1.16 Curvilinear Coordinates

1.16.1 The What and Why of Curvilinear Coordinate Systems

Up until now, a rectangular Cartesian coordinate system has been used, and a set of orthogonal unit base vectors $\mathbf{e}_i$ has been employed as the basis for representation of vectors and tensors. More general coordinate systems, called **curvilinear coordinate systems**, can also be used. An example is shown in Fig. 1.16.1: a Cartesian system shown in Fig. 1.16.1a with basis vectors $\mathbf{e}_i$ and a curvilinear system is shown in Fig. 1.16.1b with basis vectors $\mathbf{g}_i$. Some important points are as follows:

1. The Cartesian space can be generated from the coordinate axes x_i; the generated lines (the dotted lines in Fig. 1.16.1) are perpendicular to each other. The base vectors $\mathbf{e}_i$ lie along these lines (they are tangent to them). In a similar way, the curvilinear space is generated from coordinate curves Θ_i; the base vectors $\mathbf{g}_i$ are tangent to these curves.
2. The Cartesian base vectors $\mathbf{e}_i$ are orthogonal to each other and of unit size; in general, the basis vectors $\mathbf{g}_i$ are not orthogonal to each other and are not of unit size.
3. The Cartesian basis is independent of position; the curvilinear basis changes from point to point in the space (the base vectors may change in orientation and/or magnitude).

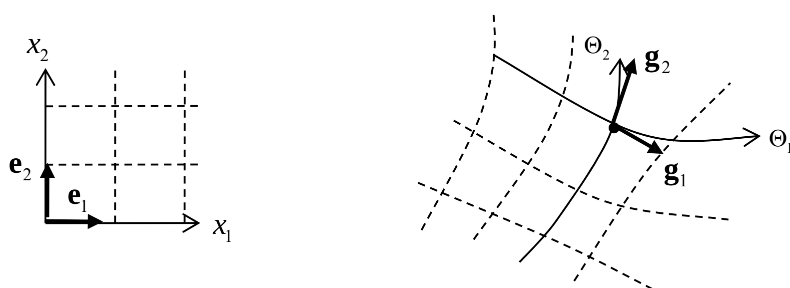

Figure 1.16.1: A Cartesian coordinate system and a curvilinear coordinate system

An example of a curvilinear system is the commonly-used cylindrical coordinate system, shown in Fig. 1.16.2. Here, the curvilinear coordinates $\Theta_1, \Theta_2, \Theta_3$ are the familiar r, θ, z. This cylindrical system is itself a special case of curvilinear coordinates in that the base vectors are always orthogonal to each other. However, unlike the Cartesian system, the orientations of the $\mathbf{g}_1, \mathbf{g}_2$ (r, θ) base vectors change as one moves about the cylinder axis.

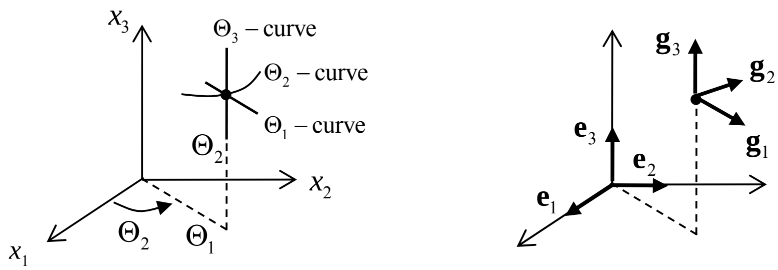

Figure 1.16.2: Cylindrical Coordinates

The question arises: why would one want to use a curvilinear system? There are two main reasons:

1. The problem domain might be of a particular shape, for example a spherical cell, or a soil specimen that is roughly cylindrical. In that case, it is often easier to solve the problems posed by first describing the problem geometry in terms of a curvilinear, e.g. spherical or cylindrical, coordinate system.
2. It may be easier to solve the problem using a Cartesian coordinate system, but a description of the problem in terms of a curvilinear coordinate system allows one to see aspects of the problem which are not obvious in the Cartesian system: it allows for a deeper understanding of the problem.

To give an idea of what is meant by the second point here, consider a simple mechanical deformation of a "square" into a "parallelogram", as shown in Figure 1.16.3. This can be viewed as a deformation of the actual coordinate system, from the Cartesian system aligned with the square, to the "curved" system (actually straight lines, but now not perpendicular) aligned with the edges of the parallelogram. The relationship between the sets of base vectors, the $\mathbf{e}_i$ and the $\mathbf{g}_i$, is intimately connected to the actual physical deformation taking place. In our study of curvilinear coordinates, we will examine this relationship, and also the relationship between the Cartesian coordinates x_i and the curvilinear coordinates Θ_i, and this will give us a very deep knowledge of the essence of the deformation which is taking place. This notion will become more clear when we look at kinematics, convected coordinates and related topics in the next chapter.

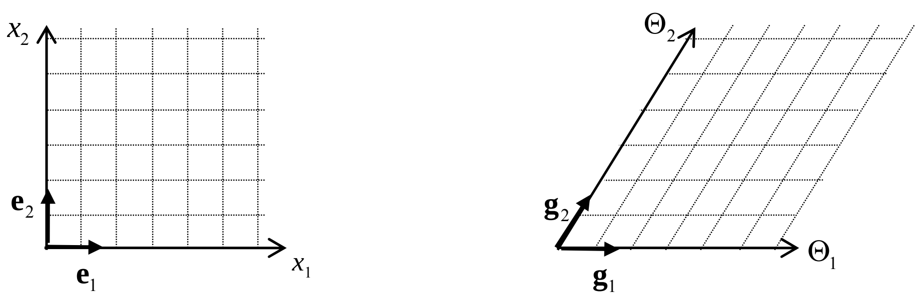

Figure 1.16.3: Deformation of a Square into a Parallelogram

1.16.2 Vectors in Curvilinear Coordinates

The description of scalars in curvilinear coordinates is no different to that in Cartesian coordinates, as they are independent of the basis used. However, the description of vectors is not so straightforward, or obvious, and it will be useful here to work carefully through an example two dimensional problem: consider again the Cartesian coordinate system and the **oblique coordinate system** (which delineates a "parallelogram"-type space), Fig. 1.16.4. These systems have, respectively, base vectors $\mathbf{e}_i$ and $\mathbf{g}_i$, and coordinates x_i and Θ_i. (We will take the $\mathbf{g}_i$ to be of unit size for the purposes of this example.) The base vector $\mathbf{g}_2$ makes an angle α with the horizontal, as shown.

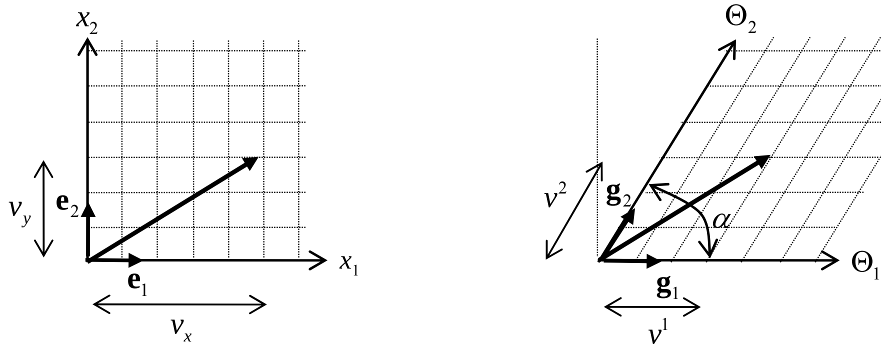

Figure 1.16.4: A Cartesian coordinate system and an oblique coordinate system

Let a vector $\mathbf{v}$ have Cartesian components v_x, v_y, so that it can be described in the Cartesian coordinate system by

$$\mathbf{v} = v_x \mathbf{e}_1 + v_y \mathbf{e}_2 \tag{1.16.1}$$

Let the *same* vector $\mathbf{v}$ have components v^1, v^2 (the reason for using superscripts, when we have always used subscripts hitherto, will become clearer below), so that it can be described in the oblique coordinate system by

$$\mathbf{v} = v^1 \mathbf{g}_1 + v^2 \mathbf{g}_2 \tag{1.16.2}$$

Using some trigonometry, one can see that these components are related through

$$\begin{aligned} v^1 &= v_x - \frac{1}{\tan\alpha} v_y \\ v^2 &= +\frac{1}{\sin\alpha} v_y \end{aligned} \tag{1.16.3}$$

Now we come to a very important issue: in our work on vector and tensor analysis thus far, a number of important and useful "rules" and relations have been derived. These rules have been *independent of the coordinate system* used. One example is that the magnitude of a vector $\mathbf{v}$ is given by the square root of the dot product: $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$. A natural question to ask is: does this rule work for our oblique coordinate system? To see, first let us evaluate the length squared directly from the Cartesian system:

$$|\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v} = v_x^2 + v_y^2 \tag{1.16.4}$$

Following the same logic, we can evaluate

$$\begin{aligned}(v^1)^2 + (v^2)^2 &= \left(v_x - \frac{1}{\tan\alpha}v_y\right)^2 + \left(\frac{1}{\sin\alpha}v_y\right)^2 \\ &= v_x^2 - 2\frac{1}{\tan\alpha}v_x v_y + \left(\frac{1}{\tan^2\alpha} + \frac{1}{\sin^2\alpha}\right)v_y^2\end{aligned} \tag{1.16.5}$$

It is clear from this calculation that our "sum of the squares of the vector components" rule which worked in Cartesian coordinates does not now give us the square of the vector length in curvilinear coordinates.

