BIRZEIT UNIVERSITY

Faculty of Science

Physics Department

Physics 212

Harmonic Analysis

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– Abstract:

The aim of this experiment is to determine the frequencies and amplitudes of the Fourier components of a square wave, or any other periodic waveform. Using Fourier theorem, the components of the square wave were calculated. By exploitation the beats phenomena, we could compare between the observed frequencies of the harmonic frequencies with the frequencies expected from the calculations.

The main results of this experiment are:

$$V_{S_1}(t) = \frac{3}{2} + \sum_{\substack{n=1 \\ n:odd}}^{\infty} \frac{6}{n\pi} \sin(120n\pi t)$$

Where,

$$A_0 = 3\,, \qquad A_n = 0\,, \qquad B_n = \begin{cases} 0 \,, & n: even \\ \dfrac{6}{n\pi} \,, & n: odd \end{cases}\,, \qquad \omega_0 = 120\pi\ rad/s$$

Experimental and their expected theoretical results:

m	$f_{S_{2exp}}(Hz)$	A_{exp}	$(A_m/A_1)_{exp}$	$f_{S_{2theo}}(Hz)$	A_{theo}	$(A_m/A_1)_{theo}$
1	59.73	18.7	1.000	60.00	19.1	1.000
3	178.6	6.2	0.332	180.0	6.4	0.333
5	297.5	3.4	0.182	300.0	3.8	0.200
7	416.8	2.7	0.144	420.0	2.7	0.143
9	536.1	2.0	0.107	540.0	2.1	0.111
11	655.1	1.8	0.096	660.0	1.7	0.091
13	774.1	1.4	0.075	780.0	1.5	0.077
15	893.3	1.3	0.070	900.0	1.3	0.067

– Theory:

Fourier Theorem is a mathematical theorem stating that a periodic function $F(t)$ which is reasonably continuous may be expressed as the sum of a series of sine or cosine terms (called the Fourier series), each of which has specific amplitude and phase coefficients known as Fourier coefficients.

Consider a function $F(t)$ that is periodic with period T and angular frequency ω_0. Then, we can write Fourier series as following:

$$F(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}[A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)]$$

Where,

$$A_0 = \frac{2}{T}\int_{-T/2}^{T/2} F(t)dt$$

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} F(t)\cos(n\omega_0 t)\,dt$$

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} F(t)\sin(n\omega_0 t)\,dt$$

When two sound waves of different frequency approach your ear, the alternating constructive and destructive interference causes the sound to be alternatively soft and loud - a phenomenon which is called "beating" or producing beats.

In order to measure the amplitudes and frequencies of the Fourier components of the waveform, we will use the phenomenon of beats. Consider the superposition of two sinusoidal waves, one with frequency ω_n and the other with frequency ω. We will assume that the waves have some phase difference and that both have amplitude A.

