Birzeit University
Physics Department

PHYS212
Experiment 6
Harmonic Analysis

Student Name: Hamza AlHasan

ID: 1181636

Partner Name: Mahmoud Aldweik

ID: 1181683

Instructor: Dr.Wael Karain

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1 Abstract

The aim of this experiment is to determine the frequencies and amplitudes of the Fourier components of a square wave, or any other periodic waveform. Using Fourier theorem, the components of the square wave were calculated. By exploitation the beats phenomena, we could compare between the observed frequencies of the harmonic frequencies with the frequencies expected from the calculations. The circuit containing a diode bridge rectifier and a galvanometer is used, with two signal generators.

2 Theory

2.1 Even and Odd Functions

Let f be defined on an interval, finite or infinite, and centered at $x = 0$. We say that f is an even function if:

$$f(-x) = f(x) \tag{1}$$

and an odd function if:

$$f(-x) = -f(x) \tag{2}$$

for all x in that interval. That is, the graph of f is **symmetric** about $x = 0$ if f is even, and **antisymmetric** about $x = 0$ if f is odd. For Examples, cosine function is even, and sine is odd.

There are several useful algebraic properties of even and odd functions, such as the following:

$$\begin{aligned}
even + even &= even \\
even \times even &= even \\
odd + odd &= odd \\
odd \times odd &= even \\
even \times odd &= odd
\end{aligned}$$

In addition, two useful integral properties are as follows. If f is even, then:

$$\int_{-A}^{A} f(x)dx = 2\int_{0}^{A} f(x)dx \tag{3}$$

and if f is odd, then:

$$\int_{-A}^{A} f(x)dx = 0 \tag{4}$$

Note carefully that a given function is not necessarily even or odd; it may be both even and odd (zero function), or it may be neither.

Next, suppose that for a given function f there exists a positive constant T such that:

$$f(x + T) = f(x) \tag{5}$$

for every x in the domain of f. Then, we say that f is a **periodic** function of x, with period T.

Notice that if f is periodic with period T, it is necessarily periodic with period 2T, 3T, 4T... as well. However, for all these possible periods, if there exists a smallest one, that period is called the **fundamental period**.[1]

2.2 Fourier Series

Let us first consider Fourier analysis for periodic functions. The Fourier theorem states that a periodic function can be constructed by summing a series of sinusoidal waves of different amplitudes, frequencies and phases. This series can possibly involve an infinite number of sine and/or cosine terms. One term in the series appears at the fundamental frequency (ω_0), while the other terms oscillate at frequencies which are harmonics (i.e. multiples) of the fundamental frequency ($\omega_n = n\omega_0$). Therefore, we can mathematically express a periodic function $F(t)$ in Fourier component form as follows:

$$F(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega t) + \sum_{n=1}^{\infty} B_n \sin(n\omega t) \tag{6}$$

where A_0 is the DC component of the periodic function, $A_n \cos(n\omega t) + B_n \sin(n\omega t)$ is the nth harmonic component of the function and $A_n \& B_n$ are known as the Fourier coefficients or amplitudes. The fundamental angular frequency is $\omega = \frac{2\pi}{T}$, where T is the period of the function.

The coefficients are calculated as follows:

$$A_0 = \frac{1}{T}\int_{-T/2}^{T/2} F(t)dt \tag{7}$$

$$A_n = \frac{1}{2T}\int_{-T/2}^{T/2} F(t)\cos(n\omega t)dt \tag{8}$$

$$B_n = \frac{1}{2T}\int_{-T/2}^{T/2} F(t)\sin(n\omega t)dt \tag{9}$$

As an example, the construction of a periodic voltage square wave form is illustrated in the next section.

2.3 Fourier Series For a Square Wave

Suppose the following voltage square wave of period T:

$$V(t) = \begin{cases} -V_0 & , -T/2 < t < 0 \\ +V_0 & , 0 < t < T/2 \end{cases}$$

$$V(t+T) = V(t)$$

The Fourier series expression for this square wave is as following, from equation(6):

$$V(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega t) + \sum_{n=1}^{\infty} B_n \sin(n\omega t)$$