To find the general rule which works in both (all) coordinate systems, we have to complicate matters somewhat: introduce a *second* set of base vectors into our oblique system. The first set of base vectors, the $\mathbf{g}_1$ and $\mathbf{g}_2$ aligned with the coordinate directions Θ_1 and Θ_2 of Fig. 1.16.4, are termed **covariant** base vectors. Our second set of vectors, which will be termed **contravariant** base vectors, will be denoted by superscripts: $\mathbf{g}^1$ and $\mathbf{g}^2$, and will be aligned with a new set of coordinate directions, Θ^1 and Θ^2.

The new set is defined as follows: the base vector $\mathbf{g}^1$ is perpendicular to $\mathbf{g}_2$ $(\mathbf{g}^1\cdot\mathbf{g}_2 = 0)$, and the base vector $\mathbf{g}^2$ is perpendicular to $\mathbf{g}_1$ $(\mathbf{g}_1\cdot\mathbf{g}^2 = 0)$, Fig. 1.16.5a. (The base vectors' orientation with respect to each other follows the "right-hand rule" familiar with Cartesian bases; this will be discussed further below when the general 3D case is examined.) Further, we ensure that

$$\mathbf{g}_1\cdot\mathbf{g}^1 = 1, \qquad \mathbf{g}_2\cdot\mathbf{g}^2 = 1 \tag{1.16.6}$$

With $\mathbf{g}_1 = \mathbf{e}_1, \mathbf{g}_2 = \cos\alpha\mathbf{e}_1 + \sin\alpha\mathbf{e}_2$, these conditions lead to

$$\mathbf{g}^1 = \mathbf{e}_1 - \frac{1}{\tan\alpha}\mathbf{e}_2, \quad \mathbf{g}^2 = \frac{1}{\sin\alpha}\mathbf{e}_2 \tag{1.16.7}$$

and $|\mathbf{g}^1| = |\mathbf{g}^2| = 1/\sin\alpha$.

A good trick for remembering which are the covariant and which are the contravariant is that the third letter of the word tells us whether the word is associated with subscripts or with superscripts. In "covariant", the "v" is pointing down, so we use subscripts; for "contravariant", the "n" is (with a bit of imagination) pointing up, so we use superscripts.

Let the components of the vector **v** using this new contravariant basis be v_1 and v_2, Fig. 1.16.5b, so that

$$\mathbf{v} = v_1 \mathbf{g}^1 + v_2 \mathbf{g}^2 \tag{1.16.8}$$

Note the positon of the subscripts and superscripts in this expression: when the base vectors are contravariant ("superscripts"), the associated vector components are covariant ("subscripts"); compare this with the alternative expression for **v** using the covariant basis, Eqn. 1.16.2, $\mathbf{v} = v^1 \mathbf{g}_1 + v^2 \mathbf{g}_2$, which has covariant base vectors and contravariant vector components.

When **v** is written with covariant components, Eqn. 1.16.8, it is called a **covariant vector**. When **v** is written with contravariant components, Eqn. 1.16.2, it is called a **contravariant vector**. This is not the best of terminology, since it gives the impression that the vector is intrinsically covariant or contravariant, when it is in fact only a matter of which base vectors are being used to describe the vector. For this reason, this terminology will be avoided in what follows.

Examining Fig. 1.16.5b, one can see that $\left| v_1 \mathbf{g}^1 \right| = v_x / \sin\alpha$ and $\left| v_2 \mathbf{g}^2 \right| = v_x / \tan\alpha + v_y$, so that

$$\begin{aligned} v_1 &= v_x \\ v_2 &= \cos\alpha \, v_x + \sin\alpha \, v_y \end{aligned} \tag{1.16.9}$$

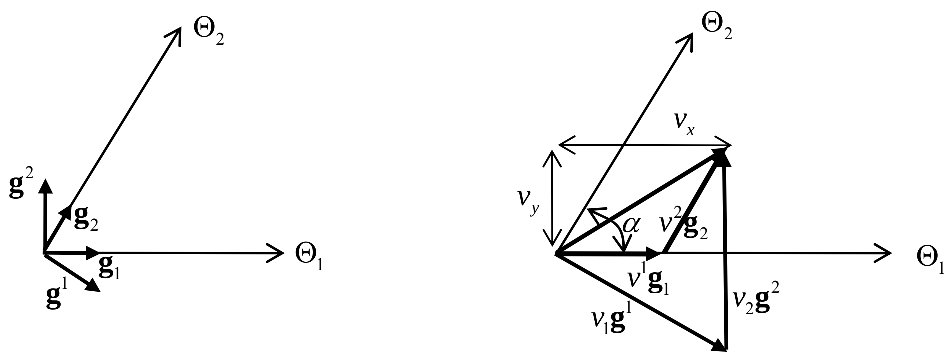

Figure 1.16.5: 2 sets of basis vectors; (a) covariant and contravariant base vectors, (b) covariant and contravariant components of a vector

Now one can evaluate the quantity

$$\begin{aligned} v_1 v^1 + v_2 v^2 &= \left(v_x\right)\left(v_x - \frac{1}{\tan\alpha} v_y\right) + \left(\cos\alpha \, v_x + \sin\alpha \, v_y\right)\left(\frac{1}{\sin\alpha} v_y\right) \\ &= v_x^2 + v_y^2 \end{aligned} \tag{1.16.10}$$

Thus multiplying the covariant and contravariant components together gives the length squared of the vector; this had to be so given how we earlier defined the two sets of base vectors:

$$\begin{aligned}|\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v} &= \left(v_1\mathbf{g}^1 + v_2\mathbf{g}^2\right)\cdot\left(v^1\mathbf{g}_1 + v^2\mathbf{g}_2\right) \\ &= v_1v^1\left(\mathbf{g}^1\cdot\mathbf{g}_1\right) + v_2v^2\left(\mathbf{g}^2\cdot\mathbf{g}_2\right) + v_1v^2\left(\mathbf{g}^1\cdot\mathbf{g}_2\right) + v_2v^1\left(\mathbf{g}^2\cdot\mathbf{g}_1\right) \\ &= v_1v^1 + v_2v^2\end{aligned} \qquad (1.16.11)$$

In general, the dot product of two vectors **u** and **v** in the general curvilinear coordinate system is defined through (the fact that the latter equality holds is another consequence of our choice of base vectors, as can be seen by re-doing the calculation of Eqn. 1.16.11 with 2 different vectors, and their different, covariant and contravariant, representations)

$$\mathbf{u}\cdot\mathbf{v} = u_1v^1 + u_2v^2 = u^1v_1 + u^2v_2 \qquad (1.16.12)$$

Cartesian Coordinates as Curvilinear Coordinates

The Cartesian coordinate system is a special case of the more general curvilinear coordinate system, where the covariant and contravariant bases are identically the same and the covariant and contravariant components of a vector are identically the same, so that one does not have to bother with carefully keeping track of whether an index is subscript or superscript – we just use subscripts for everything because it is easier.

More formally, in our two-dimensional space, our covariant base vectors are $\mathbf{g}_1 = \mathbf{e}_1, \mathbf{g}_2 = \mathbf{e}_2$. With the contravariant base vectors orthogonal to these, $\mathbf{g}^1\cdot\mathbf{g}_2 = 0$, $\mathbf{g}_1\cdot\mathbf{g}^2 = 0$, and with Eqn. 1.16.6, $\mathbf{g}_1\cdot\mathbf{g}^1 = 1, \mathbf{g}_2\cdot\mathbf{g}^2 = 1$, the contravariant basis is $\mathbf{g}^1 = \mathbf{e}_1, \mathbf{g}^2 = \mathbf{e}_2$. A vector **v** can then be represented as

$$\mathbf{v} = v_1\mathbf{g}^1 + v_2\mathbf{g}^2 = v^1\mathbf{g}_1 + v^2\mathbf{g}_2 \qquad (1.16.13)$$

which is nothing other than $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$, with $v_1 = v^1, v_2 = v^2$. The dot product is, formally,

$$\mathbf{u}\cdot\mathbf{v} = u_1v^1 + u_2v^2 = u^1v_1 + u^2v_2 \qquad (1.16.14)$$

which we choose to write as the equivalent $\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2$.

1.16.3 General Curvilinear Coordinates

We now define more generally the concepts discussed above.

A Cartesian coordinate system is defined by the fixed base vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and the coordinates (x^1, x^2, x^3), and any point p in space is then determined by the position vector $\mathbf{x} = x^i \mathbf{e}_i$ (see Fig. 1.16.6)[1]. This can be expressed in terms of curvilinear coordinates $(\Theta^1, \Theta^2, \Theta^3)$ by the transformation (and inverse transformation)

$$\begin{aligned} \Theta^i &= \Theta^i\left(x^1, x^2, x^3\right) \\ x^i &= x^i(\Theta^1, \Theta^2, \Theta^3) \end{aligned} \tag{1.16.15}$$

For example, the transformation equations for the oblique coordinate system of Fig. 1.16.4 are

$$\begin{aligned} &\Theta^1 = x^1 - \frac{1}{\tan\alpha} x^2, \quad \Theta^2 = \frac{1}{\sin\alpha} x^2, \quad \Theta^3 = x^3 \\ &x^1 = \Theta^1 + \cos\alpha\,\Theta^2, \quad x^2 = \sin\alpha\,\Theta^2, \quad x^3 = \Theta^3 \end{aligned} \tag{1.16.16}$$

If Θ^1 is varied while holding Θ^2 and Θ^3 constant, a space curve is generated called a Θ^1 **coordinate curve**. Similarly, Θ^2 and Θ^3 coordinate curves may be generated. Three **coordinate surfaces** intersect in pairs along the coordinate curves. On each surface, one of the curvilinear coordinates is constant.