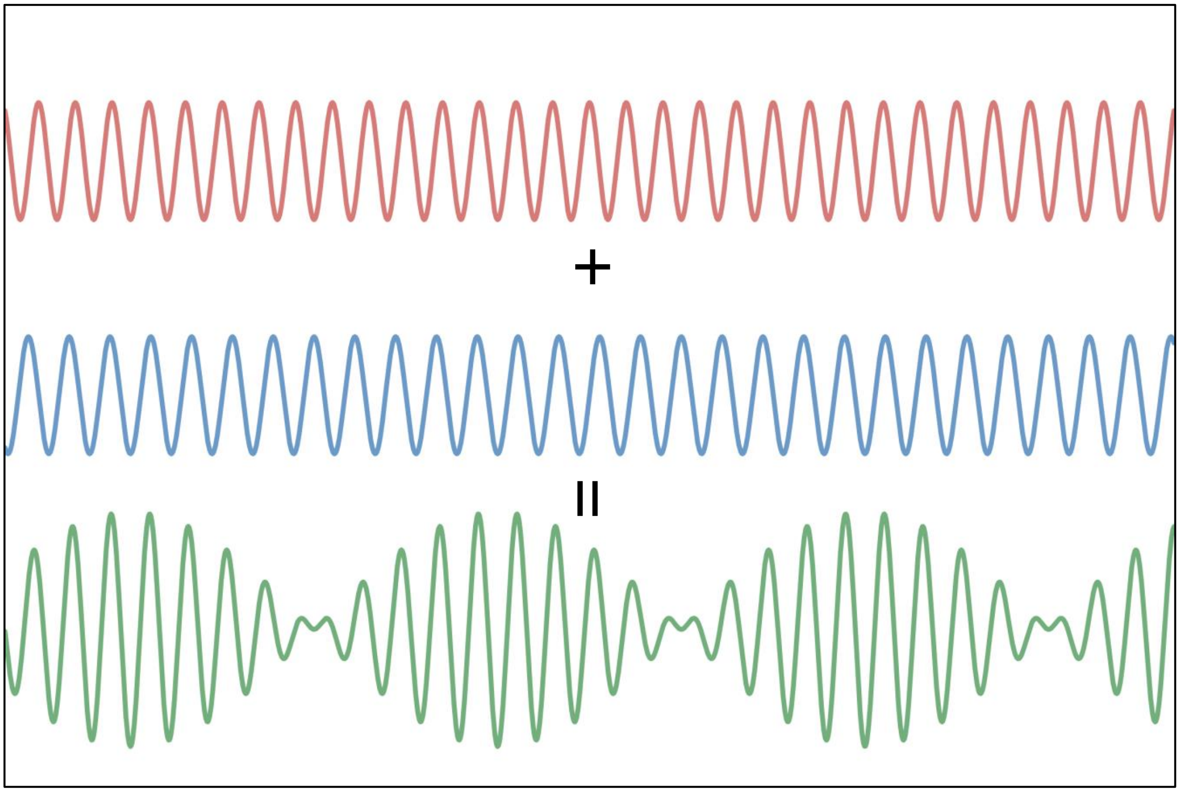
$$S(t) = A\sin(\omega_n t + \phi_n) + A\sin(\omega t)$$

Using the identity,

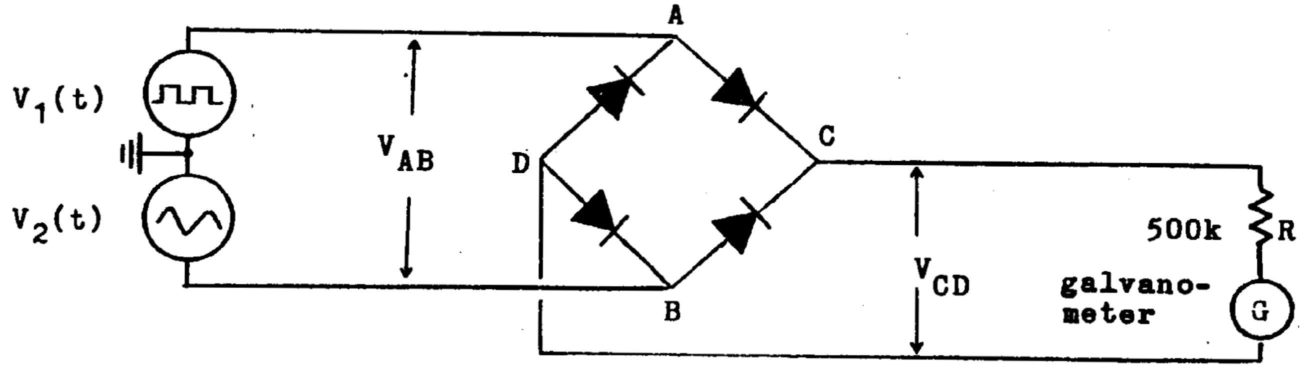
$$\sin a + \sin b = 2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)$$

Then we get,

$$S(t) = 2A\cos\left(\frac{\omega_n - \omega}{2}t + \frac{\phi_n}{2}\right)\sin\left(\frac{\omega_n + \omega}{2}t + \frac{\phi_n}{2}\right)$$



– Procedure:



Prepare the apparatus: The previous circuit, two signal generators.

1. Connect the previuos circuit. Signal generator S_1 should be set to give a square wave at a frequency of approximately 60h Hz. Signal generator S_2 be used to provide the sine wave.
2. Adjust the amplitudes of the two signals so that the galvanometer spot swings through almost the full scale. Keep the amplitudes of the two signals approximately equal.
3. Record the highest amplitude of the oscillation and the frequency of the sine wave.
4. Increase the frequency of the sine wave. Check for beats at integer multiples of $f_{S_2} = m f_{S_1}$. Then record the highest amplitude. You will see that multipling by even number would be useless, so record the odd multipling numbers.