Where:

$$\omega = \frac{2\pi}{T}$$

$$\begin{aligned} A_0 &= \frac{1}{T}\int_{-T/2}^{T/2} V(t)dt \\ &= \frac{1}{T}\left[\int_{-T/2}^{0} -V_0 dt + \int_{0}^{T/2} +V_0 dt\right] \\ &= \frac{1}{T}\left[V_0 t\Big|_0^{-T/2} + V_0 t\Big|_0^{T/2}\right] \\ &= \frac{1}{T}\left[\frac{-V_0 T}{2} + \frac{V_0 T}{2}\right] \\ &= 0 \end{aligned}$$

$$
\begin{aligned}
A_n &= \frac{1}{2T}\int_{-T/2}^{T/2} V(t)\cos(n\omega t)dt \\
&= \frac{1}{2T}\left[\int_{-T/2}^{0} -V_0\cos(n\omega t)dt + \int_{0}^{T/2} V_0\cos(n\omega t)dt\right] \\
&= \frac{1}{2T}\left[\frac{V_0\sin(n\omega t)}{n\omega}\Big|_{0}^{-T/2} + \frac{V_0\sin(n\omega t)}{n\omega}\Big|_{0}^{T/2}\right] \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
B_n &= \frac{1}{2T}\int_{-T/2}^{T/2} V(t)\sin(n\omega t)dt \\
&= \frac{1}{2T}\left[\int_{-T/2}^{0} -V_0\sin(n\omega t)dt + \int_{0}^{T/2} V_0\sin(n\omega t)dt\right] \\
&= \frac{1}{2T}\left[\frac{V_0\cos(n\omega t)}{n\omega}\Big|_{-T/2}^{0} + \frac{V_0\cos(n\omega t)}{n\omega}\Big|_{T/2}^{0}\right] \\
&= \frac{1}{2T}\left[(\frac{V_0}{n\omega} - \frac{V_0(-1)^n}{n\omega}) + (\frac{V_0}{n\omega} - \frac{V_0(-1)^n}{n\omega})\right] \\
&= f\,\left[\frac{V_0}{n\omega} - \frac{V_0(-1)^n}{n\omega}\right] \\
&= \frac{V_0}{2\pi}\left[\frac{1-(-1)^n}{n}\right]
\end{aligned}
$$

Finally, substitute A_0, A_n, and B_n in the original equation, we get:

$$
V(t) = \sum_{n=1}^{\infty} \frac{V_0}{2\pi}\left(\frac{1-(-1)^n}{n}\right)\sin(n\omega t) \tag{10}
$$

Note that $V(t) = 0$ for even n.

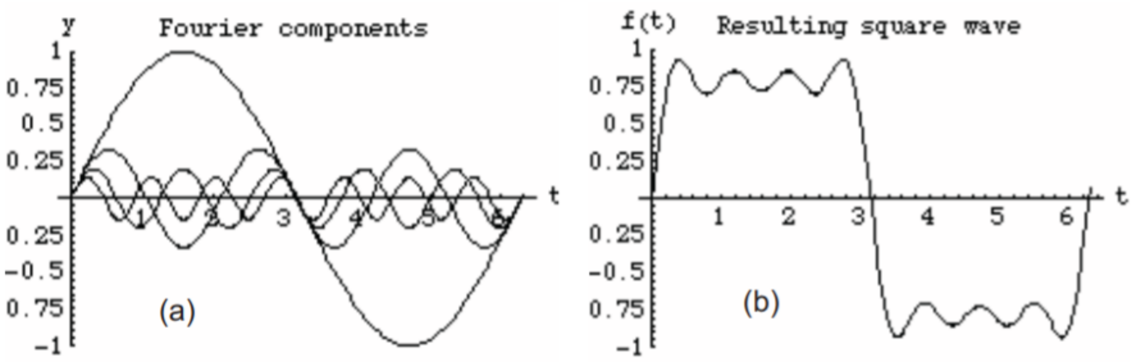


Figure 1: (a) The fundamental wave has amplitude=1, and the rest are harmonics of the fundamental. (b) Summing the waveforms in (a) results in an approximate square wave.

Figure(1) illustrates an example of the Fourier components of a square wave with amplitude=1. It shows the sum of only few terms in the series. However, if we sum up more terms of the series, a better approximation to the square wave is obtained. If we sum an infinite number of terms, we obtain the ideal square wave with perfectly sharp edges.