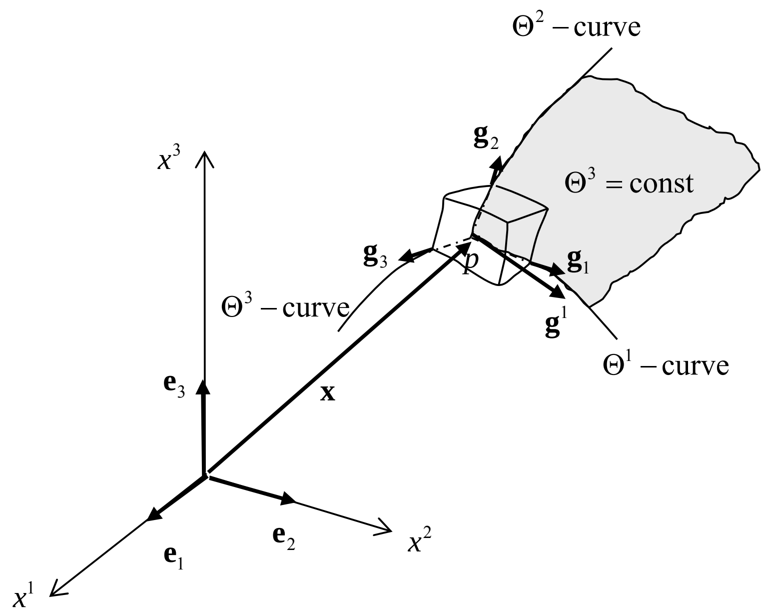

Figure 1.16.6: curvilinear coordinate system and coordinate curves

In order to be able to solve for the Θ^i given the x^i, and to solve for the x^i given the Θ^i, it is necessary and sufficient that the following determinants are non-zero – see Appendix 1.B.2 (the first here is termed the **Jacobian** J of the transformation):

[1] superscripts are used for the Cartesian system here and in much of what follows for notational consistency (see later)

$$J \equiv \det\left[\frac{\partial x^i}{\partial \Theta^j}\right] = \left|\frac{\partial x^i}{\partial \Theta^j}\right|, \qquad \det\left[\frac{\partial \Theta^i}{\partial x^j}\right] = \left|\frac{\partial \Theta^i}{\partial x^j}\right| = \frac{1}{J}, \tag{1.16.17}$$

the last equality following from (1.15.2, 1.10.18d).

Clearly Eqns 1.16.15a can be inverted to get Eqn. 1.16.15b, and vice versa, but just to be sure, we can check that the Jacobian and inverse are non-zero:

$$J = \begin{vmatrix} 1 & \cos\alpha & 0 \\ 0 & \sin\alpha & 0 \\ 0 & 0 & 1 \end{vmatrix} = \sin\alpha, \qquad \frac{1}{J} = \begin{vmatrix} 1 & -\frac{1}{\tan\alpha} & 0 \\ 0 & \frac{1}{\sin\alpha} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{\sin\alpha}, \tag{1.16.18}$$

The Jacobian is zero, i.e. the transformation is singular, only when $\alpha = 0$, i.e. when the parallelogram is shrunk down to a line.

1.16.4 Base Vectors in the Moving Frame

Covariant Base Vectors

From §1.6.2, writing $\mathbf{x} = \mathbf{x}(\Theta^i)$, tangent vectors to the coordinate curves at $\mathbf{x}$ are given by[2]

$$\boxed{\mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial \Theta^i} = \frac{\partial x^m}{\partial \Theta^i}\mathbf{e}_m} \qquad \textbf{Covariant Base Vectors} \tag{1.16.19}$$

with inverse $\mathbf{e}_i = \left(\partial \Theta^m / \partial x^i\right)\mathbf{g}_m$. The $\mathbf{g}_i$ emanate from the point *p* and are directed towards the site of increasing coordinate Θ^i. They are called **covariant base vectors**. Increments in the two coordinate systems are related through

$$d\mathbf{x} = \frac{d\mathbf{x}}{\partial \Theta^i} d\Theta^i = \mathbf{g}_i d\Theta^i$$

Note that the triple scalar product $\mathbf{g}_1 \cdot (\mathbf{g}_2 \times \mathbf{g}_3)$, Eqns. 1.3.17-18, is equivalent to the determinant in 1.16.17,

$$\mathbf{g}_1 \cdot (\mathbf{g}_2 \times \mathbf{g}_3) = \begin{vmatrix} (\mathbf{g}_1)_1 & (\mathbf{g}_1)_2 & (\mathbf{g}_1)_3 \\ (\mathbf{g}_2)_1 & (\mathbf{g}_2)_2 & (\mathbf{g}_2)_3 \\ (\mathbf{g}_3)_1 & (\mathbf{g}_3)_2 & (\mathbf{g}_3)_3 \end{vmatrix} = J = \det\left[\frac{\partial x^i}{\partial \Theta^j}\right] \tag{1.16.20}$$

[2] in the Cartesian system, with the coordinate curves parallel to the coordinate axes, these equations reduce trivially to $\mathbf{e}_i = \left(\partial x^m / \partial x^i\right)\mathbf{e}_m = \delta_{mi}\mathbf{e}_m$

so that the condition that the determinant does not vanish is equivalent to the condition that the vectors $\mathbf{g}_i$ are linearly independent, and so the $\mathbf{g}_i$ can form a basis.

For example, from Eqns. 1.16.16b, the covariant base vectors for the oblique coordinate system of Fig. 1.16.4, are

$$\begin{aligned}
\mathbf{g}_1 &= \frac{\partial x^1}{\partial \Theta^1}\mathbf{e}_1 + \frac{\partial x^2}{\partial \Theta^1}\mathbf{e}_2 + \frac{\partial x^3}{\partial \Theta^1}\mathbf{e}_3 = \mathbf{e}_1 \\
\mathbf{g}_2 &= \frac{\partial x^1}{\partial \Theta^2}\mathbf{e}_1 + \frac{\partial x^2}{\partial \Theta^2}\mathbf{e}_2 + \frac{\partial x^3}{\partial \Theta^2}\mathbf{e}_3 = \cos\alpha\,\mathbf{e}_1 + \sin\alpha\,\mathbf{e}_2 \\
\mathbf{g}_3 &= \frac{\partial x^1}{\partial \Theta^3}\mathbf{e}_1 + \frac{\partial x^2}{\partial \Theta^3}\mathbf{e}_2 + \frac{\partial x^3}{\partial \Theta^3}\mathbf{e}_3 = \mathbf{e}_3
\end{aligned} \tag{1.16.21}$$

Contravariant Base Vectors

Unlike in Cartesian coordinates, where $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, the covariant base vectors do not necessarily form an orthonormal basis, and $\mathbf{g}_i \cdot \mathbf{g}_j \neq \delta_{ij}$. As discussed earlier, in order to deal with this complication, a second set of base vectors are introduced, which are defined as follows: introduce three **contravariant base vectors** $\mathbf{g}^i$ such that each vector is normal to one of the three coordinate surfaces through the point *p*. From §1.6.4, the normal to the coordinate surface $\Theta^1\left(x^1, x^2, x^3\right) = \text{const}$ is given by the gradient vector $\operatorname{grad}\Theta^1$, with Cartesian representation

$$\operatorname{grad}\Theta^1 = \frac{\partial \Theta^1}{\partial x^m}\mathbf{e}^m \tag{1.16.22}$$

and, in general, one may define the contravariant base vectors through

$$\boxed{\mathbf{g}^i = \frac{\partial \Theta^i}{\partial x^m}\mathbf{e}^m} \qquad \textbf{Contravariant Base Vectors} \tag{1.16.23}$$

The contravariant base vector $\mathbf{g}^1$ is shown in Fig. 1.16.6.

As with the covariant base vectors, the triple scalar product $\mathbf{g}^1 \cdot \left(\mathbf{g}^2 \times \mathbf{g}^3\right)$ is equivalent to the determinant in 1.16.17,

$$\mathbf{g}^1 \cdot \left(\mathbf{g}^2 \times \mathbf{g}^3\right) = \begin{vmatrix} \left(\mathbf{g}^1\right)_1 & \left(\mathbf{g}^1\right)_2 & \left(\mathbf{g}^1\right)_3 \\ \left(\mathbf{g}^2\right)_1 & \left(\mathbf{g}^2\right)_2 & \left(\mathbf{g}^2\right)_3 \\ \left(\mathbf{g}^3\right)_1 & \left(\mathbf{g}^3\right)_2 & \left(\mathbf{g}^3\right)_3 \end{vmatrix} = \frac{1}{J} = \det\left[\frac{\partial \Theta^j}{\partial x^i}\right] \tag{1.16.24}$$

and again the condition that the determinant does not vanish is equivalent to the condition that the vectors $\mathbf{g}^i$ are linearly independent, and so the contravariant vectors also form a basis.

