Solutions Manual for

The Physics of Vibrations and Waves – 6th Edition

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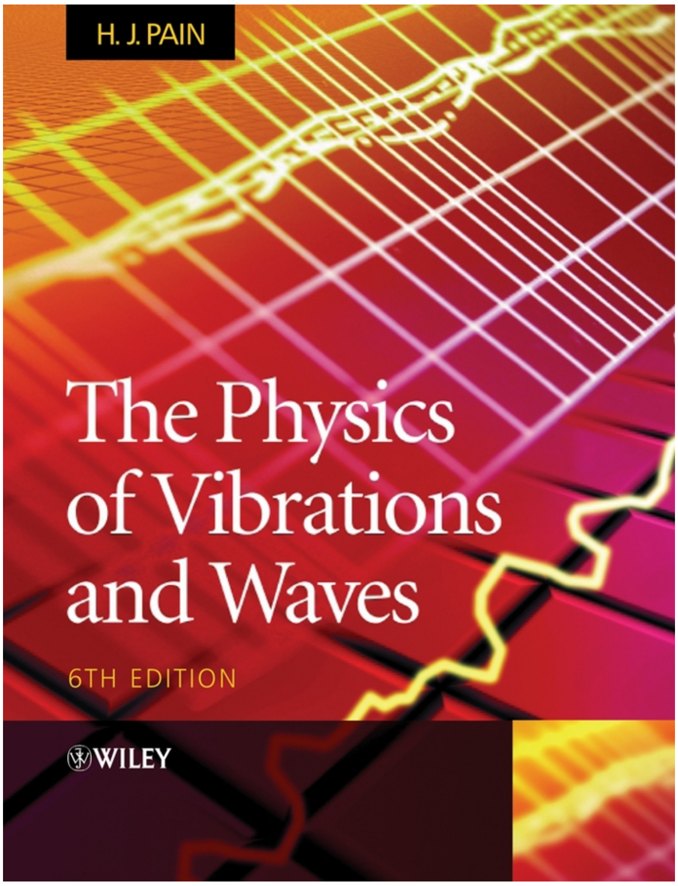


SOLUTIONS TO CHAPTER 1

1.1

In Figure 1.1(a), the restoring force is given by:

$$F = -mg\sin\theta$$

By substitution of relation $\sin\theta = x/l$ into the above equation, we have:

$$F = -mg\,x/l$$

so the stiffness is given by:

$$s = -F/x = mg/l$$

so we have the frequency given by:

$$\omega^2 = s/m = g/l$$

Since θ is a very small angle, i.e. $\theta = \sin\theta = x/l$, or $x = l\theta$, we have the restoring force given by:

$$F = -mg\theta$$

Now, the equation of motion using angular displacement θ can by derived from Newton's second law:

$$F = m\ddot{x}$$

i.e. $$-mg\theta = ml\ddot{\theta}$$

i.e. $$\ddot{\theta} + \frac{g}{l}\theta = 0$$

which shows the frequency is given by:

$$\omega^2 = g/l$$

In Figure 1.1(b), restoring couple is given by $-C\theta$, which has relation to moment of inertia I given by:

$$-C\theta = I\ddot{\theta}$$

i.e. $$\ddot{\theta} + \frac{C}{I}\theta = 0$$

which shows the frequency is given by:

$$\omega^2 = C/I$$

In Figure 1.1(d), the restoring force is given by:

$$F = -2T\, x/l$$

so Newton's second law gives:

$$F = m\ddot{x} = -2Tx/l$$

i.e. $$\ddot{x} + 2Tx/lm = 0$$

which shows the frequency is given by:

$$\omega^2 = \frac{2T}{lm}$$

In Figure 1.1(e), the displacement for liquid with a height of x has a displacement of $x/2$ and a mass of ρAx, so the stiffness is given by:

$$s = \frac{G}{x/2} = \frac{2\rho Axg}{x} = 2\rho Ag$$

Newton's second law gives:

$$-G = m\ddot{x}$$

i.e. $$-2\rho Axg = \rho Al\ddot{x}$$

i.e. $$\ddot{x} + \frac{2g}{l}x = 0$$

which show the frequency is given by:

$$\omega^2 = 2g/l$$

In Figure 1.1(f), by taking logarithms of equation $pV^{\gamma} = \text{constant}$, we have:

$$\ln p + \gamma \ln V = \text{constant}$$

so we have: $$\frac{dp}{p} + \gamma \frac{dV}{V} = 0$$

i.e. $$dp = -\gamma p \frac{dV}{V}$$

The change of volume is given by $dV = Ax$, so we have:

$$dp = -\gamma p \frac{Ax}{V}$$

The gas in the flask neck has a mass of ρAl, so Newton's second law gives:

$$Adp = m\ddot{x}$$

i.e. $$-\gamma p\frac{A^2x}{V}=\rho Al\ddot{x}$$

i.e. $$\ddot{x}+\frac{\gamma pA}{l\rho V}x=0$$

which show the frequency is given by:

$$\omega^2=\frac{\gamma pA}{l\rho V}$$

In Figure 1.1 (g), the volume of liquid displaced is Ax, so the restoring force is $-\rho gAx$. Then, Newton's second law gives:

$$F=-\rho gAx=m\ddot{x}$$

i.e. $$\ddot{x}+\frac{g\rho A}{m}x=0$$

which shows the frequency is given by:

$$\omega^2=g\rho A/m$$

1.2

Write solution $x=a\cos(\omega t+\phi)$ in form: $x=a\cos\phi\cos\omega t-a\sin\phi\sin\omega t$ and compare with equation (1.2) we find: $A=a\cos\phi$ and $B=-a\sin\phi$. We can also find, with the same analysis, that the values of A and B for solution $x=a\sin(\omega t-\phi)$ are given by: $A=-a\sin\phi$ and $B=a\cos\phi$, and for solution $x=a\cos(\omega t-\phi)$ are given by: $A=a\cos\phi$ and $B=a\sin\phi$.

Try solution $x=a\cos(\omega t+\phi)$ in expression $\ddot{x}+\omega^2x$, we have:

$$\ddot{x}+\omega^2x=-a\omega^2\cos(\omega t+\phi)+\omega^2a\cos(\omega t+\phi)=0$$

Try solution $x=a\sin(\omega t-\phi)$ in expression $\ddot{x}+\omega^2x$, we have:

$$\ddot{x}+\omega^2x=-a\omega^2\sin(\omega t-\phi)+\omega^2a\sin(\omega t-\phi)=0$$

Try solution $x=a\cos(\omega t-\phi)$ in expression $\ddot{x}+\omega^2x$, we have:

$$\ddot{x}+\omega^2x=-a\omega^2\cos(\omega t-\phi)+\omega^2a\cos(\omega t-\phi)=0$$

1.3

(a) If the solution $x = a\sin(\omega t + \phi)$ satisfies $x = a$ at $t = 0$, then, $x = a\sin\phi = a$ i.e. $\phi = \pi/2$. When the pendulum swings to the position $x = +a/\sqrt{2}$ for the first time after release, the value of ωt is the minimum solution of equation $a\sin(\omega t + \pi/2) = +a/\sqrt{2}$, i.e. $\omega t = \pi/4$. Similarly, we can find: for $x = a/2$, $\omega t = \pi/3$ and for $x = 0$, $\omega t = \pi/2$.

If the solution $x = a\cos(\omega t + \phi)$ satisfies $x = a$ at $t = 0$, then, $x = a\cos\phi = a$ i.e. $\phi = 0$. When the pendulum swings to the position $x = +a/\sqrt{2}$ for the first time after release, the value of ωt is the minimum solution of equation $a\cos\omega t = +a/\sqrt{2}$, i.e. $\omega t = \pi/4$. Similarly, we can find: for $x = a/2$, $\omega t = \pi/3$ and for $x = 0$, $\omega t = \pi/2$.

If the solution $x = a\sin(\omega t - \phi)$ satisfies $x = a$ at $t = 0$, then, $x = a\sin(-\phi) = a$ i.e. $\phi = -\pi/2$. When the pendulum swings to the position $x = +a/\sqrt{2}$ for the first time after release, the value of ωt is the minimum solution of equation $a\sin(\omega t + \pi/2) = +a/\sqrt{2}$, i.e. $\omega t = \pi/4$. Similarly, we can find: for $x = a/2$, $\omega t = \pi/3$ and for $x = 0$, $\omega t = \pi/2$.

If the solution $x = a\cos(\omega t - \phi)$ satisfies $x = a$ at $t = 0$, then, $x = a\cos(-\phi) = a$ i.e. $\phi = 0$. When the pendulum swings to the position $x = +a/\sqrt{2}$ for the first time after release, the value of ωt is the minimum solution of equation $a\cos\omega t = +a/\sqrt{2}$, i.e. $\omega t = \pi/4$. Similarly, we can find: for $x = a/2$, $\omega t = \pi/3$ and for $x = 0$, $\omega t = \pi/2$.

(b) If the solution $x = a\sin(\omega t + \phi)$ satisfies $x = -a$ at $t = 0$, then, $x = a\sin\phi = -a$ i.e. $\phi = -\pi/2$. When the pendulum swings to the position

$x = +a/\sqrt{2}$ for the first time after release, the value of ωt is the minimum solution of equation $a\sin(\omega t - \pi/2) = +a/\sqrt{2}$, i.e. $\omega t = 3\pi/4$. Similarly, we can find: for $x = a/2$, $\omega t = 2\pi/3$ and for $x = 0$, $\omega t = \pi/2$.

If the solution $x = a\cos(\omega t + \phi)$ satisfies $x = -a$ at $t = 0$, then, $x = a\cos\phi = -a$ i.e. $\phi = \pi$. When the pendulum swings to the position $x = +a/\sqrt{2}$ for the first time after release, the value of ωt is the minimum solution of equation $a\cos(\omega t + \pi) = +a/\sqrt{2}$, i.e. $\omega t = 3\pi/4$. Similarly, we can find: for $x = a/2$, $\omega t = 2\pi/3$ and for $x = 0$, $\omega t = \pi/2$.