2.4 Beats

Consider an interference, that results from the superposition of two waves having slightly different frequencies. In this case, when the two waves are observed at a point in space, they are periodically in and out of phase. That is, there is a temporal (time) alternation between constructive and destructive interference. As a consequence, this phenomenon is referred as interference in time or temporal interference. For example, if two tuning forks of slightly different frequencies are struck, one hears a sound of periodically varying amplitude. This phenomenon is called **beating**.

"**Beating** is the periodic variation in amplitude at a given point due to the superposition of two waves having slightly different frequencies." [2]

Consider the superposition of two sinusoidal waves, one with frequency ω_n and the other with frequency ω. We will assume that the waves have some phase difference and that both have amplitude A. The resultant wave is as following:

$$S(t) = A\sin(\omega_n t + \phi_n) + A\sin(\omega t)$$

By using the trigonometric identity:

$$\sin(a) + \sin(b) = 2\sin(\frac{a+b}{2})\cos(\frac{a-b}{2})$$

we obtain:

$$S(t) = 2A\cos(\frac{\omega_n - \omega}{2}t + \frac{\phi_n}{2})\sin(\frac{\omega_n + \omega}{2}t + \frac{\phi_n}{2})$$

The sum of two sinusoidal waves of equal magnitude is equivalent to a product of two sinusoidal waves, one with frequency $(\omega_n - \omega)/2$ and the other with frequency $(\omega_n + \omega)/2$. In general, this will be some complicated waveform; however, consider the case in which ω_n and ω differ by a relatively small amount. The term $\cos\left(\frac{\omega_n - \omega}{2}t + \frac{\phi_n}{2}\right)$ will be a low frequency oscillation, while the term $\sin\left(\frac{\omega_n + \omega}{2}t + \frac{\phi_n}{2}\right)$ will be a high frequency oscillation. Their product is shown in figure(2). The result is a rapidly oscillating function with relatively slow variations in the overall amplitude. These low frequency variations in the amplitude are called beats. As the difference between ω_n and ω becomes small, the period of the beats becomes long.[4]

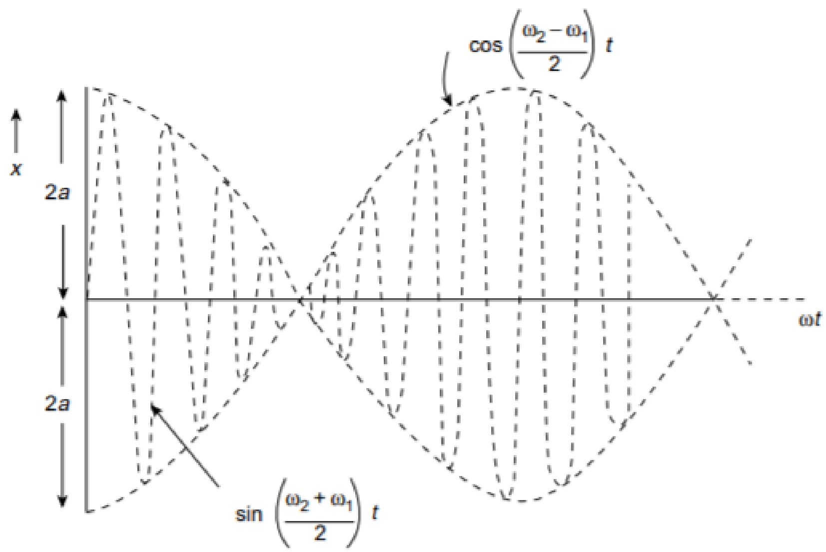


Figure 2: The superposition of two sinusoidal waves having same amplitude and slightly different frequencies[3]

3 Experimental Setup

3.1 Apparatus

The amplitudes and the frequencies of the waveform of the square wave (after rewriting it using Fourier series) are measured by mixing a sinusoidal signal of known frequency ω, and amplitude equal to the square wave amplitude. So, two signal generators are used; one for a square wave (S_1), and the other for the sinusoidal signal (S_2).

Mixing of the waveform and the sinusoidal signal is accomplished using the circuit shown in Figure(3).

The diode bridge rectifier is used to let the current flows through the load in the same direction, regardless of the sign of the input voltage.