From Eqns. 1.16.16a, the contravariant base vectors for the oblique coordinate system are

$$\begin{aligned}
\mathbf{g}^1 &= \frac{\partial \Theta^1}{\partial x^1}\mathbf{e}_1 + \frac{\partial \Theta^1}{\partial x^2}\mathbf{e}_2 + \frac{\partial \Theta^1}{\partial x^3}\mathbf{e}_3 = \mathbf{e}_1 - \frac{1}{\tan\alpha}\mathbf{e}_2 \\
\mathbf{g}^2 &= \frac{\partial \Theta^2}{\partial x^1}\mathbf{e}_1 + \frac{\partial \Theta^2}{\partial x^2}\mathbf{e}_2 + \frac{\partial \Theta^2}{\partial x^3}\mathbf{e}_3 = \frac{1}{\sin\alpha}\mathbf{e}_2 \\
\mathbf{g}^3 &= \frac{\partial \Theta^3}{\partial x^1}\mathbf{e}_1 + \frac{\partial \Theta^3}{\partial x^2}\mathbf{e}_2 + \frac{\partial \Theta^3}{\partial x^3}\mathbf{e}_3 = \mathbf{e}_3
\end{aligned} \qquad (1.16.25)$$

1.16.5 Metric Coefficients

It follows from the definitions of the covariant and contravariant vectors that {▲Problem 1}

$$\boxed{\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j} \qquad (1.16.26)$$

This relation implies that each base vector $\mathbf{g}^i$ is orthogonal to two of the reciprocal base vectors $\mathbf{g}_i$. For example, $\mathbf{g}^1$ is orthogonal to both $\mathbf{g}_2$ and $\mathbf{g}_3$. Eqn. 1.16.26 is the defining relationship between **reciprocal pairs** of general bases. Of course the $\mathbf{g}^i$ were chosen precisely because they satisfy this relation. Here, δ_i^j is again the Kronecker delta[3], with a value of 1 when $i = j$ and zero otherwise.

One needs to be careful to distinguish between subscripts and superscripts when dealing with arbitrary bases, but the rules to follow are straightforward. For example, each free index which is not summed over, such as *i* or *j* in 1.16.26, must be either a subscript or superscript on both sides of an equation. Hence the new notation for the Kronecker delta symbol.

Unlike the orthogonal base vectors, the dot product of a covariant/contravariant base vector with another base vector is not necessarily one or zero. Because of their importance in curvilinear coordinate systems, the dot products are given a special symbol: define the **metric coefficients** to be

$$\boxed{\begin{aligned} g_{ij} &= \mathbf{g}_i \cdot \mathbf{g}_j \\ g^{ij} &= \mathbf{g}^i \cdot \mathbf{g}^j \end{aligned}} \qquad \textbf{Metric Coefficients} \qquad (1.16.27)$$

For example, the metric coefficients for the oblique coordinate system of Fig. 1.16.4 are

$$\begin{gathered}
g_{11} = 1, \quad g_{12} = g_{21} = \cos\alpha, \quad g_{22} = 1 \\
g^{11} = \frac{1}{\sin^2\alpha}, \quad g^{12} = g^{21} = -\frac{\cos\alpha}{\sin^2\alpha}, \quad g^{22} = \frac{1}{\sin^2\alpha}
\end{gathered} \qquad (1.16.28)$$

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[3] although in this context it is called the *mixed* Kronecker delta

The following important and useful relations may be derived by manipulating the equations already introduced: {▲Problem 2}

$$\mathbf{g}_i = g_{ij}\mathbf{g}^j$$
$$\mathbf{g}^i = g^{ij}\mathbf{g}_j \qquad (1.16.29)$$

and {▲Problem 3}

$$g^{ij}g_{kj} = \delta^i_k \equiv g^i_k \qquad (1.16.30)$$

Note here another rule about indices in equations involving general bases: summation can only take place over a dummy index if one is a subscript and the other is a superscript – they are paired off as with the *j*'s in these equations.

The metric coefficients can be written explicitly in terms of the curvilinear components:

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \frac{\partial x^k}{\partial \Theta^i}\frac{\partial x^k}{\partial \Theta^j}, \qquad g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j = \frac{\partial \Theta^i}{\partial x^k}\frac{\partial \Theta^j}{\partial x^k} \qquad (1.16.31)$$

Note here also a rule regarding derivatives with general bases: the index *i* on the right hand side of 1.16.31a is a superscript of Θ but it is in the denominator of a quotient and so is regarded as a subscript to the entire symbol, matching the subscript *i* on the **g** on the left hand side [4].

One can also write 1.16.31 in the matrix form

$$\left[g_{ij}\right] = \left[\frac{\partial x^k}{\partial \Theta^i}\right]^{\mathrm{T}}\left[\frac{\partial x^k}{\partial \Theta^j}\right], \qquad \left[g^{ij}\right] = \left[\frac{\partial \Theta^i}{\partial x^k}\right]\left[\frac{\partial \Theta^j}{\partial x^k}\right]^{\mathrm{T}} \qquad (1.16.32)$$

and, from 1.10.16a,b,

$$\det\left[g_{ij}\right] = \left(\det\left[\frac{\partial x^i}{\partial \Theta^j}\right]\right)^2 = J^2, \qquad \det\left[g^{ij}\right] = \left(\det\left[\frac{\partial \Theta^i}{\partial x^j}\right]\right)^2 = \frac{1}{J^2} \qquad (1.16.33)$$

These determinants play an important role, and are denoted by *g*:

$$g = \det\left[g_{ij}\right] = \frac{1}{\det\left[g^{ij}\right]}, \qquad \sqrt{g} = J \qquad (1.16.34)$$

Note:

- The matrix $\left[\partial x^k / \partial \Theta^i\right]$ is called the Jacobian *matrix* **J**, so $\mathbf{J}^{\mathrm{T}}\mathbf{J} = \left[g_{ij}\right]$

[4] the rule for pairing off indices has been broken in Eqn. 1.16.31 for clarity; more precisely, these equations should be written as $g_{ij} = \left(\partial x^m / \partial \Theta^i\right)\left(\partial x^n / \partial \Theta^j\right)\delta_{mn}$ and $g^{ij} = \left(\partial \Theta^i / \partial x^m\right)\left(\partial \Theta^j / \partial x^n\right)\delta^{mn}$

1.16.6 Scale Factors

The covariant and contravariant base vectors are not necessarily unit vectors. the unit vectors are, with $|\mathbf{g}_i| = \sqrt{\mathbf{g}_i \cdot \mathbf{g}_i}$, $|\mathbf{g}^i| = \sqrt{\mathbf{g}^i \cdot \mathbf{g}^i}$, :

$$\hat{\mathbf{g}}_i = \frac{\mathbf{g}_i}{|\mathbf{g}_i|} = \frac{\mathbf{g}_i}{\sqrt{g_{ii}}}, \qquad \hat{\mathbf{g}}^i = \frac{\mathbf{g}^i}{|\mathbf{g}^i|} = \frac{\mathbf{g}^i}{\sqrt{g^{ii}}} \qquad \text{(no sum)} \tag{1.16.35}$$

The lengths of the covariant base vectors are denoted by *h* and are called the **scale factors**:

$$h_i = |\mathbf{g}_i| = \sqrt{g_{ii}} \qquad \text{(no sum)} \tag{1.16.36}$$

1.16.7 Line Elements and The Metric

Consider a differential line element, Fig. 1.16.7,

$$d\mathbf{x} = dx^i \mathbf{e}_i = d\Theta^i \mathbf{g}_i \tag{1.16.37}$$

The square of the length of this line element, denoted by $(\Delta s)^2$ and called the **metric** of the space, is then

$$(\Delta s)^2 = d\mathbf{x} \cdot d\mathbf{x} = (d\Theta^i \mathbf{g}_i) \cdot (d\Theta^j \mathbf{g}_j) = g_{ij} d\Theta^i d\Theta^j \tag{1.16.38}$$

This relation $(\Delta s)^2 = g_{ij} d\Theta^i d\Theta^j$ is called the **fundamental differential quadratic form**. The g_{ij}'s can be regarded as a set of scale factors for converting increments in Θ^i to changes in length.

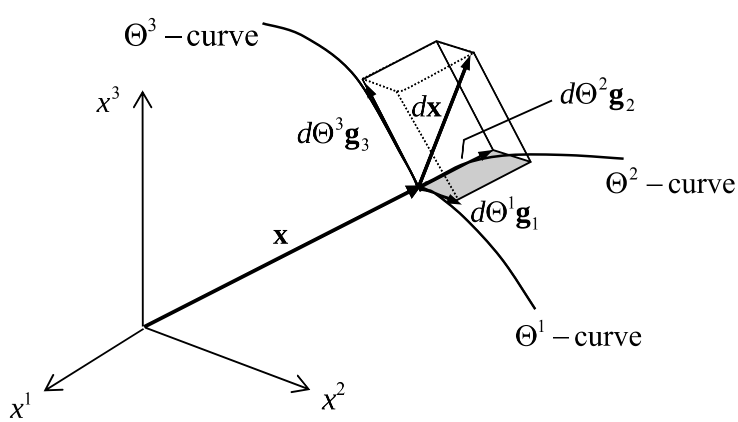

Figure 1.16.7: a line element in space

For a two dimensional space,

$$\begin{aligned}(\Delta s)^2 &= g_{11}d\Theta^1 d\Theta^1 + g_{12}d\Theta^1 d\Theta^2 + g_{21}d\Theta^2 d\Theta^1 + g_{22}d\Theta^2 d\Theta^2 \\ &= g_{11}\left(d\Theta^1\right)^2 + 2g_{12}d\Theta^1 d\Theta^2 + g_{22}\left(d\Theta^2\right)^2\end{aligned} \tag{1.16.39}$$

so that, for the oblique coordinate system of Fig. 1.16.4, from 1.16.28,

$$(\Delta s)^2 = \left(d\Theta^1\right)^2 + 2\cos\alpha\, d\Theta^1 d\Theta^2 + \left(d\Theta^2\right)^2 \tag{1.16.40}$$

This relation can be verified by applying Pythagoras' theorem to the geometry of Figure 1.16.8.