If the solution $x = a\sin(\omega t - \phi)$ satisfies $x = -a$ at $t = 0$, then, $x = a\sin(-\phi) = -a$ i.e. $\phi = \pi/2$. When the pendulum swings to the position $x = +a/\sqrt{2}$ for the first time after release, the value of ωt is the minimum solution of equation $a\sin(\omega t - \pi/2) = +a/\sqrt{2}$, i.e. $\omega t = 3\pi/4$. Similarly, we can find: for $x = a/2$, $\omega t = 2\pi/3$ and for $x = 0$, $\omega t = \pi/2$.

If the solution $x = a\cos(\omega t - \phi)$ satisfies $x = -a$ at $t = 0$, then, $x = a\cos(-\phi) = -a$ i.e. $\phi = \pi$. When the pendulum swings to the position $x = +a/\sqrt{2}$ for the first time after release, the value of ωt is the minimum solution of equation $a\cos(\omega t - \pi) = +a/\sqrt{2}$, i.e. $\omega t = 3\pi/4$. Similarly, we can find: for $x = a/2$, $\omega t = 2\pi/3$ and for $x = 0$, $\omega t = \pi/2$.

1.4

The frequency of such a simple harmonic motion is given by:

$$\omega_0 = \sqrt{\frac{s}{m_e}} = \sqrt{\frac{e^2}{4\pi\varepsilon_0 r^3 m_e}} = \sqrt{\frac{(1.6\times10^{-19})^2}{4\times\pi\times8.85\times10^{-12}\times(0.05\times10^{-9})^3\times9.1\times10^{-31}}} \approx 4.5\times10^{16}[rad\cdot s^{-1}]$$

Its radiation generates an electromagnetic wave with a wavelength λ given by:

$$\lambda = \frac{2\pi c}{\omega_0} = \frac{2\times\pi\times 3\times 10^8}{4.5\times 10^{16}} \approx 4.2\times 10^{-8}[m] = 42[nm]$$

Therefore such a radiation is found in X-ray region of electromagnetic spectrum.

1.5

(a) If the mass m is displaced a distance of x from its equilibrium position, either the upper or the lower string has an extension of $x/2$. So, the restoring force of the mass is given by: $F = -sx/2$ and the stiffness of the system is given by: $s' = -F/x = s/2$. Hence the frequency is given by $\omega_a^2 = s'/m = s/2m$.

(b) The frequency of the system is given by: $\omega_b^2 = s/m$

(c) If the mass m is displaced a distance of x from its equilibrium position, the restoring force of the mass is given by: $F = -sx - sx = -2sx$ and the stiffness of the system is given by: $s' = -F/x = 2s$. Hence the frequency is given by $\omega_c^2 = s'/m = 2s/m$.

Therefore, we have the relation: $\omega_a^2 : \omega_b^2 : \omega_c^2 = s/2m : s/m : 2s/m = 1:2:4$

1.6

At time $t = 0$, $x = x_0$ gives:

$$a\sin\phi = x_0 \qquad (1.6.1)$$

$\dot{x} = v_0$ gives:

$$a\omega\cos\phi = v_0 \qquad (1.6.2)$$

From (1.6.1) and (1.6.2), we have

$$\tan\phi = \omega x_0/v_0 \quad \text{and} \quad a = (x_0^2 + v_0^2/\omega^2)^{1/2}$$

1.7

The equation of this simple harmonic motion can be written as: $x = a\sin(\omega t + \phi)$.

The time spent in moving from x to $x + dx$ is given by: $dt = dx/|v_t|$, where v_t is the velocity of the particle at time t and is given by: $v_t = \dot{x} = a\omega\cos(\omega t + \phi)$.

Noting that the particle will appear twice between x and $x+dx$ within one period of oscillation. We have the probability η of finding it between x to $x+dx$ given by: $\eta = \frac{2dt}{T}$ where the period is given by: $T = \frac{2\pi}{\omega}$, so we have:

$$\eta = \frac{2dt}{T} = \frac{2\omega dx}{2\pi a\omega\cos(\omega t+\phi)} = \frac{dx}{\pi a\cos(\omega t+\phi)} = \frac{dx}{\pi a\sqrt{1-\sin^2(\omega t+\phi)}} = \frac{dx}{\pi\sqrt{a^2-x^2}}$$

1.8

Since the displacements of the equally spaced oscillators in y direction is a sine curve, the phase difference $\delta\phi$ between two oscillators a distance x apart given is proportional to the phase difference 2π between two oscillators a distance λ apart by: $\delta\phi/2\pi = x/\lambda$, i.e. $\delta\phi = 2\pi x/\lambda$.

1.9

The mass loses contact with the platform when the system is moving downwards and the acceleration of the platform equals the acceleration of gravity. The acceleration of a simple harmonic vibration can be written as: $a = A\omega^2\sin(\omega t+\phi)$, where A is the amplitude, ω is the angular frequency and ϕ is the initial phase. So we have:

$$A\omega^2\sin(\omega t+\phi) = g$$

i.e.

$$A = \frac{g}{\omega^2\sin(\omega t+\phi)}$$

Therefore, the minimum amplitude, which makes the mass lose contact with the platform, is given by:

$$A_{\min} = \frac{g}{\omega^2} = \frac{g}{4\pi^2 f^2} = \frac{9.8}{4\times\pi^2\times 5^2} \approx 0.01[m]$$

1.10

The mass of the element dy is given by: $m' = mdy/l$. The velocity of an element dy of its length is proportional to its distance y from the fixed end of the spring, and is given by: $v' = yv/l$. where v is the velocity of the element at the other end of the spring, i.e. the velocity of the suspended mass M. Hence we have the kinetic energy of this element given by:

$$KE_{dy} = \frac{1}{2}m'v'^2 = \frac{1}{2}\left(\frac{m}{l}dy\right)\left(\frac{y}{l}v\right)^2$$

The total kinetic energy of the spring is given by:

$$KE_{spring} = \int_0^l KE_{dy}dy = \int_0^l \frac{1}{2}\left(\frac{m}{l}dy\right)\left(\frac{y}{l}v\right)^2 = \frac{mv^2}{2l^3}\int_0^l y^2dy = \frac{1}{6}mv^2$$

The total kinetic energy of the system is the sum of kinetic energies of the spring and the suspended mass, and is given by:

$$KE_{tot} = \frac{1}{6}mv^2 + \frac{1}{2}Mv^2 = \frac{1}{2}(M + m/3)v^2$$

which shows the system is equivalent to a spring with zero mass with a mass of $M + m/3$ suspended at the end. Therefore, the frequency of the oscillation system is given by:

$$\omega^2 = \frac{s}{M + m/3}$$

1.11

In Figure 1.1(a), the restoring force of the simple pendulum is $-mg\sin\theta$, then, the stiffness is given by: $s = mg\sin\theta/x = mg/l$. So the energy is given by:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}sx^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{mg}{l}x^2$$

The equation of motion is by setting $dE/dt = 0$, i.e.:

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{mg}{l}x^2\right) = 0$$

i.e. $$\ddot{x} + \frac{g}{l}x = 0$$

In Figure 1.1(b), the displacement is the rotation angle θ, the mass is replaced by the moment of inertia I of the disc and the stiffness by the restoring couple C of the wire. So the energy is given by:

$$E = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}C\theta^2$$

The equation of motion is by setting $dE/dt = 0$, i.e.:

$$\frac{d}{dt}\left(\frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}C\theta^2\right) = 0$$

i.e. $$\ddot{\theta}+\frac{C}{I}\theta=0$$

In Figure 1.1(c), the energy is directly given by:

$$E=\frac{1}{2}mv^2+\frac{1}{2}sx^2$$

The equation of motion is by setting $dE/dt=0$, i.e.:

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2+\frac{1}{2}sx^2\right)=0$$

i.e. $$\ddot{x}+\frac{s}{m}x=0$$

In Figure 1.1(c), the restoring force is given by: $-2Tx/l$, then the stiffness is given by: $s=2T/l$. So the energy is given by:

$$E=\frac{1}{2}mv^2+\frac{1}{2}sx^2=\frac{1}{2}m\dot{x}^2+\frac{1}{2}\frac{2T}{l}x^2=\frac{1}{2}m\dot{x}^2+\frac{T}{l}x^2$$

The equation of motion is by setting $dE/dt=0$, i.e.:

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2+\frac{T}{l}x^2\right)=0$$

i.e. $$\ddot{x}+\frac{2T}{lm}x=0$$

In Figure 1.1(e), the liquid of a volume of ρAl is displaced from equilibrium position by a distance of $l/2$, so the stiffness of the system is given by $s=2\rho gAl/l=2\rho gA$. So the energy is given by:

$$E=\frac{1}{2}mv^2+\frac{1}{2}sx^2=\frac{1}{2}\rho Al\dot{x}^2+\frac{1}{2}2\rho gAx^2=\frac{1}{2}\rho Al\dot{x}^2+\rho gAx^2$$

The equation of motion is by setting $dE/dt=0$, i.e.:

$$\frac{d}{dt}\left(\frac{1}{2}\rho Al\dot{x}^2+\rho gAx^2\right)=0$$

i.e. $$\ddot{x}+\frac{2g}{l}x=0$$

In Figure 1.1(f), the gas of a mass of ρAl is displaced from equilibrium position by a distance of x and causes a pressure change of $dp = -\gamma pAx/V$, then, the stiffness of the system is given by $s = -Adp/x = \gamma pA^2/V$. So the energy is given by:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}sx^2 = \frac{1}{2}\rho Al\dot{x}^2 + \frac{1}{2}\frac{\gamma pA^2x^2}{V}$$

The equation of motion is by setting $dE/dt = 0$, i.e.:

$$\frac{d}{dt}\left(\frac{1}{2}\rho Al\dot{x}^2 + \frac{1}{2}\frac{\gamma pA^2x^2}{V}\right) = 0$$

i.e.

$$\ddot{x} + \frac{\gamma pA}{l\rho V}x = 0$$

In Figure 1.1(g), the restoring force of the hydrometer is $-\rho gAx$, then the stiffness of the system is given by $s = \rho gAx/x = \rho gA$. So the energy is given by:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}sx^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\rho gAx^2$$

The equation of motion is by setting $dE/dt = 0$, i.e.:

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}\rho gAx^2\right) = 0$$

i.e.

$$\ddot{x} + \frac{A\rho g}{m}x = 0$$

1.12

The displacement of the simple harmonic oscillator is given by:

$$x = a\sin\omega t \qquad (1.12.1)$$

so the velocity is given by:

$$\dot{x} = a\omega\cos\omega t \qquad (1.12.2)$$

From (1.12.1) and (1.12.2), we can eliminate t and get:

$$\frac{x^2}{a^2} + \frac{\dot{x}^2}{a^2\omega^2} = \sin^2\omega t + \cos^2\omega t = 1 \qquad (1.12.3)$$

which is an ellipse equation of points $(x, \dot{x})$.