- **Data:**

$$V_{S_1}(t) = \begin{cases} 0 \quad , & -T/2 < t < 0 \\ 3 \quad , & 0 \le t \le T/2 \end{cases}$$

$$V(t+T) = V(t)$$

$$f_{S_1} = 60Hz$$

$$V_{S_2}(t) = A\sin(\omega t)$$

$f_{S_2}(Hz)$	A
59.73	18.7
178.6	6.2
297.5	3.4
416.8	2.7
536.1	2.0
655.1	1.8
774.1	1.4
893.3	1.3

– Calculations:

$$V_{S_1}(t) = \begin{cases} 0 \quad , & -T/2 < t < 0 \\ 3 \quad , & 0 \le t \le T/2 \end{cases}$$

$$V_{S_1}(t+T) = V_{S_1}(t)$$

$$f_{S_1} = 60Hz$$

$$T = \frac{1}{f_{S_1}} = 1/60\ s$$

$$V_{S_1}(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}[A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)]$$

$$\omega_0 = 2\pi f_{S_1} = 2\pi(60) = 120\pi$$

$$A_0 = \frac{2}{T}\int_{-T/2}^{T/2} V_{S_1}(t)dt = \frac{2}{1/60}\int_{-1/120}^{1/120} V_{S_1}(t)dt = 120\int_0^{1/120} 3dt = 3$$

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} V_{S_1}(t)\cos(120n\pi t)\,dt = \frac{2}{1/60}\int_{-1/120}^{1/120} V_{S_1}(t)\cos(120n\pi t)\,dt$$

$$= 120\int_0^{1/120} 3\cos(120n\pi t)\,dt = \frac{3}{n\pi}\sin(120n\pi t)\Big|_0^{1/120} = 0$$

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} V_{S_1}(t)\sin(120n\pi t)\,dt = \frac{2}{1/60}\int_{-1/120}^{1/120} V_{S_1}(t)\sin(120n\pi t)\,dt$$

$$= 120\int_0^{1/120} 3\sin(120n\pi t)\,dt = -\frac{3}{n\pi}\cos(120n\pi t)\Big|_0^{1/120}$$

$$= -\frac{3}{n\pi}((-1)^n - 1) = \begin{cases} 0 \quad , & n: even \\ \dfrac{6}{n\pi} \quad , & n: odd \end{cases}$$

$$\Longrightarrow V_{S_1}(t) = \frac{3}{2} + \sum_{\substack{n=1 \\ n:odd}}^{\infty} \frac{6}{n\pi}\sin(120n\pi t)$$

$$V_{AB} = V_A - V_B = V_{S_1} - V_{S_2}$$

$$V_{AB} = V_{S_1}(t) - V_{S_2}(t) = \frac{3}{2} + \left(\sum_{\substack{n=1 \\ n:odd}}^{\infty} \frac{6}{n\pi} \sin(120n\pi t) \right) - A\sin(\omega t)$$

$$= \frac{3}{2} + \left(\sum_{\substack{n=1 \\ n\neq m \\ n:odd}}^{\infty} \frac{6}{n\pi} \sin(120n\pi t) \right) + \frac{6}{m\pi} \sin(120m\pi t) - A\sin(\omega t)$$

$$= \frac{3}{2} + \left(\sum_{\substack{n=1 \\ n\neq m \\ n:odd}}^{\infty} \frac{6}{n\pi} \sin(120n\pi t) \right) + \left(\frac{6}{m\pi} - A\right) \sin(120m\pi t)$$

$$+ A(\sin(120m\pi t) - \sin(\omega t))$$

$$= \frac{3}{2} + \left(\sum_{\substack{n=1 \\ n\neq m \\ n:odd}}^{\infty} \frac{6}{n\pi} \sin(120n\pi t) \right) + \left(\frac{6}{m\pi} - A\right) \sin(120m\pi t)$$

$$+ 2A\cos\left(\frac{120m\pi - \omega}{2} t\right) \sin\left(\frac{120m\pi + \omega}{2} t\right)$$

When ($\omega \approx 120m\pi$) the beats will have a long period.

