, dx = \frac{1}{2L+1}\left[(L+1)\left(\frac{2}{2L+3}\right)\right]$

But, if $L-1 = n$, the first integral on RHS is zero, and the second integral is $\frac{2}{2n+1} = \frac{2}{2L-1}$.

So, $\int_{-1}^{1} x\, P_L(x)\, P_n(x)\, dx = \frac{1}{2L+1}\left[L \cdot \frac{2}{2L-1}\right]$

Otherwise, i.e, if $n \neq L+1$ and $n \neq L-1$, then our main integral is zero.

Therefore, $\int_{-1}^{1} x\, P_L(x)\, P_n(x)\, dx = \begin{cases} \frac{2L}{(2L+1)(2L-1)}, & n = L-1 \\ \frac{2L+2}{(2L+1)(2L+3)}, & n = L+1 \\ \text{zero}, & n \neq L-1 \text{ and } n \neq L+1 \end{cases}$

$$\int_{-1}^{1} x^2 P_L(x)\, P_n(x)\, dx$$

Hamza

Using the recursion relation:

$$x P_n = \frac{1}{2n+1}\left[(n+1)P_{n+1} + n P_{n-1}\right]$$

Multiply both siddes by $x P_L(x)$ and take the integral from $x=-1$ to $x=1$:

$$\int_{-1}^{1} x^2 P_n P_L\, dx = \frac{1}{2n+1}\left[(n+1)\underbrace{\int_{-1}^{1} x P_L P_{n+1}\, dx}_{\mathrm{I}} + n\underbrace{\int_{-1}^{1} x P_L P_{n-1}\, dx}_{\mathrm{II}}\right]$$

From the integral solved before,

$$\mathrm{I} = \begin{cases} \dfrac{2L}{(2L+1)(2L-1)}, & n = L-2 \\[2ex] \dfrac{2L+2}{(2L+1)(2L+3)}, & n = L \\[2ex] \text{zero}, & \text{otherwise} \end{cases} \qquad \mathrm{II} = \begin{cases} \dfrac{2L}{(2L+1)(2L-1)}, & n = L \\[2ex] \dfrac{2L+2}{(2L+1)(2L+3)}, & n = L+2 \\[2ex] \text{zero}, & \text{otherwise} \end{cases}$$

This means:

at $n = L-2$: $\displaystyle\int_{-1}^{1} x^2 P_n P_L\, dx = \frac{n+1}{2n+1}\cdot\frac{2L}{(2L+1)(2L-1)} = \frac{2L(L-1)}{(2L-3)(2L+1)(2L-1)}$

at $n = L+2$: $\displaystyle\int_{-1}^{1} x^2 P_n P_L\, dx = \frac{n}{2n+1}\cdot\frac{2L+2}{(2L+1)(2L+3)} = \frac{(L+2)(2L+2)}{(2L+5)(2L+1)(2L+3)}$

at $n = L$: $\displaystyle\int_{-1}^{1} x^2 P_n P_L\, dx = \frac{1}{2n+1}\left[(n+1)\cdot\frac{2L+2}{(2L+1)(2L+3)} + n\cdot\frac{2L}{(2L+1)(2L-1)}\right]$

$$= \frac{(L+1)(2L+2)}{(2L+1)(2L+1)(2L+3)} + \frac{2L^2}{(2L+1)^2(2L-1)}$$

otherwise : $\displaystyle\int_{-1}^{1} x^2 P_n P_L\, dx = \text{zero}.$

The result:

$$\int_{-1}^{1} x^2 P_n P_L\, dx = \begin{cases} \dfrac{2L(L-1)}{(2L-3)(2L+1)(2L-1)}, & n = L-2 \\[2ex] \dfrac{(L+2)(2L+2)}{(2L+5)(2L+1)(2L+3)}, & n = L+2 \\[2ex] \dfrac{(L+1)(2L+2)}{(2L+1)^2(2L+3)} + \dfrac{2L^2}{(2L+1)^2(2L-1)}, & n = L \\[2ex] \text{zero}, & \text{otherwise} \end{cases}$$

5 $\delta(x-a) = \begin{cases} 0, & x \neq a \\ \infty, & x = a \end{cases}$

$$\int_{x_1}^{x_2} f(x)\,\delta(x-a)\,dx = f(a) \quad \text{if } a \in (x_1, x_2)$$

Hamza

Fourier Series: - 2π periodic -

$$\delta(x-a) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x-a)\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty} (1)\cdot\delta(x-a)\,dx = \frac{1}{\pi}\cdot 1 = \frac{1}{\pi}$$

here, we are sure that $a \in (-\infty, \infty)$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x-a)\cos nx\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \delta(x-a)\cos nx\,dx = \frac{1}{\pi}\cdot\cos(an)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x-a)\sin nx\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \delta(x-a)\sin nx\,dx = \frac{1}{\pi}\sin(an)$$

$$\Rightarrow \delta(x-a) = \frac{1}{\pi} + \sum_{n=1}^{\infty}\left[\frac{\cos(na)}{\pi}\cos(nx) + \frac{\sin(na)}{\pi}\sin(nx)\right]$$