The energy of the simple harmonic oscillator is given by:

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}sx^2 \qquad (1.12.4)$$

Write (1.12.3) in form $\dot{x}^2 = \omega^2(a^2 - x^2)$ and substitute into (1.12.4), then we have:

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}sx^2 = \frac{1}{2}m\omega^2(a^2 - x^2) + \frac{1}{2}sx^2$$

Noting that the frequency ω is given by: $\omega^2 = s/m$, we have:

$$E = \frac{1}{2}s(a^2 - x^2) + \frac{1}{2}sx^2 = \frac{1}{2}sa^2$$

which is a constant value.

1.13

The equations of the two simple harmonic oscillations can be written as:

$$y_1 = a\sin(\omega t + \phi) \qquad \text{and} \qquad y_2 = a\sin(\omega t + \phi + \delta)$$

The resulting superposition amplitude is given by:

$$R = y_1 + y_2 = a[\sin(\omega t + \phi) + \sin(\omega t + \phi + \delta)] = 2a\sin(\omega t + \phi + \delta/2)\cos(\delta/2)$$

and the intensity is given by:

$$I = R^2 = 4a^2\cos^2(\delta/2)\sin^2(\omega t + \phi + \delta/2)$$

i.e.

$$I \propto 4a^2\cos^2(\delta/2)$$

Noting that $\sin^2(\omega t + \phi + \delta/2)$ varies between 0 and 1, we have:

$$0 \le I \le 4a^2\cos^2(\delta/2)$$

1.14

$$\left(\frac{x}{a_1}\sin\phi_2 - \frac{y}{a_2}\sin\phi_1\right)^2 + \left(\frac{y}{a_2}\cos\phi_1 - \frac{x}{a_1}\cos\phi_2\right)^2$$

$$= \frac{x^2}{a_1^2}\sin^2\phi_2 + \frac{y^2}{a_2^2}\sin^2\phi_1 - \frac{2xy}{a_1a_2}\sin\phi_1\sin\phi_2 + \frac{y^2}{a_2^2}\cos^2\phi_1 + \frac{x^2}{a_1^2}\cos^2\phi_2 - \frac{2xy}{a_1a_2}\cos\phi_1\cos\phi_2$$

$$= \frac{x^2}{a_1^2}(\sin^2\phi_2 + \cos^2\phi_2) + \frac{y^2}{a_2^2}(\sin^2\phi_1 + \cos^2\phi_1) - \frac{2xy}{a_1a_2}(\sin\phi_1\sin\phi_2 + \cos\phi_1\cos\phi_2)$$

$$= \frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1a_2}\cos(\phi_1 - \phi_2)$$

On the other hand, by substitution of :

$$\frac{x}{a_1} = \sin\omega t\cos\phi_1 + \cos\omega t\sin\phi_1$$

$$\frac{y}{a_2} = \sin\omega t \cos\phi_2 + \cos\omega t \sin\phi_2$$

into expression $\left(\frac{x}{a_1}\sin\phi_2 - \frac{y}{a_2}\sin\phi_1\right)^2 + \left(\frac{y}{a_2}\cos\phi_1 - \frac{x}{a_1}\cos\phi_2\right)^2$, we have:

$$\begin{aligned}
&\left(\frac{x}{a_1}\sin\phi_2 - \frac{y}{a_2}\sin\phi_1\right)^2 + \left(\frac{y}{a_2}\cos\phi_1 - \frac{x}{a_1}\cos\phi_2\right)^2 \\
&= \sin^2\omega t(\sin\phi_2\cos\phi_1 - \sin\phi_1\cos\phi_2)^2 + \cos^2\omega t(\cos\phi_1\sin\phi_2 - \cos\phi_2\sin\phi_1)^2 \\
&= (\sin^2\omega t + \cos^2\omega t)\sin^2(\phi_2 - \phi_1) \\
&= \sin^2(\phi_2 - \phi_1)
\end{aligned}$$

From the above derivation, we have:

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1 a_2}\cos(\phi_1 - \phi_2) = \sin^2(\phi_2 - \phi_1)$$

1.15

By elimination of t from equation $x = a\sin\omega t$ and $y = b\cos\omega t$, we have:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which shows the particle follows an elliptical path. The energy at any position of x, y on the ellipse is given by:

$$\begin{aligned}
E &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}sx^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}sy^2 \\
&= \frac{1}{2}ma^2\omega^2\cos^2\omega t + \frac{1}{2}ma^2\omega^2\sin^2\omega t + \frac{1}{2}mb^2\omega^2\sin^2\omega t + \frac{1}{2}mb^2\omega^2\cos^2\omega t \\
&= \frac{1}{2}ma^2\omega^2 + \frac{1}{2}mb^2\omega^2 \\
&= \frac{1}{2}m\omega^2(a^2 + b^2)
\end{aligned}$$

The value of the energy shows it is a constant and equal to the sum of the separate energies of the simple harmonic vibrations in x direction given by $\frac{1}{2}m\omega^2a^2$ and in y direction given by $\frac{1}{2}m\omega^2b^2$.

At any position of x, y on the ellipse, the expression of $m(x\dot{y} - y\dot{x})$ can be written as:

$$m(x\dot{y} - y\dot{x}) = m(-ab\omega\sin^2\omega t - ab\omega\cos^2\omega t) = -abm\omega(\sin^2\omega t + \cos^2\omega t) = -abm\omega$$

which is a constant. The quantity $abm\omega$ is the angular momentum of the particle.

1.16

All possible paths described by equation 1.3 fall within a rectangle of $2a_1$ wide and $2a_2$ high, where $a_1 = x_{\max}$ and $a_2 = y_{\max}$, see Figure 1.8.

When $x = 0$ in equation (1.3) the positive value of $y = a_2\sin(\phi_2 - \phi_1)$. The value of $y_{\max} = a_2$. So $y_{x=0}/y_{\max} = \sin(\phi_2 - \phi_1)$ which defines $\phi_2 - \phi_1$.

1.17

In the range $0 \le \phi \le \pi$, the values of $\cos\phi_i$ are $-1 \le \cos\phi_i \le +1$. For n random values of ϕ_i, statistically there will be $n/2$ values $-1 \le \cos\phi_i \le 0$ and $n/2$ values $0 \le \cos\phi_i \le 1$. The positive and negative values will tend to cancel each other and the sum of the n values: $\sum_{\substack{i=1 \\ i\ne j}}^{n}\cos\phi_i \to 0$, similarly $\sum_{j=1}^{n}\cos\phi_j \to 0$. i.e.

$$\sum_{\substack{i=1 \\ i\ne j}}^{n}\cos\phi_i \sum_{j=1}^{n}\cos\phi_j \to 0$$

1.18

The exponential form of the expression:

$$a\sin\omega t + a\sin(\omega t + \delta) + a\sin(\omega t + 2\delta) + \cdots + a\sin[\omega t + (n-1)\delta]$$

is given by:

$$ae^{i\omega t} + ae^{i(\omega t+\delta)} + ae^{i(\omega t+2\delta)} + \cdots + ae^{i[\omega t+(n-1)\delta]}$$

From the analysis in page 28, the above expression can be rearranged as:

$$ae^{i\left[\omega t+\left(\frac{n-1}{2}\right)\delta\right]}\frac{\sin n\delta/2}{\sin\delta/2}$$

with the imaginary part:

$$a\sin\left[\omega t + \left(\frac{n-1}{2}\right)\delta\right]\frac{\sin n\delta/2}{\sin\delta/2}$$

which is the value of the original expression in sine term.

1.19

From the analysis in page 28, the expression of z can be rearranged as:

$$z = ae^{i\omega t}(1 + e^{i\delta} + e^{i2\delta} + \cdots e^{i(n-1)\delta}) = ae^{i\omega t}\frac{\sin n\delta/2}{\sin\delta/2}$$

The conjugate of z is given by:

$$z^* = ae^{-i\omega t}\frac{\sin n\delta/2}{\sin\delta/2}$$

so we have:

$$zz^* = ae^{i\omega t}\frac{\sin n\delta/2}{\sin\delta/2}\cdot ae^{-i\omega t}\frac{\sin n\delta/2}{\sin\delta/2} = a^2\frac{\sin^2 n\delta/2}{\sin^2\delta/2}$$

SOLUTIONS TO CHAPTER 2

2.1

The system is released from rest, so we know its initial velocity is zero, i.e.

$$\left.\frac{dx}{dt}\right|_{t=0} = 0$$

(2.1.1)

Now, rearrange the expression for the displacement in the form:

$$x = \frac{F+G}{2}e^{(-p+q)t} + \frac{F-G}{2}e^{(-p-q)t}$$

(2.1.2)

Then, substitute (2.1.2) into (2.1.1), we have

$$\left.\frac{dx}{dt}\right|_{t=0} = \left[(-p+q)\frac{F+G}{2}e^{(-p+q)t} + (-p-q)\frac{F-G}{2}e^{(-p-q)t}\right]_{t=0} = 0$$

i.e.

$$qG = pF$$

(2.1.3)