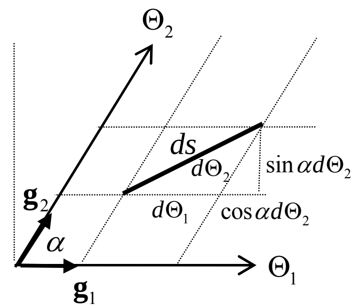

Figure 1.16.8: Length of a line element

1.16.8 Line, Surface and Volume Elements

Here we list expressions for the area of a surface element ΔS and the volume of a volume element ΔV, in terms of the increments in the curvilinear coordinates $\Delta\Theta^1, \Delta\Theta^2, \Delta\Theta^3$. These are particularly useful for the evaluation of surface and volume integrals in curvlinear coordinates.

Surface Area and Volume Elements

The surface area ΔS_1 of a face of the elemental parallelepiped on which Θ_1 is constant (to which $\mathbf{g}^1$ is normal) is, using 1.7.6,

$$\begin{aligned}\Delta S_1 &= \left|(\Delta\Theta^2 \mathbf{g}_2)\times(\Delta\Theta^3 \mathbf{g}_3)\right| \\ &= \left|\mathbf{g}_2 \times \mathbf{g}_3\right|\Delta\Theta^2\Delta\Theta^3 \\ &= \sqrt{(\mathbf{g}_2\times\mathbf{g}_3)\cdot(\mathbf{g}_2\times\mathbf{g}_3)}\,\Delta\Theta^2\Delta\Theta^3 \\ &= \sqrt{(\mathbf{g}_2\cdot\mathbf{g}_2)(\mathbf{g}_3\cdot\mathbf{g}_3)-(\mathbf{g}_2\cdot\mathbf{g}_3)(\mathbf{g}_2\cdot\mathbf{g}_3)}\,\Delta\Theta^2\Delta\Theta^3 \\ &= \sqrt{g_{22}g_{33}-(g_{23})^2}\,\Delta\Theta^2\Delta\Theta^3 \\ &= \sqrt{g g^{11}}\,\Delta\Theta^2\Delta\Theta^3\end{aligned} \tag{1.16.41}$$

and similarly for the other surfaces. For a two dimensional space, one has

$$
\begin{aligned}
\Delta S &= \left|(\Delta\Theta^1 \mathbf{g}_1) \times (\Delta\Theta^2 \mathbf{g}_2)\right| \\
&= \sqrt{g_{11}g_{22} - (g_{12})^2}\,\Delta\Theta^1\Delta\Theta^2 \\
&= \sqrt{g}\,\Delta\Theta^1\Delta\Theta^2 \quad \left(= J\Delta\Theta^1\Delta\Theta^2\right)
\end{aligned}
\tag{1.16.42}
$$

The volume ΔV of the parallelepiped involves the triple scalar product 1.16.20:

$$
\Delta V = \left(\mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3\right)\Delta\Theta^1\Delta\Theta^2\Delta\Theta^3 = \sqrt{g}\,\Delta\Theta^1\Delta\Theta^2\Delta\Theta^3 \quad \left(= J\Delta\Theta^1\Delta\Theta^2\Delta\Theta^3\right) \tag{1.16.43}
$$

1.16.9 Orthogonal Curvilinear Coordinates

In the special case of **orthogonal curvilinear coordinates**, one has

$$
g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \delta_{ij}\left|\mathbf{g}_i\right|\left|\mathbf{g}_j\right| = \delta_{ij} h_i h_j, \qquad \left[g_{ij}\right] = \begin{bmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{bmatrix} \tag{1.16.44}
$$

The contravariant base vectors are collinear with the covariant, but the vectors are of different magnitudes:

$$
\mathbf{g}_i = h_i \hat{\mathbf{g}}_i, \quad \mathbf{g}^i = \frac{1}{h_i}\hat{\mathbf{g}}_i \tag{1.16.45}
$$

It follows that

$$
\begin{aligned}
&(\Delta s)^2 = h_1^2 d\Theta_1^2 + h_2^2 d\Theta_2^2 + h_3^2 d\Theta_3^2 \\[1em]
&\Delta S_1 = h_2 h_3 \Delta\Theta_2 \Delta\Theta_3 \\
&\Delta S_2 = h_3 h_1 \Delta\Theta_3 \Delta\Theta_1 \\
&\Delta S_3 = h_1 h_2 \Delta\Theta_1 \Delta\Theta_2 \\[1em]
&\Delta V = h_1 h_2 h_3 \Delta\Theta_1 \Delta\Theta_2 \Delta\Theta_3
\end{aligned}
\tag{1.16.46}
$$

Examples

1. Cylindrical Coordinates

Consider the cylindrical coordinates, $(r, \theta, z) = \left(\Theta^1, \Theta^2, \Theta^3\right)$, *cf.* §1.6.10, Fig. 1.16.9:

$$
\begin{aligned}
x^1 &= \Theta^1 \cos\Theta^2 & \Theta^1 &= \sqrt{\left(x^1\right)^2 + \left(x^2\right)^2} \\
x^2 &= \Theta^1 \sin\Theta^2 , & \Theta^2 &= \tan^{-1}\left(x^2 / x^1\right) \\
x^3 &= \Theta^3 & \Theta^3 &= x^3
\end{aligned}
$$

with

$$
\Theta^1 \geq 0, \quad 0 \leq \Theta^2 < 2\pi, \quad -\infty < \Theta^3 < \infty
$$

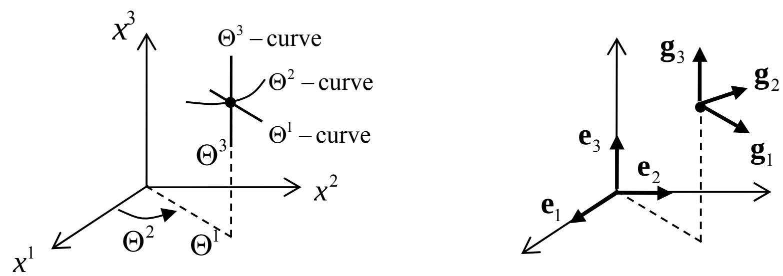

Figure 1.16.9: Cylindrical Coordinates

From Eqns. 1.16.19 (compare with 1.6.29), 1.16.27,

$$
\begin{aligned}
\mathbf{g}_1 &= +\cos\Theta^2 \mathbf{e}_1 + \sin\Theta^2 \mathbf{e}_2 \\
\mathbf{g}_2 &= -\Theta^1 \sin\Theta^2 \mathbf{e}_1 + \Theta^1 \cos\Theta^2 \mathbf{e}_2 , \\
\mathbf{g}_3 &= \mathbf{e}_3
\end{aligned}
\qquad
g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \left(\Theta^1\right)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

and, from 1.16.17,

$$
J = \det\left[\frac{\partial x^i}{\partial \Theta^j}\right] = \Theta^1
$$

so that there is a one-to-one correspondence between the Cartesian and cylindrical coordinates at all point except for $\Theta^1 = 0$ (which corresponds to the axis of the cylinder). These points are called **singular points** of the transformation. The unit vectors and scale factors are {▲Problem 11}

$$
\begin{aligned}
h_1 &= |\mathbf{g}_1| = 1 \quad (=1) & \hat{\mathbf{g}}_1 &= \mathbf{g}_1 \quad (= \mathbf{e}_r) \\
h_2 &= |\mathbf{g}_2| = \Theta^1 \, (= r) & \hat{\mathbf{g}}_2 &= \frac{\mathbf{g}_2}{\Theta^1} (= \mathbf{e}_\theta) \\
h_3 &= |\mathbf{g}_3| = 1 \quad (=1) & \hat{\mathbf{g}}_3 &= \mathbf{g}_3 \quad (= \mathbf{e}_z)
\end{aligned}
$$

The line, surface and volume elements are

Metric: $\quad (\Delta s)^2 = \left(d\Theta^1\right)^2 + \left(\Theta^1 d\Theta^2\right)^2 + \left(d\Theta^3\right)^2 \quad \left(= dr^2 + (r d\theta)^2 + dz^2\right)$

$$\Delta S_1 = \Theta^1 \Delta\Theta^2 \Delta\Theta^3$$

Surface Element: $\Delta S_2 = \quad \Delta\Theta^3 \Delta\Theta^1$

$$\Delta S_3 = \Theta^1 \Delta\Theta^1 \Delta\Theta^2$$

Volume Element: $\Delta V = \Theta^1 \Delta\Theta^1 \Delta\Theta^2 \Delta\Theta^3 \quad (= r\Delta r \Delta\theta \Delta z)$

2. Spherical Coordinates

Consider the spherical coordinates, $(r, \theta, \phi) = (\Theta^1, \Theta^2, \Theta^3)$, *cf.* §1.6.10, Fig. 1.16.10:

$$\begin{aligned} x^1 &= \Theta^1 \sin\Theta^2 \cos\Theta^3 \\ x^2 &= \Theta^1 \sin\Theta^2 \sin\Theta^3 , \\ x^3 &= \Theta^1 \cos\Theta^2 \end{aligned} \qquad \begin{aligned} \Theta_1 &= \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \\ \Theta_2 &= \tan^{-1}\left(\sqrt{(x^1)^2 + (x^2)^2} / (x^3)^2 \right) \\ \Theta_3 &= \tan^{-1}\left((x^2)^2 / (x^1)^2 \right) \end{aligned}$$

with

$$\Theta_1 \geq 0, \quad 0 \leq \Theta^2 \leq \pi, \quad 0 \leq \Theta^3 < 2\pi$$