$\omega = 2\pi f_{S_2} \approx 120m\pi$

$f_{S_2} \approx 60m \quad , \quad m: odd$

To find the highest amplitude, derive V_{AB} respect to t to find the maximaization.

$$V_{AB} = \frac{3}{2} + \left(\sum_{\substack{n=1 \\ n:odd}}^{\infty} \frac{6}{n\pi} \sin(120n\pi t) \right) - A \sin(\omega t)$$

$$\frac{dV_{AB}}{dt} = \left(120 \sum_{\substack{n=1 \\ n:odd}}^{\infty} 6 \cos(120n\pi t) \right) - \omega A \cos(\omega t) = 0$$

$$\Longrightarrow 120 \sum_{\substack{n=1 \\ n:odd}}^{\infty} 6 \cos(120n\pi t) = \omega A \cos(\omega t)$$

$$\Longrightarrow 120 \sum_{\substack{n=1 \\ n:odd}}^{\infty} 6 \cos(120n\pi t) = \omega A \cos(\omega t)$$

$$\Longrightarrow \omega = 120n\pi \quad , \quad \omega A = (120)(6)$$

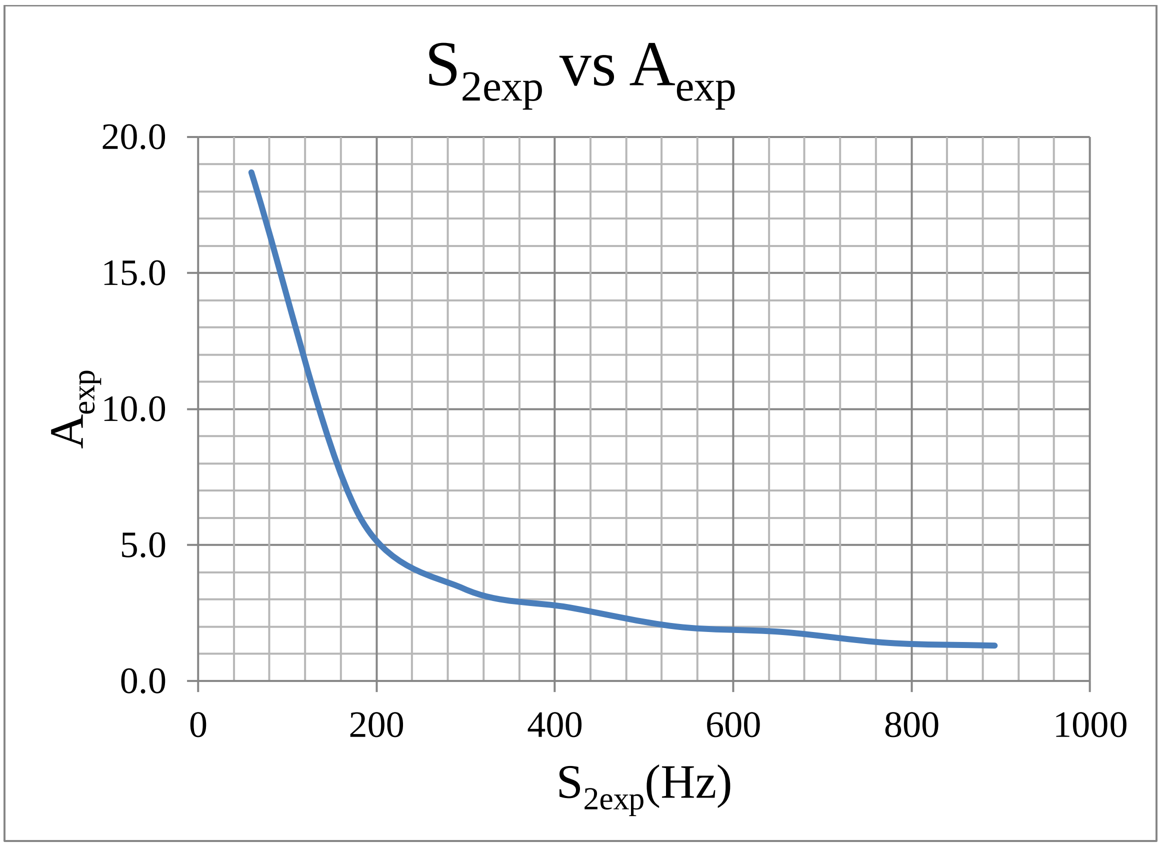
$$\Longrightarrow A = \frac{6}{n\pi}$$

Scale factor = 10

$$\Longrightarrow A = \frac{60}{n\pi}$$

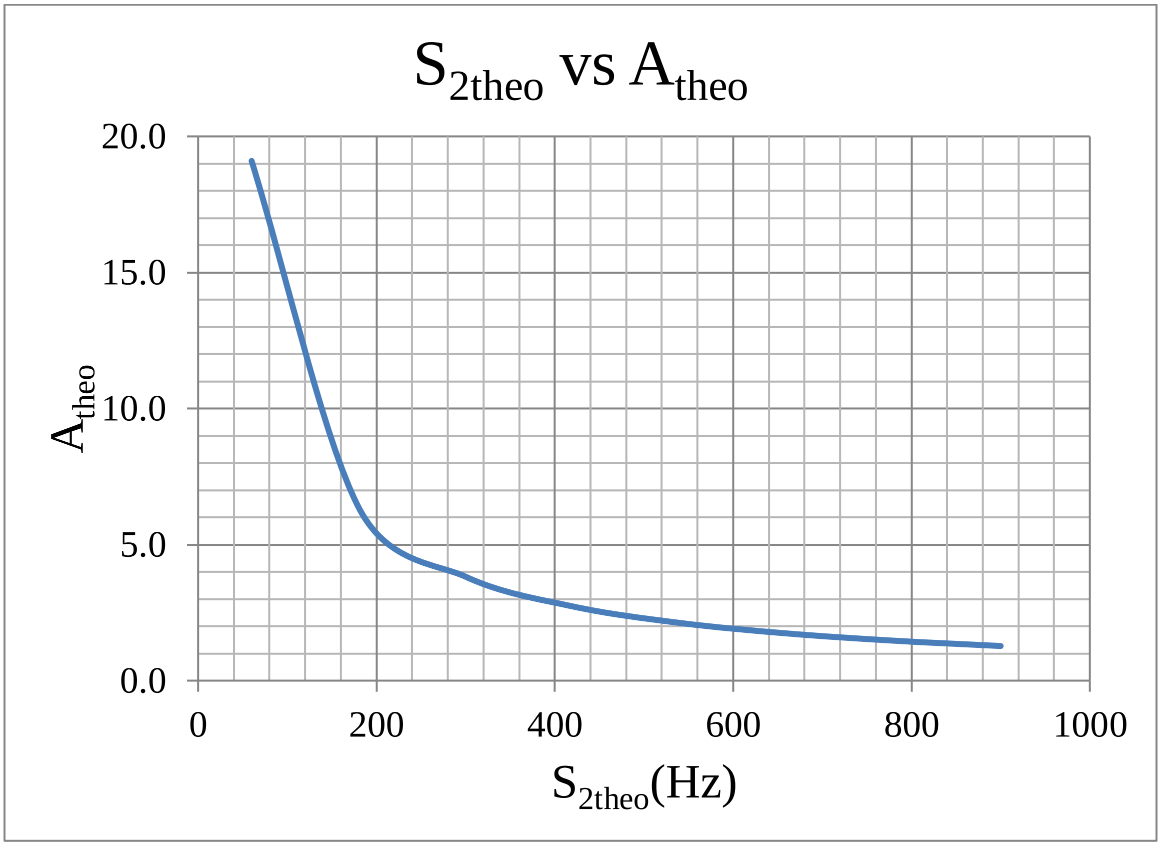
Experimental Calculations:

m	$f_{S_{2exp}}(Hz)$	A_{exp}	$(A_m/A_1)_{exp}$
1	59.73	18.7	1.000
3	178.6	6.2	0.332
5	297.5	3.4	0.182
7	416.8	2.7	0.144
9	536.1	2.0	0.107
11	655.1	1.8	0.096
13	774.1	1.4	0.075
15	893.3	1.3	0.070