By substitution of the expressions of q and p into equation (2.1.3), we have the ratio given by:

$$\frac{G}{F} = \frac{r}{\left(r^2 - 4ms\right)^{1/2}}$$

2.2

The first and second derivatives of x are given by:

$$\dot{x} = \left[B - \frac{r}{2m}(A+Bt)\right]e^{-rt/2m}$$

$$\ddot{x} = \left[-\frac{rB}{m} + \frac{r^2}{4m^2}(A+Bt)\right]e^{-rt/2m}$$

We can verify the solution by substitution of x, $\dot{x}$ and $\ddot{x}$ into equation:

$$m\ddot{x} + r\dot{x} + sx = 0$$

then we have equation:

$$\left(s - \frac{r^2}{4m}\right)(A+Bt) = 0$$

which is true for all t, provided the first bracketed term of the above equation is zero, i.e.

$$s - \frac{r^2}{4m} = 0$$

i.e. $$r^2/4m^2 = s/m$$

2.3

The initial displacement of the system is given by:

$$x = e^{-rt/2m}\left(C_1 e^{i\omega' t} + C_2 e^{-i\omega' t}\right) = A\cos\phi \quad \text{at} \quad t = 0$$

So:

$$C_1 + C_2 = A\cos\phi \tag{2.3.1}$$

Now let the initial velocity of the system to be:

$$\dot{x} = \left(-\frac{r}{2m} + i\omega'\right)C_1 e^{(-r/2m + i\omega')t} + \left(-\frac{r}{2m} - i\omega'\right)C_2 e^{(-r/2m - i\omega')t} = -\omega' A\sin\phi \quad \text{at} \quad t = 0$$

i.e. $$-\frac{r}{2m}A\cos\phi + i\omega'\left(C_1 - C_2\right) = -\omega' A\sin\phi$$

If r/m is very small or $\phi \approx \pi/2$, the first term of the above equation approximately equals zero, so we have:

$$C_1 - C_2 = iA\sin\phi \tag{2.3.2}$$

From (2.3.1) and (2.3.2), C_1 and C_2 are given by:

$$C_1 = \frac{A(\cos\phi + i\sin\phi)}{2} = \frac{A}{2}e^{i\phi}$$
$$C_2 = \frac{A(\cos\phi - i\sin\phi)}{2} = \frac{A}{2}e^{-i\phi}$$

2.4

Use the relation between current and charge, $I = \dot{q}$, and the voltage equation:

$$q/C + IR = 0$$

we have the equation:

$$R\dot{q} + q/C = 0$$

solve the above equation, we get:

$$q = C_1 e^{-t/RC}$$

where C_1 is arbitrary in value. Use initial condition, we get $C_1 = q_0$,

i.e. $$q = q_0 e^{-t/RC}$$

which shows the relaxation time of the process is RC s.

2.5

(a) $$\omega_0^2 - \omega'^2 = 10^{-6}\omega_0^2 = r^2/4m^2 \quad => \quad \omega_0 m/r = 500$$

The condition also shows $\omega' \approx \omega_0$, so:

$$Q = \omega' m/r \approx \omega_0 m/r = 500$$

Use $\tau' = \dfrac{2\pi}{\omega'}$, we have:

$$\delta = \frac{r}{2m}\tau' = \frac{\pi r}{m\omega'} = \frac{\pi}{Q} = \frac{\pi}{500}$$

(b) The stiffness of the system is given by:

$$s = \omega_0^2 m = 10^{12} \times 10^{-10} = 100[Nm^{-1}]$$

and the resistive constant is given by:

$$r = \frac{\omega_0 m}{Q} = \frac{10^6 \times 10^{-10}}{500} = 2 \times 10^{-7}[N \cdot sm^{-1}]$$

(c) At $t = 0$ and maximum displacement, $\dot{x} = 0$, energy is given by:

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}sx^2 = \frac{1}{2}sx_{\max}^2 = \frac{1}{2}\times 100 \times 10^{-4} = 5\times 10^{-3}[J]$$

Time for energy to decay to e^{-1} of initial value is given by:

$$t = \frac{m}{r} = \frac{10^{-10}}{2\times 10^{-7}} = 0.5[ms]$$

(d) Use definition of Q factor:

$$Q = 2\pi \frac{E}{-\Delta E}$$

where, E is energy stored in system, and $-\Delta E$ is energy lost per cycle, so energy loss in the first cycle, $-\Delta E_1$, is given by:

$$-\Delta E_1 = -\Delta E = 2\pi \frac{E}{Q} = 2\pi \times \frac{5\times 10^{-3}}{500} = 2\pi \times 10^{-5}[J]$$

2.6

The frequency of a damped simple harmonic oscillation is given by:

$$\omega'^2 = \omega_0^2 - \frac{r^2}{4m^2} \Rightarrow \omega'^2 - \omega_0^2 = \frac{r^2}{4m^2} \Rightarrow \Delta\omega = \omega' - \omega_0 = \frac{r^2}{4m^2(\omega' + \omega_0)}$$

Use $\omega' \approx \omega$ and $Q = \dfrac{\omega_0 m}{r}$ we find fractional change in the resonant frequency is given by:

$$\frac{\Delta\omega}{\omega_0} = \frac{\omega' - \omega_0}{\omega_0} \approx \frac{r^2}{8m^2\omega_0^2} = \left(8Q^2\right)^{-1}$$

2.7

See page 71 of text. Analysis is the same as that in the text for the mechanical case except that inductance L replaces mass m, resistance R replaces r and stiffness s is replaced by $1/C$, where C is the capacitance. A large Q value requires a small R.

2.8

Electrons per unit area of the plasma slab is given by:

$$q = -nle$$

When all the electrons are displaced a distance x, giving a restoring electric field: $E = nex/\varepsilon_0$, the restoring force per unit area is given by:

$$F = qE = -\frac{xn^2e^2l}{\varepsilon_0}$$

Newton's second law gives:

$$\text{restoring force per unit area} = \text{electrons mass per unit area} \times \text{electrons acceleration}$$

i.e.

$$F = -\frac{xn^2e^2l}{\varepsilon_0} = nlm_e \times \ddot{x}$$

i.e. $$\ddot{x} + \frac{ne^2}{m\varepsilon_0}x = 0$$

From the above equation, we can see the displacement distance of electrons, x, oscillates with angular frequency:

$$\omega_e^2 = \frac{ne^2}{m_e\varepsilon_0}$$

2.9

As the string is shortened work is done against: (a) gravity $(mg\cos\theta)$ and (b) the centrifugal force $(mv^2/r = ml\dot{\theta}^2)$ along the time of shortening. Assume that during shortening there are many swings of constant amplitude so work done is:

$$A = -(mg\overline{\cos\theta} + ml\overline{\dot{\theta}^2})\Delta l$$

where the bar denotes the average value. For small θ, $\cos\theta = 1 - \theta^2/2$ so:

$$A = -mg\Delta l + (mg\,\overline{\theta^2}/2 - ml\overline{\dot{\theta}^2})$$

The term $-mg\Delta l$ is the elevation of the equilibrium position and does not affect the energy of motion so the energy change is:

$$\Delta E = (mg\,\overline{\theta^2}/2 - ml\overline{\dot{\theta}^2})\Delta l$$

Now the pendulum motion has energy:

$$E = \frac{m}{2} l^2 \dot{\theta}^2 + mgl(1 - \cos\theta),$$

that is, kinetic energy plus the potential energy related to the rest position, for small θ this becomes:

$$E = \frac{ml^2\dot{\theta}^2}{2} + \frac{mgl\theta^2}{2}$$

which is that of a simple harmonic oscillation with linear amplitude $l\theta_0$.

Taking the solution $\theta = \theta_0 \cos\omega t$ which gives $\overline{\theta^2} = \theta_0^2/2$ and $\overline{\dot{\theta}^2} = \omega^2\,\theta_0^2/2$ with $\omega = \sqrt{g/l}$ we may write:

$$E = \frac{ml^2\omega^2\theta_0^2}{2} = \frac{mgl\theta_0^2}{2}$$

and

$$\Delta E = \left(\frac{ml\omega^2\theta_0^2}{4} - \frac{ml\omega^2\theta_0^2}{2}\right)\Delta l = -\frac{ml\omega^2\theta_0^2}{4}\cdot\Delta l$$

so:

$$\frac{\Delta E}{E} = -\frac{1}{2}\frac{\Delta l}{l}$$

Now $\omega = 2\pi\nu = \sqrt{g/l}$ so the frequency ν varies with $l^{-1/2}$ and

$$\frac{\Delta\nu}{\nu} = -\frac{1}{2}\frac{\Delta l}{l} = \frac{\Delta E}{E}$$

so:

$$\frac{E}{\nu} = \text{constant}$$

SOLUTIONS TO CHAPTER 3

3.1

The solution of the vector form of the equation of motion for the forced oscillator:

$$m\ddot{\mathbf{x}} + r\dot{\mathbf{x}} + s\mathbf{x} = F_0 e^{i\omega t}$$

is given by:

$$\mathbf{x} = \frac{iF_0 e^{i(\omega t - \phi)}}{\omega Z_m} = -\frac{iF}{\omega Z_m}\cos(\omega t - \phi) + \frac{F}{\omega Z_m}\sin(\omega t - \phi)$$