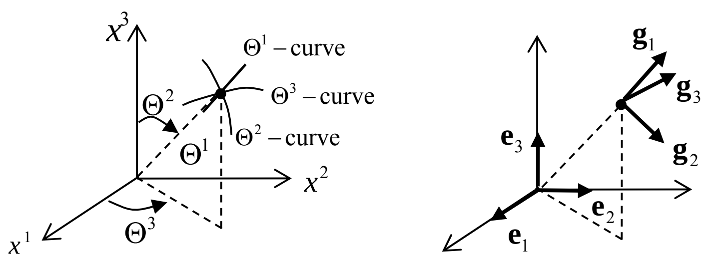

Figure 1.16.10: Spherical Coordinates

From Eqns. 1.16.19 (compare with 1.6.36), 1.16.27,

$$\begin{aligned} \mathbf{g}_1 &= +\sin\Theta^2 \cos\Theta^3 \mathbf{e}_1 + \sin\Theta^2 \sin\Theta^3 \mathbf{e}_2 + \cos\Theta^2 \mathbf{e}_3 \\ \mathbf{g}_2 &= \Theta^1 \left(+\cos\Theta^2 \cos\Theta^3 \mathbf{e}_1 + \cos\Theta^2 \sin\Theta^3 \mathbf{e}_2 - \sin\Theta^2 \mathbf{e}_3 \right), \\ \mathbf{g}_3 &= \Theta^1 \sin\Theta^2 \left(-\sin\Theta^3 \mathbf{e}_1 + \cos\Theta^3 \mathbf{e}_2 \right) \end{aligned} \quad g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (\Theta^1)^2 & 0 \\ 0 & 0 & (\Theta^1 \sin\Theta^2)^2 \end{bmatrix}$$

and, from 1.16.17,

$$J = \det\left[\frac{\partial x^i}{\partial \Theta^j} \right] = (\Theta^1)^2 \sin\Theta^2$$

so that there is a one-to-one correspondence between the Cartesian and spherical coordinates at all point except for the singular points along the x^3 axis.

The unit vectors and scale factors are {▲Problem 11}

$$\begin{array}{llll} h_1 = |\mathbf{g}_1| = 1 & (=1) & \hat{\mathbf{g}}_1 = \mathbf{g}_1 & (=\mathbf{e}_r) \\ h_2 = |\mathbf{g}_2| = \Theta^1 & (=r) & \hat{\mathbf{g}}_2 = \dfrac{\mathbf{g}_2}{\Theta^1} & (=\mathbf{e}_\theta) \\ h_3 = |\mathbf{g}_3| = \Theta^1 \sin\Theta^2 & (=r\sin\theta) & \hat{\mathbf{g}}_3 = \dfrac{\mathbf{g}_3}{\Theta^1 \sin\Theta^2} & (=\mathbf{e}_\phi) \end{array}$$

The line, surface and volume elements are

Metric:

$$(\Delta s)^2 = (d\Theta^1)^2 + (\Theta^1 d\Theta^2)^2 + (\Theta^1 \sin\Theta^2 d\Theta^3)^2$$
$$\left(= dr^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2\right)$$

Surface Element:

$$\Delta S_1 = (\Theta^1)^2 \sin\Theta^2 \Delta\Theta^2 \Delta\Theta^3$$
$$\Delta S_2 = \Theta^1 \sin\Theta^2 \Delta\Theta^3 \Delta\Theta^1$$
$$\Delta S_3 = \Theta^1 \Delta\Theta^1 \Delta\Theta^2$$

Volume Element:

$$\Delta V = (\Theta^1)^2 \sin\Theta^2 \Delta\Theta^1 \Delta\Theta^2 \Delta\Theta^3 \quad \left(= r^2 \sin\theta \Delta r \Delta\theta \Delta\phi\right)$$

1.16.10 Vectors in Curvilinear Coordinates

A vector can now be represented in terms of *either* basis:

$$\mathbf{u} = u_i(\Theta^1, \Theta^2, \Theta^3)\mathbf{g}^i = u^i(\Theta^1, \Theta^2, \Theta^3)\mathbf{g}_i \qquad (1.16.47)$$

The u_i are the **covariant components** of **u** and u^i are the **contravariant components** of **u**. Thus the covariant components are the coefficients of the contravariant base vectors and *vice versa* – subscripts denote covariance while superscripts denote contravariance.

Analogous to the orthonormal case, where $\mathbf{u}\cdot\mathbf{e}_i = u_i$ {▲Problem 4}:

$$\mathbf{u}\cdot\mathbf{g}_i = u_i, \quad \mathbf{u}\cdot\mathbf{g}^i = u^i \qquad (1.16.48)$$

Note the following useful formula involving the metric coefficients, for raising or lowering the index on a vector component, relating the covariant and contravariant components, {▲Problem 5}

$$u^i = g^{ij}u_j, \quad u_i = g_{ij}u^j \qquad (1.16.49)$$

Physical Components of a Vector

The contravariant and covariant components of a vector do not have the same physical significance in a curvilinear coordinate system as they do in a rectangular Cartesian system; in fact, they often have different dimensions. For example, the differential $d\mathbf{x}$ of the position vector has in cylindrical coordinates the contravariant components

$(dr, d\theta, dz)$, that is, $d\mathbf{x} = d\Theta^1\mathbf{g}_1 + d\Theta^2\mathbf{g}_2 + d\Theta^3\mathbf{g}_3$ with $\Theta^1 = r$, $\Theta^2 = \theta$, $\Theta^3 = z$. Here, $d\theta$ does not have the same dimensions as the others. The **physical components** in this example are $(dr, rd\theta, dz)$.

The physical components $u^{\langle i \rangle}$ of a vector $\mathbf{u}$ are defined to be the components along the *covariant* base vectors, referred to unit vectors. Thus,

$$\begin{aligned}\mathbf{u} &= u^i\mathbf{g}_i \\ &= \sum_{i=1}^{3} u^i h_i \hat{\mathbf{g}}_i \equiv u^{\langle i \rangle}\hat{\mathbf{g}}_i\end{aligned} \tag{1.16.50}$$

and

$$\boxed{u^{\langle i \rangle} = h_i u^i = \sqrt{g_{ii}}\,u^i} \quad \text{(no sum)} \quad \textbf{Physical Components of a Vector} \tag{1.16.51}$$

From the above, the physical components of a vector $\mathbf{v}$ in the cylindrical coordinate system are v^1, $\Theta^1 v^2$, v^3 and, for the spherical system, Θ^1, $\Theta^1 v^2$, $\Theta^1 \sin\Theta^2 v^3$.

The Vector Dot Product

The dot product of two vectors can be written in one of two ways: {▲Problem 6}

$$\boxed{\mathbf{u}\cdot\mathbf{v} = u_i v^i = u^i v_i} \quad \textbf{Dot Product of Two Vectors} \tag{1.16.52}$$

The Vector Cross Product

The triple scalar product is an important quantity in analysis with general bases, particularly when evaluating cross products. From Eqns. 1.16.20, 1.16.24, 1.16.24,

$$\begin{aligned}g &= [\mathbf{g}_1\cdot\mathbf{g}_2\times\mathbf{g}_3]^2 = \det[g_{ij}] \\ &= \frac{1}{[\mathbf{g}^1\cdot\mathbf{g}^2\times\mathbf{g}^3]^2} = \frac{1}{\det[g^{ij}]}\end{aligned} \tag{1.16.53}$$

Introducing permutation symbols e_{ijk}, e^{ijk}, one can in general write[5]

$$e_{ijk} \equiv \mathbf{g}_i\cdot\mathbf{g}_j\times\mathbf{g}_k = \varepsilon_{ijk}\sqrt{g}, \qquad e^{ijk} \equiv \mathbf{g}^i\cdot\mathbf{g}^j\times\mathbf{g}^k = \varepsilon^{ijk}\frac{1}{\sqrt{g}}$$

---

[5] assuming the base vectors form a *right* handed set, otherwise a negative sign needs to be included

where $\varepsilon_{ijk} = \varepsilon^{ijk}$ is the Cartesian permutation symbol (Eqn. 1.3.10). The cross product of the base vectors can now be written in terms of the reciprocal base vectors as (note the similarity to the Cartesian relation 1.3.12) {▲Problem 7}

$$\boxed{\begin{aligned}\mathbf{g}_i \times \mathbf{g}_j &= e_{ijk}\mathbf{g}^k \\ \mathbf{g}^i \times \mathbf{g}^j &= e^{ijk}\mathbf{g}_k\end{aligned}} \qquad \textbf{Cross Products of Base Vectors} \qquad (1.16.54)$$

Further, from 1.3.19,

$$e^{ijk}e_{pqr} = \varepsilon^{ijk}\varepsilon_{pqr}, \qquad e^{ijk}e_{pqk} = \delta^i_p\delta^j_q - \delta^j_p\delta^i_q \qquad (1.16.55)$$