Theoritical Calculations:

m	$f_{S_{2theo}}(Hz)$	A_{theo}	$(A_m/A_1)_{theo}$
1	60.00	19.1	1.000
3	180.0	6.4	0.333
5	300.0	3.8	0.200
7	420.0	2.7	0.143
9	540.0	2.1	0.111
11	660.0	1.7	0.091
13	780.0	1.5	0.077
15	900.0	1.3	0.067



– Results:

$$V_{S_1}(t) = \frac{3}{2} + \sum_{\substack{n=1 \\ n:odd}}^{\infty} \frac{6}{n\pi} \sin(120n\pi t)$$

Where,

$A_0 = 3$

$A_n = 0$

$$B_n = \begin{cases} 0 \quad , & n: even \\ \frac{6}{n\pi} \quad , & n: odd \end{cases}$$

$\omega_0 = 120\pi \; rad/s$

Experimental and their expected theoretical results:

m	$f_{S_{2exp}}(Hz)$	A_{exp}	$(A_m/A_1)_{exp}$	$f_{S_{2theo}}(Hz)$	A_{theo}	$(A_m/A_1)_{theo}$
1	59.73	18.7	1.000	60.00	19.1	1.000
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15	893.3	1.3	0.070	900.0	1.3	0.067

– Discussion:

In this experiment, two power sources are used. The first one was has a shape of square wave V_{S_1}. The second wave was a sinusoidal wave has a sine shape V_{S_2}.

Using Fourier theorem, any continuous periodic function can be expressed as Fourier series as the following:

$$F(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}[A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)]$$

Where,

$$A_0 = \frac{2}{T}\int_{-T/2}^{T/2} F(t)dt$$

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} F(t)\cos(n\omega_0 t)\,dt$$

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} F(t)\sin(n\omega_0 t)\,dt$$

The square wave which used,

$$V_{S_1}(t) = \begin{cases} 0 \ , & -T/2 < t < 0 \\ 3 \ , & 0 \le t \le T/2 \end{cases}$$

$$V(t+T) = V(t)\,, \qquad f_{S_1} = 60Hz$$

This wave after converting to Fourier series took the form:

$$V_{S_1}(t) = \frac{3}{2} + \sum_{\substack{n=1 \\ n:odd}}^{\infty} \frac{6}{n\pi}\sin(120n\pi t)$$

Where,

$$A_0 = 3\,, \qquad A_n = 0\,, \qquad B_n = \begin{cases} 0 \;, & n{:}\,even \\ \dfrac{6}{n\pi} \;, & n{:}\,odd \end{cases}\,, \qquad \omega_0 = 120\pi\ rad/s$$

It has infinite terms of sine wave, since the square wave is extremely sharp wave, and more combinations of waves means more sharpness in the shape of the wave.

To find the frequencies which the amplitude would be maximum, we used the beats phenomena.

The second wave (the sinusoidal wave) was:

$$V_{S_2}(t) = A\sin(\omega t)$$

And,

$$V_{AB} = V_A - V_B = V_{S_1} - V_{S_2}$$

Then after some mathematical calculations,

$$V_{AB} = \frac{3}{2} + \left(\sum_{\substack{n=1 \\ n\neq m \\ n:odd}}^{\infty} \frac{6}{n\pi}\sin(120n\pi t) \right) + \left(\frac{6}{m\pi} - A\right)\sin(120m\pi t)$$
$$+ 2A\cos\left(\frac{120m\pi - \omega}{2}t\right)\sin\left(\frac{120m\pi + \omega}{2}t\right)$$

Where the beats phenomenon is appearing because the term:

$$\left[2A\cos\left(\frac{120m\pi - \omega}{2}t\right)\sin\left(\frac{120m\pi + \omega}{2}t\right)\right]$$

Since, when the cosine term controls the envelop which the wave lies. On the other words, if ($\omega \approx 120m\pi$) the cosine term approaches to value of one, then the amplitude would reach to its maximum value.