Since $F_0 e^{i\omega t}$ represents its imaginary part: $F_0 \sin\omega t$, the value of x is given by the imaginary part of the solution, i.e.:

$$x = -\frac{F}{\omega Z_m}\cos(\omega t - \phi)$$

The velocity is given by:

$$v = \dot{x} = \frac{F}{Z_m}\sin(\omega t - \phi)$$

3.2

The transient term of a forced oscillator decay with $e^{-rt/2m}$ to e^{-k} at time t, i.e.:

$$-rt/2m = -k$$

so, we have the resistance of the system given by:

$$r = 2mk/t \qquad (3.2.1)$$

For small damping, we have

$$\omega \approx \omega_0 = \sqrt{s/m} \qquad (3.2.2)$$

We also have steady state displacement given by:

$$x = x_0 \sin(\omega t - \phi)$$

where the maximum displacement is:

$$x_0 = \frac{F_0}{\omega\sqrt{r^2 + (\omega m - s/m)^2}} \qquad (3.2.3)$$

By substitution of (3.2.1) and (3.2.2) into (3.2.3), we can find the average rate of growth of the oscillations given by:

$$\frac{x_0}{t} = \frac{F_0}{2km\omega_0}$$

3.3

Write the equation of an undamped simple harmonic oscillator driven by a force of frequency ω in the vector form, and use $F_0 e^{i\omega t}$ to represent its imaginary part $F_0 \sin \omega t$, we have:

$$m\ddot{\mathbf{x}} + s\mathbf{x} = F_0 e^{i\omega t} \qquad (3.3.1)$$

We try the steady state solution $\mathbf{x} = \mathbf{A}e^{i\omega t}$ and the velocity is given by:

$$\dot{\mathbf{x}} = i\omega \mathbf{A}e^{i\omega t} = i\omega \mathbf{x}$$

so that:

$$\ddot{\mathbf{x}} = i^2\omega^2\mathbf{x} = -\omega^2\mathbf{x}$$

and equation (3.3.1) becomes:

$$(-\mathbf{A}\omega^2 m + \mathbf{A}s)e^{i\omega t} = F_0 e^{i\omega t}$$

which is true for all t when

$$-\mathbf{A}\omega^2 m + \mathbf{A}s = F_0$$

i.e.

$$\mathbf{A} = \frac{F_0}{s - \omega^2 m}$$

i.e.

$$\mathbf{x} = \frac{F_0}{s - \omega^2 m} e^{i\omega t}$$

The value of x is the imaginary part of vector $\mathbf{x}$, given by:

$$x = \frac{F_0}{s - \omega^2 m} \sin \omega t$$

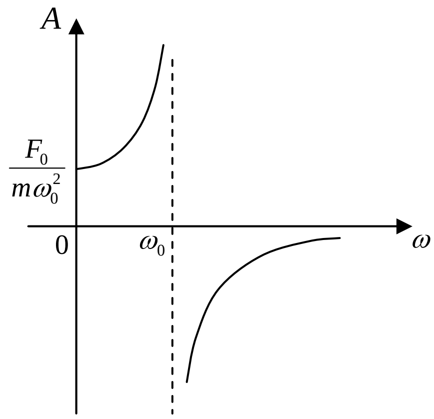
i.e.

$$x = \frac{F_0 \sin \omega t}{m(\omega_0^2 - \omega^2)} \qquad \text{where} \quad \omega_0^2 = \frac{s}{m}$$

Hence, the amplitude of x is given by:

$$A = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

and its behaviour as a function of frequency is shown in the following graph:



By solving the equation:

$$m\ddot{x} + sx = 0$$

we can easily find the transient term of the equation of the motion of an undamped simple harmonic oscillator driven by a force of frequency ω is given by:

$$x = C\cos\omega_0 t + D\sin\omega_0 t$$

where, $\omega_0 = \sqrt{s/m}$, C and D are constant. Finally, we have the general solution for the displacement given by the sum of steady term and transient term:

$$x = \frac{F_0 \sin\omega t}{m(\omega_0^2 - \omega^2)} + C\cos\omega_0 t + D\sin\omega_0 t \qquad (3.3.2)$$

3.4

In equation (3.3.2), $x = 0$ at $t = 0$ gives:

$$x\big|_{t=0} = \left[\frac{F_0 \sin\omega t}{m(\omega_0^2 - \omega^2)} + A\cos\omega_0 t + B\sin\omega_0 t\right]_{t=0} = A = 0 \qquad (3.4.1)$$

In equation (3.3.2), $\dot{x} = 0$ at $t = 0$ gives:

$$\left.\frac{dx}{dt}\right|_{t=0} = \left[\frac{\omega F_0 \cos\omega t}{m(\omega_0^2 - \omega^2)} - \omega_0 A\sin\omega_0 t + \omega_0 B\cos\omega_0 t\right]_{t=0} = \frac{\omega F_0}{m(\omega_0^2 - \omega^2)} + \omega_0 B = 0$$

i.e.

$$B = -\frac{F_0\omega}{m\omega_0(\omega_0^2 - \omega^2)} \qquad (3.4.2)$$

By substitution of (3.4.1) and (3.4.2) into (3.3.2), we have:

$$x = \frac{F_0}{m}\frac{1}{(\omega_0^2 - \omega^2)}\left(\sin\omega t - \frac{\omega}{\omega_0}\sin\omega_0 t\right) \qquad (3.4.3)$$

By substitution of $\omega = \omega_0 + \Delta\omega$ into (3.4.3), we have:

$$x = -\frac{F_0}{m(\omega_0+\omega)\Delta\omega}\left(\sin\omega_0 t\cos\Delta\omega t + \sin\Delta\omega t\cos\omega_0 t - \frac{\omega}{\omega_0}\sin\omega_0 t\right) \qquad (3.4.4)$$

Since $\Delta\omega/\omega_0 << 1$ and $\Delta\omega t << 1$, we have:

$$\omega \approx \omega_0,\ \ \sin\Delta\omega t \approx \Delta\omega t,\text{ and }\ \cos\Delta\omega t \approx 1$$

Then, equation (3.4.4) becomes:

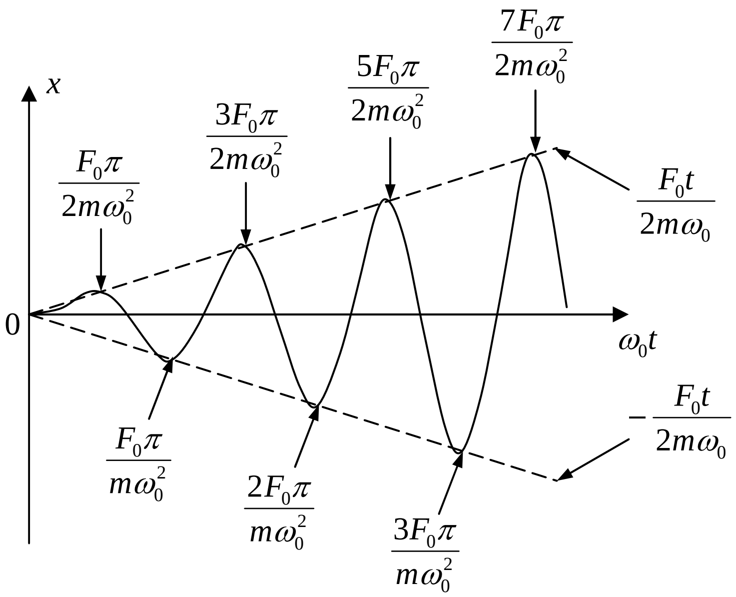
$$x = -\frac{F_0}{2m\omega_0\Delta\omega}\left(\sin\omega_0 t + \Delta\omega t\cos\omega_0 t - \frac{\omega}{\omega_0}\sin\omega_0 t\right)$$

i.e.
$$x = -\frac{F_0}{2m\omega_0\Delta\omega}\left(-\frac{\Delta\omega}{\omega_0}\sin\omega_0 t + \Delta\omega t\cos\omega_0 t\right)$$

i.e.
$$x = \frac{F_0}{2m\omega_0}\left(\frac{\sin\omega_0 t}{\omega_0} - t\cos\omega_0 t\right)$$

i.e.
$$x = \frac{F_0}{2m\omega_0^2}(\sin\omega_0 t - \omega_0 t\cos\omega_0 t)$$

The behaviour of displacement x as a function of $\omega_0 t$ is shown in the following



graph:

3.5

The general expression of displacement of a simple damped mechanical oscillator driven by a force $F_0\cos\omega t$ is the sum of transient term and steady state term, given by:

$$\mathbf{x} = Ce^{-\frac{rt}{2m}+i\omega_t t} - \frac{iF_0 e^{i(\omega t-\phi)}}{\omega Z_m}$$

where, C is constant, $Z_m = \sqrt{r^2 + (\omega m - s/\omega)^2}$, $\omega_t = \sqrt{s/m - r^2/4m^2}$ and

$\phi = \tan^{-1}\left(\frac{\omega m - s/\omega}{r}\right)$, so the general expression of velocity is given by:

$$\mathbf{v} = \dot{\mathbf{x}} = C\left(-\frac{r}{2m} + i\omega_t\right)e^{-\frac{rt}{2m}+i\omega_t t} + \frac{F_0}{Z_m}e^{i(\omega t-\phi)}$$

and the general expression of acceleration is given by:

$$\dot{\mathbf{v}} = C\left(\frac{r^2}{4m^2} - \omega_t^2 - \frac{i\omega_t r}{m}\right)e^{-\frac{rt}{2m}+i\omega_t t} + \frac{i\omega F_0}{Z_m}e^{i(\omega t-\phi)}$$

i.e. $$\dot{\mathbf{v}} = C\left(\frac{r^2 - 2ms}{2m^2} - \frac{i\omega_t r}{m}\right)e^{-\frac{rt}{2m}+i\omega_t t} + \frac{i\omega F_0}{Z_m}e^{i(\omega t-\phi)} \qquad (3.5.1)$$

From (3.5.1), we find the amplitude of acceleration at steady state is given by:

$$\dot{v} = \frac{\omega F_0}{Z_m} = \frac{\omega F_0}{\sqrt{r^2 + (\omega m - s/\omega)^2}}$$