The Cross Product

The cross product of vectors can be written as {▲Problem 8}

$$\boxed{\begin{aligned}\mathbf{u}\times\mathbf{v} = e_{ijk}u^i v^j \mathbf{g}^k &= \sqrt{g}\begin{vmatrix}\mathbf{g}^1 & \mathbf{g}^2 & \mathbf{g}^3 \\ u^1 & u^2 & u^3 \\ v^1 & v^2 & v^3\end{vmatrix} \\ = e^{ijk}u_i v_j \mathbf{g}_k &= \frac{1}{\sqrt{g}}\begin{vmatrix}\mathbf{g}_1 & \mathbf{g}_2 & \mathbf{g}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3\end{vmatrix}\end{aligned}} \qquad \textbf{Cross Product of Two Vectors} \qquad (1.16.56)$$

1.16.11 Tensors in Curvilinear Coordinates

Tensors can be represented in any of four ways, depending on which combination of base vectors is being utilised:

$$\mathbf{A} = A^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = A_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = A^i_{\cdot j}\mathbf{g}_i \otimes \mathbf{g}^j = A_i^{\cdot j}\mathbf{g}^i \otimes \mathbf{g}_j \qquad (1.16.56)$$

Here, A^{ij} are the **contravariant components**, A_{ij} are the **covariant components**, $A^i_{\cdot j}$ and $A_i^{\cdot j}$ are the **mixed components** of the tensor **A**. On the mixed components, the subscript is a covariant index, whereas the superscript is called a contravariant index. Note that the "first" index always refers to the first base vector in the tensor product.

An "index switching" rule for tensors is

$$A_{ij}\delta^j_k = A_{ik}, \quad A^{ij}\delta^k_j = A^{ik} \qquad (1.16.57)$$

and the rule for obtaining the components of a tensor **A** is (compare with 1.9.4), {▲Problem 9}

$$\begin{aligned}(\mathbf{A})^{ij} &\equiv A^{ij} = \mathbf{g}^i \cdot \mathbf{A}\mathbf{g}^j \\ (\mathbf{A})_{ij} &\equiv A_{ij} = \mathbf{g}_i \cdot \mathbf{A}\mathbf{g}_j \\ (\mathbf{A})^i_{\cdot j} &\equiv A^i_{\cdot j} = \mathbf{g}^i \cdot \mathbf{A}\mathbf{g}_j \\ (\mathbf{A})_i^{\cdot j} &\equiv A_i^{\cdot j} = \mathbf{g}_i \cdot \mathbf{A}\mathbf{g}^j\end{aligned} \tag{1.16.58}$$

As with the vectors, the metric coefficients can be used to lower and raise the indices on tensors, for example:

$$\begin{aligned}T^{ij} &= g^{ik} g^{jl} T_{kl} \\ T_i^{\cdot j} &= g_{ik} T^{kj}\end{aligned} \tag{1.16.59}$$

In matrix form, these expressions can be conveniently used to evaluate tensor components, e.g. (note that the matrix of metric coefficients is symmetric)

$$\left[T^{ij}\right] = \left[g^{ik}\right]\left[T_{kl}\right]\left[g^{lj}\right].$$

An example of a higher order tensor is the permutation tensor **E**, whose components are the permutation symbols introduced earlier:

$$\mathbf{E} = e_{ijk}\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k = e^{ijk}\mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}_k . \tag{1.16.60}$$

Physical Components of a Tensor

Physical components of tensors can also be defined. For example, if two vectors **a** and **b** have physical components as defined earlier, then the physical components of a tensor **T** are obtained through[6]

$$a^{\langle i \rangle} = T^{\langle ij \rangle} b^{\langle j \rangle} . \tag{1.16.61}$$

As mentioned, physical components are defined with respect to the covariant base vectors, and so the mixed components of a tensor are used, since

$$\mathbf{T}\mathbf{b} = T^i_{\cdot j}\left(\mathbf{g}_i \otimes \mathbf{g}^j\right) b^k \mathbf{g}_k = T^i_{\cdot j} b^j \mathbf{g}_i \equiv a^i \mathbf{g}_i$$

as required. Then

$$T^i_{\cdot j} \frac{b^{\langle j \rangle}}{\sqrt{g_{jj}}} = \frac{a^{\langle i \rangle}}{\sqrt{g_{ii}}} \qquad \text{(no sum on the } g\text{)}$$

and so from 1.16.51,

[6] these are called *right* physical components; *left* physical components are defined through $\mathbf{a} = \mathbf{b}\mathbf{T}$

$$\boxed{T^{\langle ij \rangle} = \frac{\sqrt{g_{ii}}}{\sqrt{g_{jj}}} T_{\cdot j}^{i}} \quad \text{(no sum)} \quad \textbf{Physical Components of a Tensor} \quad (1.16.62)$$

The Identity Tensor

The components of the identity tensor **I** in a general basis can be obtained as follows:

$$\begin{aligned}
\mathbf{u} &= u^i \mathbf{g}_i \\
&= g^{ij} u_j \mathbf{g}_i \\
&= g^{ij} (\mathbf{u} \cdot \mathbf{g}_j) \mathbf{g}_i \\
&= g^{ij} (\mathbf{g}_i \otimes \mathbf{g}_j) \mathbf{u} \\
&\equiv \mathbf{Iu}
\end{aligned}$$

Thus the contravariant components of the identity tensor are the metric coefficients g^{ij} and, similarly, the covariant components are g_{ij}. For this reason the identity tensor is also called the **metric tensor**. On the other hand, the mixed components are the Kronecker delta, δ_j^i (also denoted by g_j^i). In summary[7],

$$\begin{aligned}
(\mathbf{I})_{ij} &= g_{ij} & \mathbf{I} &= g_{ij} \left(\mathbf{g}^i \otimes \mathbf{g}^j \right) \\
(\mathbf{I})^{ij} &= g^{ij} & \mathbf{I} &= g^{ij} \left(\mathbf{g}_i \otimes \mathbf{g}_j \right) \\
(\mathbf{I})_{\cdot j}^{i} &= \delta_j^i & \mathbf{I} &= \delta_j^i \left(\mathbf{g}_i \otimes \mathbf{g}^j \right) = \mathbf{g}_i \otimes \mathbf{g}^i \\
(\mathbf{I})_{i}^{\cdot j} &= \delta_i^j & \mathbf{I} &= \delta_i^j \left(\mathbf{g}^i \otimes \mathbf{g}_j \right) = \mathbf{g}^i \otimes \mathbf{g}_i
\end{aligned} \qquad (1.16.63)$$

Symmetric Tensors

A tensor **S** is symmetric if $\mathbf{S}^{\mathrm{T}} = \mathbf{S}$, i.e. if $\mathbf{uSv} = \mathbf{vSu}$. If **S** is symmetric, then

$$S^{ij} = S^{ji}, \quad S_{ij} = S_{ji}, \quad S_{\cdot j}^{i} = S_{j}^{\cdot i} = g_{jk} g^{im} S_{\cdot m}^{k}$$

In terms of matrices,

$$\left[S^{ij} \right] = \left[S^{ij} \right]^T, \quad \left[S_{ij} \right] = \left[S_{ij} \right]^T, \quad \left[S_{\cdot j}^{i} \right] \neq \left[S_{\cdot j}^{i} \right]^{\mathrm{T}}$$

1.16.12 Generalising Cartesian Relations to the Case of General Bases

The tensor relations and definitions already derived for Cartesian vectors and tensors in previous sections, for example in §1.10, are valid also in curvilinear coordinates, for

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[7] there is no distinction between δ_i^j, δ_j^i; they are often written as g_i^j, g_j^i and there is no need to specify which index comes first, for example by $g_{\cdot i}^{j}$

example $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, $\mathrm{tr}\mathbf{A} = \mathbf{I} : \mathbf{A}$ and so on. Formulae involving the index notation may be generalised to arbitrary components by:

(1) raising or lowering the indices appropriately
(2) replacing the (ordinary) Kronecker delta δ_{ij} with the metric coefficients g_{ij}
(3) replacing the Cartesian permutation symbol ε_{ijk} with e_{ijk} in vector cross products

Some examples of this are given in Table 1.16.1 below.

Note that there is only one way of representing a scalar, there are two ways of representing a vector (in terms of its covariant or contravariant components), and there are four ways of representing a (second-order) tensor (in terms of its covariant, contravariant and both types of mixed components).