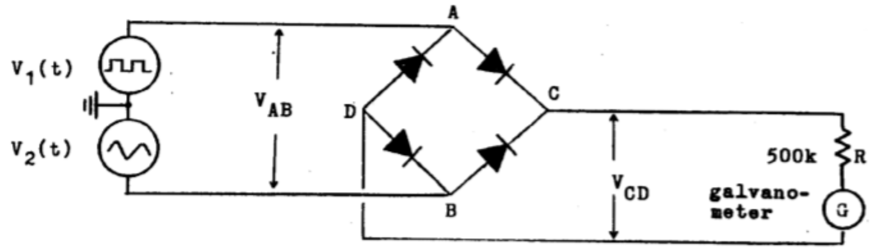


Figure 3: The circuit used

In effect, the full wave rectifier takes the absolute value of the input voltage.

The circuit also includes a galvanometer, which is a very sensitive to DC current and has low response. A resistor R is included to limit the current flowing through the galvanometer. The galvanometer is used to determine the amplitude of the mixed wave (V_{AB}). Also, it averages out the value of a signal.

Now, we want to derive a useful expression for V_{AB}, i.e, we want an expression of beats, because the galvanometer can follow it and detect its amplitude, since it responds only to the low frequency variations in the overall amplitude of the signal. First, we rewrite the equation(10), as following[1]:

$$V_1(t) = \sum_{n=1}^{\infty} A_n \sin(\omega_n t) \tag{11}$$

where:

$$\omega_n = n\omega_0 = 2n\pi f_0$$

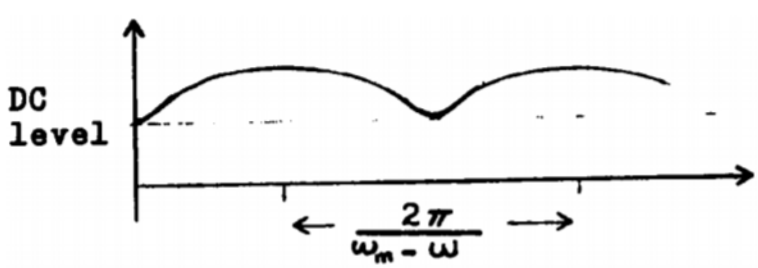
$$A_n = \frac{V_0}{2\pi}\left(\frac{1-(-1)^n}{n}\right) \tag{12}$$

[1]Where $V_1(t)$ is for the square wave, and $V_2(t)$ is for the sinusoidal wave.

Then, we write V_{AB} as following:

$$\begin{aligned}
V_{AB} &= V_A - V_B = V_1(t) - V_2(t) \\
&= \sum_{n=1}^{\infty} A_n \sin(\omega_n t) - A\sin(\omega t) \\
&= \left(\sum_{n=1, n\neq m}^{\infty} A_n \sin(\omega_n t)\right) + A_m \sin(\omega_m t) - A\sin(\omega t) \\
&= \left(\sum_{n=1, n\neq m}^{\infty} A_n \sin(\omega_n t)\right) - (A_m + A)\sin(\omega t) + [A_m \sin(\omega_m t) + A_m \sin(\omega t)] \\
&= \left(\sum_{n=1, n\neq m}^{\infty} A_n \sin(\omega_n t)\right) - (A_m + A)\sin(\omega t) \\
&\qquad\qquad + 2A_m \cos(\frac{\omega_m - \omega}{2} t)\sin(\frac{\omega_m + \omega}{2} t)
\end{aligned}$$

Here, we have the rapidly oscillating part of the signal plus a slow variation in amplitude due to the beats. The slow response of the galvanometer averages out the rapidly oscillating part of the signal (obtain a DC level), and averages out the beats. The signal that we finally detect is shown below.



We see that from the amplitude of the beats we can obtain both the amplitude and the frequency of the harmonic component of the waveform.

3.2 Procedure

- The circuit as shown in figure(3) was connected.
- The signal generator S_1 was set to give a square wave at a frequency of approximately 60 HZ. And the signal generator S_2 was set to provide a sine wave.
- The galvanometer was set on the x.03 scale.
- The signal generators were set for a peak amplitude of $\approx 3V$.
- The amplitudes of the two signals were approximately kept equal.
- The amplitude of the oscillation and the frequency of the sine wave were recorded[2].
- The frequency of the sine wave was increased to an integral multiples of the fundamental frequency (nf_0).
- These measurements were performed 7 times.

4 Data

$\boldsymbol{f_0 = 60}$ **Hz**
$\boldsymbol{A_0 = 11.9}$ **cm**

[2] The sine wave frequency was measured from the oscilloscope trace, not from the dial setting of the generator, it is only approximate, because the generator scales may not be exactly accurate.