At the frequency of maximum acceleration: $\frac{d\dot{v}}{d\omega} = 0$

i.e. $$\frac{d}{d\omega}\left[\frac{\omega F_0}{\sqrt{r^2 + (\omega m - s/\omega)^2}}\right] = 0$$

i.e. $$r^2 - 2ms + \frac{2s^2}{\omega^2} = 0$$

i.e. $$\omega^2 = \frac{2s^2}{2sm - r^2}$$

Hence, we find the expression of the frequency of maximum acceleration given by:

$$\omega = \sqrt{\frac{2s^2}{2sm - r^2}}$$

The frequency of velocity resonance is given by: $\omega = \sqrt{s/m}$, so if $r = \sqrt{sm}$, the acceleration amplitude at the frequency of velocity resonance is given by:

$$\dot{v}\Big|_{r=\sqrt{sm}} = \frac{\omega F_0}{\sqrt{r^2 + (\omega m - s/\omega)^2}} = \frac{\sqrt{s/m}F_0}{\sqrt{sm + (\sqrt{sm} - \sqrt{sm})}} = \frac{F_0}{m}$$

The limit of the acceleration amplitude at high frequencies is given by:

$$\lim_{\omega\to\infty}\dot{v} = \lim_{\omega\to\infty}\frac{\omega F_0}{\sqrt{r^2+(\omega m - s/\omega)^2}} = \lim_{\omega\to\infty}\frac{F_0}{\sqrt{\frac{r^2}{\omega^2}+\left(m-\frac{s}{\omega^2}\right)^2}} = \frac{F_0}{m}$$

So we have:

$$\dot{v}\Big|_{r=\sqrt{sm}} = \lim_{\omega\to\infty}\dot{v}$$

3.6

The displacement amplitude of a driven mechanical oscillator is given by:

$$x = \frac{F_0}{\omega\sqrt{r^2+(\omega m - s/\omega)^2}}$$

i.e.

$$x = \frac{F_0}{\sqrt{\omega^2 r^2+(\omega^2 m - s)^2}} \qquad (3.6.1)$$

The displacement resonance frequency is given by:

$$\omega = \sqrt{\frac{s}{m}-\frac{r^2}{2m^2}} \qquad (3.6.2)$$

By substitution of (3.6.2) into (3.6.1), we have:

$$x = \frac{F_0}{\sqrt{r^2\left(\frac{s}{m}-\frac{r^2}{2m^2}\right)+\left(\frac{r^2}{2m}\right)^2}}$$

i.e.

$$x = \frac{F_0}{r\sqrt{\frac{s}{m}-\frac{r^2}{4m^2}}}$$

which proves the exact amplitude at the displacement resonance of a driven mechanical oscillator may be written as:

$$x = \frac{F_0}{\omega' r}$$

where,

$$\omega'^2 = \frac{s}{m}-\frac{r^2}{4m^2}$$

3.7

(a) The displacement amplitude is given by:

$$x = \frac{F_0}{\omega\sqrt{r^2+(\omega m - s/\omega)^2}}$$

At low frequencies, we have:

$$\lim_{\omega\to 0} x = \lim_{\omega\to 0}\frac{F_0}{\omega\sqrt{r^2+(\omega m - s/\omega)^2}} = \lim_{\omega\to 0}\frac{F_0}{\sqrt{\omega^2 r^2+(\omega^2 m - s)^2}} = \frac{F_0}{s}$$

(b) The velocity amplitude is given by:

$$v = \frac{F_0}{\sqrt{r^2+(\omega m - s/\omega)^2}}$$

At velocity resonance: $\omega = \sqrt{s/m}$, so we have:

$$v_r = \left.\frac{F_0}{\sqrt{r^2+(\omega m - s/\omega)^2}}\right|_{\omega=\sqrt{s/m}} = \frac{F_0}{\sqrt{r^2+(\sqrt{sm}-\sqrt{sm})^2}} = \frac{F_0}{r}$$

(c) From problem 3.5, we have the acceleration amplitude given by:

$$\dot{v} = \frac{\omega F_0}{\sqrt{r^2+(\omega m - s/\omega)^2}}$$

At high frequency, we have:

$$\lim_{\omega\to\infty} \dot{v} = \lim_{\omega\to\infty}\frac{\omega F_0}{\sqrt{r^2+(\omega m - s/\omega)^2}} = \lim_{\omega\to\infty}\frac{F_0}{\sqrt{r^2/\omega^2+(m - s/\omega^2)^2}} = \frac{F_0}{m}$$

From (a), (b) and (c), we find $\lim_{\omega\to 0} x$, v_r and $\lim_{\omega\to\infty} \dot{v}$ are all constants, i.e. they are all frequency independent.

3.8

The expression of curve (a) in Figure 3.9 is given by:

$$x_a = -\frac{F_0 X_m}{\omega Z_m^2} = \frac{F_0 m(\omega_0^2-\omega^2)}{m^2(\omega_0^2-\omega^2)^2+\omega^2 r^2} \qquad (3.8.1)$$

where $$\omega_0 = \sqrt{s/m}$$

x_a has either maximum or minimum value when $\frac{dx_a}{d\omega} = 0$

i.e. $$\frac{d}{d\omega}\left[\frac{F_0 m(\omega_0^2-\omega^2)}{m^2(\omega_0^2-\omega^2)^2+\omega^2 r^2}\right] = 0$$

i.e. $$m^2(\omega_0^2-\omega^2)^2 - \omega_0^2 r^2 = 0$$

Then, we have two solutions of ω given by:

$$\omega_1 = \sqrt{\omega_0^2 - \frac{\omega_0 r}{m}} \qquad (3.8.2)$$

and

$$\omega_2 = \sqrt{\omega_0^2 + \frac{\omega_0 r}{m}} \qquad (3.8.3)$$

Since r is very small, rearrange the expressions of ω_1 and ω_2, we have:

$$\omega_1 = \sqrt{\omega_0^2 - \frac{\omega_0 r}{m}} = \sqrt{\left(\omega_0 - \frac{r}{2m}\right)^2 - \frac{r^2}{4m^2}} \approx \omega_0 - \frac{r}{2m}$$

$$\omega_2 = \sqrt{\omega_0^2 + \frac{\omega_0 r}{m}} = \sqrt{\left(\omega_0 + \frac{r}{2m}\right)^2 - \frac{r^2}{4m^2}} \approx \omega_0 + \frac{r}{2m}$$

The maximum and the minimum values of x_a can found by substitution of (3.8.2) and (3.8.3) into (3.8.1), so we have:

when $\omega = \omega_1$: $$x_a = \frac{F_0}{2\omega_0 r - \omega_0 r^2/m} \approx \frac{F_0}{2\omega_0 r}$$

which is the maximum value of x_a, and

when $\omega = \omega_2$: $$x_a = -\frac{F_0}{2\omega_0 r + \omega_0 r^2/m} \approx -\frac{F_0}{2\omega_0 r}$$

which is the minimum value of x_a.

3.9

The undamped oscillatory equation for a bound electron is given by:

$$\ddot{x} + \omega_0^2 x = (-eE_0/m)\cos\omega t \qquad (3.9.1)$$

Try solution $x = A\cos\omega t$ in equation (3.9.1), we have:

$$(-\omega^2 + \omega_0^2)A\cos\omega t = (-eE_0/m)\cos\omega t$$

which is true for all t provided:

$$(-\omega^2 + \omega_0^2)A = -eE_0/m$$

i.e. $$A = -\frac{eE_0}{m(\omega_0^2 - \omega^2)}$$

So, we find a solution to equation (3.9.1) given by:

$$x = -\frac{eE_0}{m(\omega_0^2 - \omega^2)}\cos\omega t \qquad (3.9.2)$$

For an electron number density n, the induced polarizability per unit volume of a medium is given by:

$$\chi_e = -\frac{nex}{\varepsilon_0 E} \tag{3.9.3}$$

By substitution of (3.9.2) and $E = E_0 \cos\omega t$ into (3.9.3), we have

$$\chi_e = -\frac{nex}{\varepsilon_0 E} = \frac{ne^2}{\varepsilon_0 m(\omega_0^2 - \omega^2)}$$

3.10

The forced mechanical oscillator equation is given by:

$$m\ddot{x} + r\dot{x} + sx = F_0 \cos\omega t$$

which can be written as:

$$m\ddot{x} + r\dot{x} + m\omega_0^2 x = F_0 \cos\omega t \tag{3.10.1}$$

where, $\omega_0 = \sqrt{s/m}$. Its solution can be written as:

$$x = \frac{F_0 r}{\omega Z_m^2}\sin\omega t - \frac{F_0 X_m}{\omega Z_m^2}\cos\omega t \tag{3.10.2}$$

where, $X_m = \omega m - s/\omega$, $Z_m = \sqrt{r^2 + (\omega m - s/\omega)^2}$, $\phi = \tan^{-1}\left(\frac{\omega m - s/\omega}{r}\right)$

By taking the displacement x as the component represented by curve (a) in Figure 3.9, i.e. by taking the second term of equation (3.10.2) as the expression of x, we have:

$$x = -\frac{F_0 X_m}{\omega Z_m^2}\cos\omega t = \frac{F_0 m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 r^2}\cos\omega t \tag{3.10.3}$$

The damped oscillatory electron equation can be written as:

$$m\ddot{x} + r\dot{x} + m\omega_0^2 x = -eE_0 \cos\omega t \tag{3.10.4}$$

Comparing (3.10.1) with (3.10.4), we can immediately find the displacement x for a damped oscillatory electron by substituting $F_0 = -eE_0$ into (3.10.3), i.e.:

$$x = -\frac{eE_0 m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 r^2}\cos\omega t \tag{3.10.5}$$

By substitution of (3.10.5) into (3.9.3), we can find the expression of χ for a damped oscillatory electron is given by:

$$\chi = -\frac{nex}{\varepsilon_0 E} = \frac{ne^2 m(\omega_0^2 - \omega^2)}{\varepsilon_0[m^2(\omega_0^2 - \omega^2)^2 + \omega^2 r^2]}$$

So we have:

$$\varepsilon_r = 1 + \chi = 1 + \frac{ne^2 m(\omega_0^2 - \omega^2)}{\varepsilon_0[m^2(\omega_0^2 - \omega^2)^2 + \omega^2 r^2]}\cos\omega t$$