	Cartesian	General Bases
$\mathbf{a}\cdot\mathbf{b}$	$a_i b_i$	$a_i b^i = a^i b_i$
$\mathbf{aB}$	$a_i B_{ij}$	$(\mathbf{aB})_j = a_i B^i_{\cdot j} = a^i B_{ij}$ $(\mathbf{aB})^j = a_i B^{ij} = a^i B_i^{\cdot j}$
$\mathbf{Ab}$	$A_{ij} b_j$	$(\mathbf{Ab})_i = A_{ij} b^j = A_i^{\cdot j} b_j$ $(\mathbf{Ab})^i = A^{ij} b_j = A^i_{\cdot j} b^j$
$\mathbf{AB}$	$A_{ik} B_{kj}$	$(\mathbf{AB})_{ij} = A_{ik} B^k_{\cdot j} = A_i^{\cdot k} B_{kj}$ $(\mathbf{AB})^{ij} = A^{ik} B_k^{\cdot j} = A^i_{\cdot k} B^{kj}$ $(\mathbf{AB})_i^{\cdot j} = A_i^{\cdot k} B_k^{\cdot j} = A_{ik} B^{kj}$ $(\mathbf{AB})^i_{\cdot j} = A^{ik} B_{kj} = A^i_{\cdot k} B^k_{\cdot j}$
$\mathbf{a}\times\mathbf{b}$	$\varepsilon_{ijk} a_i b_j$	$(\mathbf{a}\times\mathbf{b})_k = e_{ijk} a^i b^j$ $(\mathbf{a}\times\mathbf{b})^k = e^{ijk} a_i b_j$
$\mathbf{a}\otimes\mathbf{b}$	$a_i b_j$	$(\mathbf{a}\otimes\mathbf{b})_{ij} = a_i b_j$ $(\mathbf{a}\otimes\mathbf{b})^{ij} = a^i b^j$ $(\mathbf{a}\otimes\mathbf{b})_i^{\cdot j} = a_i b^j$ $(\mathbf{a}\otimes\mathbf{b})^i_{\cdot j} = a^i b_j$
$\mathbf{A}:\mathbf{B}$	$A_{ij} B_{ij}$	$A_{ij} B^{ij} = A^{ij} B_{ij} = A^i_{\cdot j} B_i^{\cdot j} = A_i^{\cdot j} B^i_{\cdot j}$
$\mathrm{tr}\mathbf{A} \equiv \mathbf{I}:\mathbf{A}$	A_{ii}	$A_i^{\cdot i} = A^i_{\cdot i}$
$\det\mathbf{A}$	$\varepsilon_{ijk} A_{i1} A_{j2} A_{k3}$	$\varepsilon_{ijk} A^i_{\cdot 1} A^j_{\cdot 2} A^k_{\cdot 3}$
$\mathbf{A}^{\mathrm{T}}$	$(\mathbf{A}^T)_{ij} = A_{ji}$	$(\mathbf{A}^{\mathrm{T}})_{ij} = A_{ji}$, $(\mathbf{A}^{\mathrm{T}})^{ij} = A^{ji}$ $(\mathbf{A}^{\mathrm{T}})^i_{\cdot j} = A_j^{\cdot i} \neq A^j_{\cdot i}$, $(\mathbf{A}^{\mathrm{T}})_i^{\cdot j} = A^j_{\cdot i} \neq A_j^{\cdot i}$

Table 1.16.1: Tensor relations in Cartesian and general curvilinear coordinates

Rectangular Cartesian (Orthonormal) Coordinate System

In an orthonormal Cartesian coordinate system, $\mathbf{g}_i = \mathbf{g}^i = \mathbf{e}_i$, $g_{ij} = \delta_{ij}$, $g = 1$, $h_i = 1$ and $e_{ijk} = \varepsilon_{ijk}$ $(= \varepsilon^{ijk})$.

1.16.13 Problems

1. Derive the fundamental relation $\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j$.
2. Show that $\mathbf{g}_i = g_{ij}\mathbf{g}^j$ [Hint: assume that one can write $\mathbf{g}_i = a_{ik}\mathbf{g}^k$ and then dot both sides with $\mathbf{g}_j$.]
3. Use the relations 1.16.29 to show that $g^{ij} g_{kj} = \delta^i_k$. Write these equations in matrix form.
4. Show that $\mathbf{u} \cdot \mathbf{g}_i = u_i$.
5. Show that $u_i = g_{ij} u^j$.
6. Show that $\mathbf{u} \cdot \mathbf{v} = u_i v^i = u^i v_i$
7. Use the relation $e_{ijk} \equiv \mathbf{g}_i \cdot \mathbf{g}_j \times \mathbf{g}_k = \varepsilon_{ijk}\sqrt{g}$ to derive the cross product relation $\mathbf{g}_i \times \mathbf{g}_j = e_{ijk}\mathbf{g}^k$. [Hint: show that $\mathbf{g}_i \times \mathbf{g}_j = (\mathbf{g}_i \times \mathbf{g}_j \cdot \mathbf{g}_k)\mathbf{g}^k$.]
8. Derive equation 1.16.56 for the cross product of vectors
9. Show that $(\mathbf{A})_{ij} = \mathbf{g}_i \cdot \mathbf{A}\mathbf{g}_j$.
10. Given $\mathbf{g}_1 = \mathbf{e}_1, \mathbf{g}_2 = \mathbf{e}_2, \mathbf{g}_3 = \mathbf{e}_1 + \mathbf{e}_3$, $\mathbf{v} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$. Find $\mathbf{g}^i, g_{ij}, e_{ijk}, v_i, v^j$ (write the metric coefficients in matrix form).
11. Derive the scale factors for the (a) cylindrical and (b) spherical coordinate systems.
12. **Parabolic Cylindrical** (orthogonal) **coordinates** are given by
$$x^1 = \tfrac{1}{2}\left((\Theta^1)^2 - (\Theta^2)^2\right), \quad x^2 = \Theta^1\Theta^2, \quad x^3 = \Theta^3$$
with
$$-\infty < \Theta^1 < \infty, \quad \Theta^2 \geq 0, \quad -\infty < \Theta^3 < \infty$$
Evaluate:
    (i) the scale factors
    (ii) the Jacobian – are there any singular points?
    (iii) the metric, surface elements, and volume element

    Verify that the base vectors $\mathbf{g}_i$ are mutually orthogonal.

    [These are intersecting parabolas in the $x^1 - x^2$ plane, all with the same axis]
13. Repeat Problem 7 for the **Elliptical Cylindrical** (orthogonal) **coordinates**:
$$x^1 = a\cosh\Theta^1 \cos\Theta^2, \quad x^2 = a\sinh\Theta^1 \sin\Theta^2, \quad x^3 = \Theta^3$$
with
$$\Theta^1 \geq 0, \quad 0 \leq \Theta^2 < 2\pi, \quad -\infty < \Theta^3 < \infty$$
[These are intersecting ellipses and hyperbolas in the $x^1 - x^2$ plane with foci at $x^1 = \pm a$.]
14. Consider the non-orthogonal curvilinear system illustrated in Fig. 1.16.11, with transformation equations

$$\Theta^1 = x^1 - \frac{1}{\sqrt{3}} x^2$$
$$\Theta^2 = \frac{2}{\sqrt{3}} x^2$$
$$\Theta^3 = x^3$$

Derive the inverse transformation equations, i.e. $x^i = x^i(\Theta^1, \Theta^2, \Theta^3)$,
the Jacobian matrices

$$\mathbf{J} = \left[\frac{\partial x^i}{\partial \Theta^j}\right], \quad \mathbf{J}^{-1} = \left[\frac{\partial \Theta^i}{\partial x^j}\right],$$

the covariant and contravariant base vectors, the matrix representation of the metric coefficients $[g_{ij}]$, $[g^{ij}]$, verify that $\mathbf{J}^{\mathrm{T}}\mathbf{J} = [g_{ij}]$, $\mathbf{J}^{-1}\mathbf{J}^{-\mathrm{T}} = [g^{ij}]$ and evaluate g.

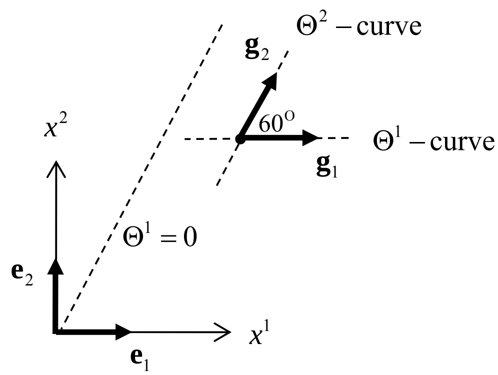

Figure 1.16.11: non-orthogoanl curvilinear coordinate system

15. Consider a (two dimensional) curvilinear coordinate system with covariant base vectors

$$\mathbf{g}_1 = \mathbf{e}_1, \quad \mathbf{g}_2 = \mathbf{e}_1 + \mathbf{e}_2$$

(a) Evaluate the contravariant base vectors and the metric coefficients g_{ij}, g^{ij}

(b) Consider the vectors

$$\mathbf{u} = \mathbf{g}_1 + 3\mathbf{g}_2, \quad \mathbf{v} = -\mathbf{g}_1 + 2\mathbf{g}_2$$

Evaluate the corresponding covariant components of the vectors. Evaluate $\mathbf{u} \cdot \mathbf{v}$ (this can be done in a number of different ways – by using the relations $u_i v^i$, $u^i v_i$, or by directly dotting the vectors in terms of the base vectors $\mathbf{g}_i$, $\mathbf{g}^i$ and using the metric coefficients)

(c) Evaluate the contravariant components of the vector $\mathbf{w} = \mathbf{A}\mathbf{u}$, given that the mixed components $A^i_{\cdot j}$ are

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Evaluate the contravariant components A^{ij} using the index lowering/raising rule 1.16.59. Re-evaluate the contravariant components of the vector **w** using these components.

16. Consider $\mathbf{A} = A^i_{\cdot j}\mathbf{g}^j \otimes \mathbf{g}_i$. Verify that any of the four versions of **I** in 1.16.63 results in $\mathbf{IA} = \mathbf{I}$.

17. Use the definitions 1.16.19, 1.16.23 to convert $A^{ij}\mathbf{g}_i \otimes \mathbf{g}_j$, $A_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$ and $A^i_{\cdot j}\mathbf{g}_i \otimes \mathbf{g}^j$ to the Cartesian bases. Hence show that $\det \mathbf{A}$ is given by the determinant of the matrix of mixed components, $\det\left[A^i_{\cdot j}\right]$, and not by $\det\left[A^{ij}\right]$ or $\det\left[A_{ij}\right]$.