Trial	$\boldsymbol{f}$**(Hz)**	$\boldsymbol{f/f_0}$	$\boldsymbol{A}$**(cm)**	$\boldsymbol{A/A_0}$
1	59.91	0.998	11.9	1.000
2	120.0	2.000	0	0
3	179.2	2.987	4.1	0.345
4	239.8	3.997	0	0
5	298.9	4.982	1.8	0.151
6	359.8	5.997	0	0
7	417.8	6.963	1.7	0.143

5 Analysis and Calculations

Using equations(11 & 12):

$$V(t) = A_1 \sin(\omega_1 t) + A_2 \sin(\omega_2 t) + A_3 \sin(\omega_3 t) + A_4 \sin(\omega_4 t) \\ + A_5 \sin(\omega_5 t) + A_6 \sin(\omega_6 t) + A_7 \sin(\omega_7 t) + ...$$

In this experiment, the first seven terms were taken and analysed.

For this seven terms, by dividing the beat frequency of them by the fundamental frequency, we obtain the following ratios (theoretically & experimentally):

$\boldsymbol{\omega_n/\omega_0}$	**Theoretically**	**Experimentally**
ω_1/ω_0	1	0.998
ω_2/ω_0	2	2.000
ω_3/ω_0	3	2.987
ω_4/ω_0	4	3.997
ω_5/ω_0	5	4.982
ω_6/ω_0	6	5.997
ω_7/ω_0	7	6.963

Now, from equation(8), the theoretical amplitude ratio is as following:

$$\frac{A_n}{A_0} = \frac{1}{n} \tag{13}$$

So by dividing the beat amplitudes (A_n) by the amplitude of the beat at the fundamental frequency (A_0), we obtain:

A_n/A_0	**Theoretically**	**Experimentally**
A_1/A_0	1	1.000
A_2/A_0	0	0
A_3/A_0	0.333	0.345
A_4/A_0	0	0
A_5/A_0	0.200	0.151
A_6/A_0	0	0
A_7/A_0	0.143	0.143

6 Discussion and Conclusion

From the table of frequency ratios, the theoretical frequency must be an integer multiple of ω_0, as shown in equation(10). Note that the theoretical and the experimental values are almost the same with an error less than 0.53% [3].

And from the table of amplitudes ratios, the even terms must be zero according to equation(8). So as shown in experimental values, $A_2 = A_4 = A_6 = 0$, which consistent with the theoretical values of them. Moreover, the values of A_1 and A_7 are the same as their theoretical values. However, there are some errors in the values of A_3 and A_5 :

[3]Since the maximum error is at trial 7: $(1 - \frac{6.963}{7}) \times 100\% = 0.53\%$

$E_3 = \frac{|0.333-0.345|}{0.333} \times 100\% = 3.60\%$

$E_5 = \frac{|0.200-0.151|}{0.200} \times 100\% = 24.5\%$

In this experiment, there are some sources of error, such as:

- The recorded values of frequency were not exactly precise in the signal generator. The values were not stable.
- Due to the error in reading the frequencies, the amplitude read from the galvanometer was affected. As a consequence, the values of the amplitudes in some cases had some errors, as shown in E_3 and the high error percentage in E_5.
- The low precise in reading amplitudes from the galvanometer spot, whose response is slow. So we didn't record the amplitude exactly.

Finally, we have shown by studying seven terms that we can write the square wave function as a sum of harmonic functions: sine and cosine. If more terms are studied, the values would be close or almost close with some errors to the theoretical calculated values. Such a process is called **Harmonic Analysis**; which is rewrite a periodic function as a sum of harmonic functions.

References

[1] D.Greenberg, M. (1998). Advanced Engineering Mathematics. 2nd ed. Delaware: Prentice Hall, pp.846-850.

[2] A. Serway, R. and W. Jewett, J. (2010). Physics for Scientists and Engineers with Modern Physics. 8th ed. USA: Brooks Cole, pp.528-530.

[3] J. Pain, H. (2005). The Physics of Vibrations and Waves. 6th ed. London: John Wiley and Sons, pp.13-15.

[4] Seelly, M., Shawabkeh, A. and Abusara, H. (2016). Phys 212: Modern Physics Lab. 2nd ed. Birzeit: Birzeit University, pp.25-36.