3.11

The instantaneous power dissipated is equal to the product of frictional force and the instantaneous velocity, i.e.:

$$P = (r\dot{x})\dot{x} = r\frac{F_0^2}{Z_m^2}\cos^2(\omega t - \phi)$$

The period for a given frequency ω is given by:

$$T = \frac{2\pi}{\omega}$$

Therefore, the energy dissipated per cycle is given by:

$$\begin{aligned} E &= \int_0^T P dt = \int_0^{2\pi/\omega} r\frac{F_0^2}{Z_m^2}\cos^2(\omega t - \phi)dt \\ &= \int_0^{2\pi/\omega} \frac{rF_0^2}{2Z_m^2}[1 - \cos 2(\omega t - \phi)]dt \\ &= \frac{2\pi}{\omega}\frac{rF_0^2}{2Z_m^2} \\ &= \frac{\pi r F_0^2}{\omega Z_m^2} \end{aligned} \qquad (3.11.1)$$

The displacement is given by:

$$x = \frac{F_0}{\omega Z_m}\sin(\omega t - \phi)$$

so we have:

$$x_{\max} = \frac{F_0}{\omega Z_m} \qquad (3.11.2)$$

By substitution of (3.11.2) into (3.11.1), we have:

$$E = \pi r \omega x_{\max}^2$$

3.12

The low frequency limit of the bandwidth of the resonance absorption curve ω_1 satisfies the equation:

$$\omega_1 m - s/\omega_1 = -r$$

which defines the phase angle given by:

$$\tan\phi_1 = \frac{\omega_1 m - s/\omega_1}{r} = -1$$

The high frequency limit of the bandwidth of the resonance absorption curve ω_2 satisfies the equation:

$$\omega_2 m - s/\omega_2 = r$$

which defines the phase angle given by:

$$\tan\phi_2 = \frac{\omega_2 m - s/\omega_2}{r} = 1$$

3.13

For a LCR series circuit, the current through the circuit is given by

$$I = I_0 e^{i\omega t}$$

The voltage across the inductance is given by:

$$L\frac{dI}{dt} = L\frac{d}{dt}I_0 e^{i\omega t} = i\omega L I_0 e^{i\omega t} = i\omega L I$$

i.e. the amplitude of voltage across the inductance is:

$$V_L = \omega L I_0 \qquad (3.13.1)$$

The voltage across the condenser is given by:

$$\frac{q}{C} = \frac{1}{C}\int I dt = \frac{1}{C}\int e^{i\omega t} dt = \frac{1}{i\omega C} I_0 e^{i\omega t} = -\frac{iI}{\omega C}$$

i.e. the amplitude of the voltage across the condenser is:

$$V_C = \frac{I_0}{\omega C} \qquad (3.13.2)$$

When an alternating voltage, amplitude V_0 is applied across LCR series circuit, current amplitude I_0 is given by: $I_0 = V_0/Z_e$, where the impedance Z_e is given by:

$$Z_m = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

At current resonance, I_0 has the maximum value:

$$I_0 = \frac{V_0}{R} \qquad (3.13.3)$$

and the resonant frequency ω_0 is given by:

$$\omega_0 L - \frac{1}{\omega_0 C} = 0 \quad \text{or} \quad \omega_0 = \frac{1}{\sqrt{LC}} \tag{3.13.4}$$

By substitution of (3.12.3) and (3.12.4) into (3.12.1), we have:

$$V_L = \frac{\omega_0 L}{R} V_0$$

By substitution of (3.12.3) and (3.12.4) into (3.12.2), we have:

$$V_C = \frac{V_0}{\omega_0 RC} = \frac{\sqrt{LC}}{RC} V_0 = \sqrt{\frac{L}{C}} \frac{V_0}{R} = \frac{L}{\sqrt{LC}} \frac{V_0}{R} = \frac{\omega_0 L}{R} V_0$$

Noting that the quality factor of an LCR series circuit is given by:

$$Q = \frac{\omega_0 L}{R}$$

so we have:

$$V_L = V_C = QV_0$$

3.14

In a resonant LCR series circuit, the potential across the condenser is given by:

$$V_C = \frac{I}{\omega C} \tag{3.14.1}$$

where, I is the current through the whole LCR series circuit, and is given by:

$$I = I_0 e^{i\omega t} \tag{3.14.2}$$

The current amplitude I_0 is given by:

$$I_0 = \frac{V_0}{Z_e} \tag{3.14.3}$$

where, V_0 is the voltage amplitude applied across the whole LCR series circuit and is a constant. Z_e is the impedance of the whole circuit, given by:

$$Z_e = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{3.14.4}$$

From (3.14.1), (3.14.2), (3.14.3), and (3.14.4) we have:

$$V_C = \frac{V_0}{C\omega\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} e^{i\omega t} = V_{C0} e^{i\omega t}$$

which has the maximum value when $\frac{dV_{C0}}{d\omega} = 0$, i.e.:

$$\frac{d}{d\omega}\frac{V_0}{C\omega\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}}=0$$

i.e.
$$R^2+\left(\omega L-\frac{1}{\omega C}\right)^2+\omega^2L^2-\frac{1}{\omega^2C^2}=0$$

i.e.
$$R^2+2\omega^2L^2-2\frac{L}{C}=0$$

i.e.
$$\omega=\sqrt{\frac{1}{LC}-\frac{R^2}{2L^2}}=\omega_0\sqrt{1-\frac{1}{2}Q_0^2}$$

where $\omega_0^2=\frac{1}{LC}$, $Q_0=\frac{\omega_0 L}{R}$

3.15

In a resonant LCR series circuit, the potential across the inductance is given by:

$$V_L=\omega LI \qquad (3.15.1)$$

where, I is the current through the whole LCR series circuit, and is given by:

$$I=I_0e^{i\omega t} \qquad (3.15.2)$$

The current amplitude I_0 is given by:

$$I_0=\frac{V_0}{Z_e} \qquad (3.15.3)$$

where, V_0 is the voltage amplitude applied across the whole LCR series circuit and is a constant. Z_e is the impedance of the whole circuit, given by:

$$Z_e=\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2} \qquad (3.15.4)$$

From (3.15.1), (3.15.2), (3.15.3), and (3.15.4) we have:

$$V_L=\frac{\omega LV_0}{\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}}e^{i\omega t}=V_{L0}e^{i\omega t}$$

which has the maximum value when $\frac{dV_{L0}}{d\omega}=0$, i.e.:

$$\frac{d}{d\omega}\frac{\omega L V_0}{\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}}=0$$

i.e.
$$R^2+\left(\omega L-\frac{1}{\omega C}\right)^2-\omega^2L^2+\frac{1}{\omega^2C^2}=0$$

i.e.
$$R^2+\frac{2}{\omega^2C^2}-2\frac{L}{C}=0$$

i.e.
$$\omega=\sqrt{\frac{1}{LC-\frac{R^2C^2}{2}}}=\sqrt{\frac{1}{LC}\frac{1}{1-\frac{R^2C}{2L}}}=\omega_0\sqrt{\frac{1}{1-\frac{R^2}{2L^2\omega_0^2}}}=\frac{\omega_0}{\sqrt{1-\frac{1}{2}Q_0^2}}$$

where $\omega_0^2=\frac{1}{LC}$, $Q_0=\frac{\omega_0 L}{R}$

3.16

Considering an electron in an atom as a lightly damped simple harmonic oscillator, we know its resonance absorption bandwidth is given by:

$$\delta\omega=\frac{r}{m} \qquad (3.16.1)$$

On the other hand, the relation between frequency and wavelength of light is given by:

$$f=\frac{c}{\lambda} \qquad (3.16.2)$$

where, c is speed of light in vacuum. From (3.16.2) we find at frequency resonance:

$$\delta f=-\frac{c}{\lambda_0^2}\delta\lambda$$

where λ_0 is the wavelength at frequency resonance. Then, the relation between angular frequency bandwidth $\delta\omega$ and the width of spectral line $\delta\lambda$ is given by:

$$|\delta\omega|=2\pi|\delta f|=\frac{2\pi c}{\lambda_0^2}|\delta\lambda| \qquad (3.16.3)$$

From (3.16.1) and (3.16.3) we have:

$$|\delta\lambda|=\frac{\lambda_0^2 r}{2\pi c m}=\frac{\lambda_0 r}{\omega_0 m}=\frac{\lambda_0}{Q}$$

So the width of the spectral line from such an atom is given by:

$$|\delta\lambda|=\frac{\lambda_0}{Q}=\frac{0.6\times10^{-6}}{5\times10^{7}}=1.2\times10^{-14}[m]$$

3.17

According to problem 3.6, the displacement resonance frequency ω_r and the corresponding displacement amplitude $x_{\max}$ are given by:

$$\omega_r = \sqrt{\omega_0^2 - \frac{r^2}{2m^2}}$$

$$x_{\max} = \left.\frac{F_0}{\omega Z_m}\right|_{\omega=\omega_r} = \frac{F_0}{\omega' r}$$

where, $Z_m = \sqrt{r^2 + (\omega m - s/\omega)^2}$, $\omega' = \sqrt{\omega_0^2 - \frac{r^2}{4m^2}}$, $\omega_0 = \sqrt{\frac{s}{m}}$

