Mechanics 1: Polar Coordinates

Polar Coordinates, and a Rotating Coordinate System. Let (r, θ) denote the polar coordinates describing the position of a particle. Let $\mathbf{r}_1$ denote a unit vector in the direction of the position vector $\mathbf{r}$, and let $\boldsymbol{\theta}_1$ denote a unit vector perpendicular to $\mathbf{r}$, and in the direction of increasing θ, see Fig. 1.

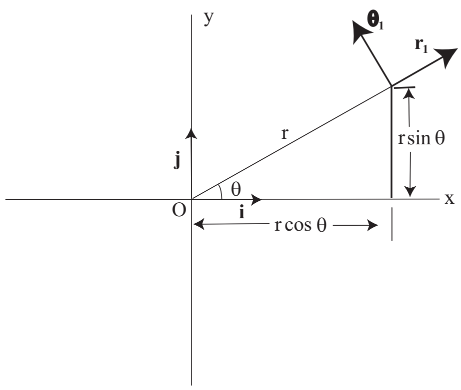

Figure 1:

First, we want to derive expressions for $\mathbf{r}_1$ and $\boldsymbol{\theta}_1$ in terms of $\mathbf{i}$ and $\mathbf{j}$. The resulting equations will tell us how to transform vectors from one coordinate system to another (a subject that we will return to later). From Fig. 1, it is easy to see that the position vector $\mathbf{r}$ is given by:

$$\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}. \tag{1}$$

Therefore we have:

Key point:

$$\mathbf{r}_1 = \frac{\mathbf{r}}{|\mathbf{r}|} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}. \tag{2}$$

It is also easy to verify that:

Key point:

$$\boldsymbol{\theta}_1 = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}. \tag{3}$$

Moreover, one can solve (2) and (3) simultaneously for $\mathbf{i}$ and $\mathbf{j}$ as functions of $\mathbf{r}_1$ and $\boldsymbol{\theta}_1$:

Key point:

$$\mathbf{i} = \cos\theta\mathbf{r}_1 - \sin\theta\boldsymbol{\theta}_1, \tag{4}$$
$$\mathbf{j} = \sin\theta\mathbf{r}_1 + \cos\theta\boldsymbol{\theta}_1. \tag{5}$$

Now there is a big difference between $\mathbf{i}, \mathbf{j}$ and $\mathbf{r}_1, \boldsymbol{\theta}_1$. As the particle moves, $\mathbf{i}, \mathbf{j}$ remain fixed in space with unit length (i.e. their derivatives with respect to t are zero), but $\mathbf{r}_1, \boldsymbol{\theta}_1$ move with the particle. We want to determine how they move by computing their derivatives with respect to t. We begin with $\mathbf{r}_1$. Differentiating (2) with respect to t gives:

Key point:

$$
\begin{aligned}
\dot{\mathbf{r}}_1 &= -\sin\theta\,\dot{\theta}\mathbf{i} + \cos\theta\,\dot{\theta}\mathbf{j}, \\
&= \dot{\theta}\left(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\right), \\
&= \dot{\theta}\boldsymbol{\theta}_1, \qquad\qquad \textbf{using } (3).
\end{aligned} \tag{6}
$$

Next we compute the derivative of $\boldsymbol{\theta}_1$ with respect to t by differentiating (3):

Key point:

$$
\begin{aligned}
\dot{\boldsymbol{\theta}}_1 &= -\cos\theta\,\dot{\theta}\mathbf{i} - \sin\theta\,\dot{\theta}\mathbf{j}, \\
&= -\dot{\theta}\left(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}\right), \\
&= -\dot{\theta}\mathbf{r}_1, \qquad\qquad \textbf{using } (2).
\end{aligned} \tag{7}
$$

Now we can compute the velocity of the particle in the coordinate system defined by $\mathbf{r}_1, \boldsymbol{\theta}_1$, where the position vector of the particle is $\mathbf{r} = r\mathbf{r}_1$:

Key point:

$$
\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{r}_1 + r\frac{d\mathbf{r}_1}{dt} = \dot{r}\mathbf{r}_1 + r\dot{\mathbf{r}}_1 = \dot{r}\mathbf{r}_1 + r\dot{\theta}\boldsymbol{\theta}_1. \tag{8}
$$

Next we compute the acceleration of the particle in this coordinate system:

Key point:

$$
\begin{aligned}
\mathbf{a} = \frac{d\mathbf{v}}{dt} &= \frac{d}{dt}\left(\dot{r}\mathbf{r}_1 + r\dot{\theta}\boldsymbol{\theta}_1\right), \\
&= \ddot{r}\mathbf{r}_1 + \dot{r}\dot{\mathbf{r}}_1 + \dot{r}\dot{\theta}\boldsymbol{\theta}_1 + r\ddot{\theta}\boldsymbol{\theta}_1 + r\dot{\theta}\dot{\boldsymbol{\theta}}_1, \\
&= \ddot{r}\mathbf{r}_1 + \dot{r}(\dot{\theta}\boldsymbol{\theta}_1) + \dot{r}\dot{\theta}\boldsymbol{\theta}_1 + r\ddot{\theta}\boldsymbol{\theta}_1 + (r\dot{\theta})(-\dot{\theta}\mathbf{r}_1), \\
&= (\ddot{r} - r\dot{\theta}^2)\mathbf{r}_1 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{\theta}_1.
\end{aligned} \tag{9}
$$

These formulae will be very useful later on.

Key point: The vectors i and j are of unit length, and their direction is fixed in space. Therefore their derivative with respect to time is zero. The vectors $\mathbf{r}_1$ and θ_1 are also of unit length, but their directions in space change. Therefore their derivatives with respect to time are non-zero, and we have computed these derivatives above. The derivatives of each vector can either be represented in the coordinate system defined by $\mathbf{i}-\mathbf{j}$, or the coordinate system defined by $\mathbf{r}_1 - \theta_1$.