The experimental values and the expected theoretical values were so closed to each other as the next table shows:

m	$f_{S_{2exp}}(Hz)$	A_{exp}	$(A_m/A_1)_{exp}$	$f_{S_{2theo}}(Hz)$	A_{theo}	$(A_m/A_1)_{theo}$
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15	893.3	1.3	0.070	900.0	1.3	0.067

The values of (A_0, A_n, B_n) were obtained from the orthogonality relation as follows:

$$\frac{1}{L}\int_{-L}^{L} \cos\left(\frac{m\pi}{L}t\right)\cos\left(\frac{n\pi}{L}t\right)dt = \begin{cases} 1 \;, & n = m \\ 0 \;, & n \neq m \end{cases}$$

$$\frac{1}{L}\int_{-L}^{L} \cos\left(\frac{m\pi}{L}t\right)\sin\left(\frac{n\pi}{L}t\right)dt = 0$$

$$\frac{1}{L}\int_{-L}^{L} \sin\left(\frac{m\pi}{L}t\right)\sin\left(\frac{n\pi}{L}t\right)dt = \begin{cases} 1 \;, & n = m \\ 0 \;, & n \neq m \end{cases}$$

For n, m nonzero integers.

The functions $\sin\left(\frac{n\pi}{L}t\right)$ and $\cos\left(\frac{n\pi}{L}t\right)$ are said to be orthogonal on the interval $[-L, L]$. We will take $L = T/2$.

$$F(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} B_n \sin(n\omega_0 t)$$

By multiplying it by $\cos(m\omega_0 t)$, then taking the integral from $t = T/2$ to $t = -T/2$:

$$\int_{-T/2}^{T/2} F(t)\cos(m\omega_0 t)dt$$

$$= \frac{1}{2}A_0 \int_{-T/2}^{T/2} \cos(m\omega_0 t)\, dt + \sum_{n=1}^{\infty} A_n \int_{-T/2}^{T/2} \cos(n\omega_0 t)\cos(m\omega_0 t)\, dt$$

$$+ \sum_{n=1}^{\infty} B_n \int_{-T/2}^{T/2} \sin(n\omega_0 t)\cos(m\omega_0 t)\, dt$$

$$\Longrightarrow \int_{-T/2}^{T/2} F(t)\cos(m\omega_0 t)\, dt = A_m \frac{T}{2}$$

$$\Longrightarrow A_m = \frac{2}{T}\int_{-T/2}^{T/2} F(t)\cos(m\omega_0 t)\, dt$$

By multiplying it by $\sin(m\omega_0 t)$, then taking the integral from $t = T/2$ to $t = -T/2$:

$$\int_{-T/2}^{T/2} F(t)\sin(m\omega_0 t)\, dt$$

$$= \frac{1}{2}A_0 \int_{-T/2}^{T/2} \sin(m\omega_0 t)\, dt + \sum_{n=1}^{\infty} A_n \int_{-T/2}^{T/2} \cos(n\omega_0 t)\sin(m\omega_0 t)\, dt$$

$$+ \sum_{n=1}^{\infty} B_n \int_{-T/2}^{T/2} \sin(n\omega_0 t)\sin(m\omega_0 t)\, dt$$

$$\Longrightarrow \int_{-T/2}^{T/2} F(t)\sin(m\omega_0 t)\, dt = B_m \frac{T}{2}$$

$$\Longrightarrow B_m = \frac{2}{T}\int_{-T/2}^{T/2} F(t)\sin(m\omega_0 t)\, dt$$

By taking the integral from $t = T/2$ to $t = -T/2$

$$\int_{-T/2}^{T/2} F(t)dt = \frac{1}{2}A_0 \int_{-T/2}^{T/2} dt + \sum_{n=1}^{\infty} A_n \int_{-T/2}^{T/2} \cos(n\omega_0 t)\, dt + \sum_{n=1}^{\infty} B_n \int_{-T/2}^{T/2} \sin(n\omega_0 t)\, dt$$

$$\Longrightarrow \int_{-T/2}^{T/2} F(t)dt = A_0 \frac{T}{2}$$

$$\Longrightarrow A_0 = \frac{2}{T}\int_{-T/2}^{T/2} F(t)dt$$

There were some systematic errors. For example, the systematic errors in the tools we cannot get rid of it. Moreover, the errors of measuring the amplitude by approximation made some errors.

– References:

1. H. Abusara, & A. Shawabkeh (2016, November). *Laboratory Manual: Modern Physics Lab* (Second Edition). *Harmonic Analysis* (pp. 25-36). Birzeit University: Faculty of Science.
2. Barry Truax (1999). *Handbook for Acoustic Ecology* (Second Edition). https://www.sfu.ca/sonic-studio-webdav/handbook/Fourier_Theorem.html.