Now, at half maximum displacement:

$$\frac{F_0}{\omega Z_m} = \frac{x_{\max}}{2} = \frac{F_0}{2\omega' r}$$

i.e. $$\omega\sqrt{r^2 + (\omega m - s/\omega)^2} = 2\omega' r$$

i.e. $$\omega^2\left[r^2 + \left(\omega m - \frac{s}{\omega}\right)^2\right] = 4\left(\frac{s}{m} - \frac{r^2}{4m^2}\right)r^2$$

i.e. $$\omega^4 + \frac{(r^2 - 2sm)}{m^2}\omega^2 + \frac{s^2}{m^2} - \frac{4sr^2}{m^3} + \frac{r^4}{m^4} = 0$$

i.e. $$\omega^4 - 2\left(\frac{s}{m} - \frac{r^2}{2m^2}\right)\omega^2 + \left(\frac{s}{m} - \frac{r^2}{2m^2}\right)^2 - \frac{3r^2}{m^2}\left(\frac{s}{m} - \frac{r^2}{4m^2}\right)^2 = 0$$

i.e. $$(\omega^2 - \omega_r^2)^2 - \left(\frac{\sqrt{3}r}{m}\omega'^2\right)^2 = 0$$

i.e. $$\omega^2 - \omega_r^2 = \pm\frac{\sqrt{3}r}{m}\omega'^2 \qquad (3.17.1)$$

If ω_1 and ω_2 are the two solutions of equation (3.17.1), and $\omega_2 > \omega_1$, then:

$$\omega_2^2 - \omega_r^2 = \frac{\sqrt{3}r}{m}\omega'^2 \qquad (3.17.2)$$

$$\omega_1^2 - \omega_r^2 = -\frac{\sqrt{3}r}{m}\omega'^2 \qquad (3.17.3)$$

Since the Q-value is high, we have:

$$Q = \frac{\omega_0 m}{r} >> 1$$

i.e. $$\omega_0^2 >> \frac{r^2}{m^2}$$

i.e. $$\omega_r \approx \omega' \approx \omega_0$$

Then, from (3.17.2) and (3.17.3) we have:

$$(\omega_2 - \omega_1)(\omega_2 + \omega_1) \approx \frac{2\sqrt{3}r}{m}\omega_0^2$$

and $$\omega_1 + \omega_2 \approx 2\omega_0$$

Therefore, the width of displacement resonance curve is given by:

$$\omega_2 - \omega_1 \approx \frac{\sqrt{3}r}{m}$$

3.18

In Figure 3.9, curve (b) corresponds to absorption, and is given by:

$$x = \frac{F_0 r}{\omega Z_m^2}\sin\omega t = \frac{F_0 \omega r}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 r^2}\sin\omega t$$

and the velocity component corresponding to absorption is given by:

$$v = \dot{x} = \frac{F_0 \omega^2 r}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 r^2}\cos\omega t$$

For Problem 3.10, the velocity component corresponding to absorption can be given by substituting $F_0 = -eE_0$ into the above equation, i.e.:

$$v = \dot{x} = -\frac{eE_0 \omega^2 r}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 r^2}\cos\omega t \qquad (3.18.1)$$

For an electron density of n, the instantaneous power supplied equal to the product of the instantaneous driving force $-neE_0\cos\omega t$ and the instantaneous velocity, i.e.:

$$P = (-neE_0\cos\omega t)\times\left(-\frac{eE_0\omega^2 r}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 r^2}\cos\omega t\right)$$
$$= \frac{ne^2E_0^2\omega^2 r}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 r^2}\cos^2\omega t$$

The average power supplied per unit volume is then given by:

$$P_{av} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} Pdt$$
$$= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \frac{ne^2E_0^2\omega^2 r}{m^2(\omega_0^2-\omega^2)^2+\omega^2r^2}\cos^2\omega t$$
$$= \frac{ne^2E_0^2}{2}\frac{\omega^2 r}{m^2(\omega_0^2-\omega^2)^2+\omega^2r^2}$$

which is also the mean rate of energy absorption per unit volume.

SOLUTIONS TO CHAPTER 4

4.1

The kinetic energy of the system is the sum of the separate kinetic energy of the two masses, i.e.:

$$E_k = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 = \frac{1}{2}m\left[\frac{1}{2}(\dot{x}+\dot{y})^2 + \frac{1}{2}(\dot{x}-\dot{y})^2\right] = \frac{1}{4}m\dot{X}^2 + \frac{1}{4}m\dot{Y}^2$$

The potential energy of the system is the sum of the separate potential energy of the two masses, i.e.:

$$E_p = \frac{1}{2}\frac{mg}{l}x^2 + \frac{1}{2}s(y-x)^2 + \frac{1}{2}\frac{mg}{l}y^2 + \frac{1}{2}s(x-y)^2$$

$$= \frac{1}{2}\frac{mg}{l}(x^2+y^2) + s(x-y)^2$$

$$= \frac{1}{2}\frac{mg}{l}\left[\frac{1}{2}(x+y)^2 + \frac{1}{2}(x-y)^2\right] + s(x-y)^2$$

$$= \frac{1}{4}\frac{mg}{l}X^2 + \left(\frac{1}{2}\frac{mg}{l} + s\right)Y^2$$

Comparing the expression of E_k and E_p with the definition of E_X and E_Y given by (4.3a) and (4.3b), we have:

$$a = \frac{1}{2}m, \quad b = \frac{mg}{4l}, \quad c = \frac{1}{4}m, \text{ and } \quad d = \frac{mg}{2l} + s$$

Noting that:

$$X_q = \left(\frac{m}{2}\right)^{1/2}(x+y) = \left(\frac{m}{2}\right)^{1/2} X$$

and
$$Y_q = \left(\frac{m}{2}\right)^{1/2}(x-y) = \left(\frac{m}{c}\right)^{1/2} Y$$

i.e.
$$X = \left(\frac{m}{2}\right)^{-1/2} X_q \quad \text{and } Y = \left(\frac{m}{2}\right)^{-1/2} Y_q$$

we have the kinetic energy of the system given by:

$$E_k = a\dot{X}^2 + c\dot{Y}^2 = \frac{1}{4}m\left[\sqrt{\frac{2}{m}}\dot{X}_q\right]^2 + \frac{1}{4}m\left[\sqrt{\frac{2}{m}}\dot{Y}_q\right]^2 = \frac{1}{2}\dot{X}_q^2 + \frac{1}{2}\dot{Y}_q^2$$

and

$$E_p = bX^2 + dY^2 = \frac{mg}{4l}\left[\sqrt{\frac{2}{m}}X_q\right]^2 + \left(\frac{mg}{2l} + s\right)\left[\sqrt{\frac{2}{m}}Y_q\right]^2 = \frac{1}{2}\frac{g}{l}X_q^2 + \left(\frac{g}{l} + \frac{2s}{m}\right)Y_q^2$$

which are the expressions given by (4.4a) and (4.4b)

4.2

The total energy of Problem 4.1 can be written as:

$$E = E_k + E_p = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}\frac{mg}{l}(x^2 + y^2) + s(x-y)^2$$

The above equation can be rearranged as the format:

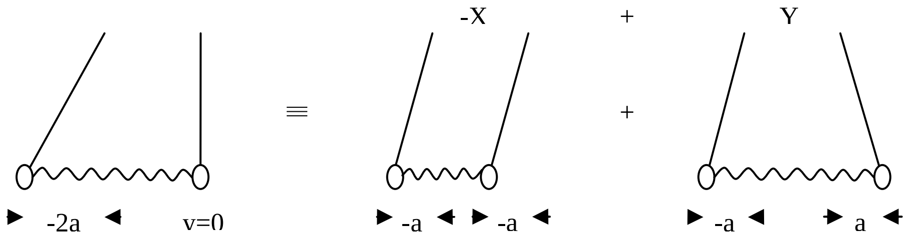
$$E = (E_{kin} + E_{pot})_x + (E_{kin} + E_{pot})_y + (E_{pot})_{xy}$$

where, $(E_{kin} + E_{pot})_x = \frac{1}{2}m\dot{x}^2 + \left(\frac{mg}{2l} + s\right)x^2$, $(E_{kin} + E_{pot})_y = \frac{1}{2}m\dot{y}^2 + \left(\frac{mg}{2l} + s\right)y^2$,

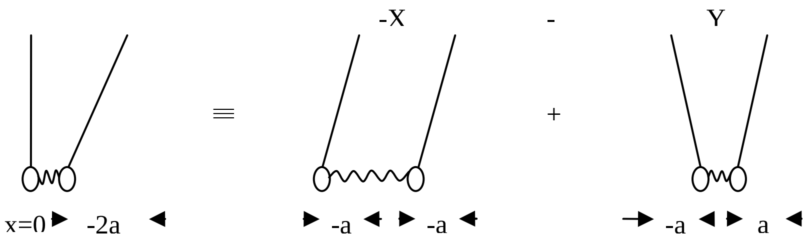
and $E_{pot} = -2sxy$

4.3

$x = -2a$, $y = 0$:



$x = 0$, $y = -2a$:



4.4

For mass m_1, Newton's second law gives:

$$m_1\ddot{x}_1 = sx$$

For mass m_2, Newton's second law gives:

$$m_2\ddot{x}_2 = -sx$$

Provided x is the extension of the spring and l is the natural length of the spring, we have:

$$x_2 - x_1 = l + x$$

By elimination of x_1 and x_2, we have:

$$-\frac{s}{m_2}x - \frac{s}{m_1}x = \ddot{x}$$

i.e.

$$\ddot{x} + \frac{m_1 + m_2}{m_1 m_2} sx = 0$$

which shows the system oscillate at a frequency:

$$\omega^2 = \frac{s}{\mu}$$

where,

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

For a sodium chloride molecule the interatomic force constant s is given by:

$$s = \omega^2 \mu = \frac{(2\pi\nu)^2 m_{Na} m_{Cl}}{m_{Na} + m_{Cl}} = \frac{4\pi^2 \times (1.14 \times 10^{13})^2 \times (23 \times 35) \times (1.67 \times 10^{-27})^2}{(23 + 35) \times 1.67 \times 10^{-27}} \approx 120[Nm^{-1}]$$

4.5

If the upper mass oscillate with a displacement of x and the lower mass oscillate with a displacement of y, the equations of motion of the two masses are given by Newton's second law as:

$$m\ddot{x} = s(y - x) - sx$$
$$m\ddot{y} = s(x - y)$$

i.e.

$$m\ddot{x} + s(x - y) - sx = 0$$
$$m\ddot{y} - s(x - y) = 0$$

Suppose the system starts from rest and oscillates in only one of its normal modes of frequency ω, we may assume the solutions:

$$x = Ae^{i\omega t}$$
$$y = Be^{i\omega t}$$

where A and B are the displacement amplitude of x and y at frequency ω.

Using these solutions, the equations of motion become:

$$[-m\omega^2 A + s(A-B) + sA]e^{i\omega t} = 0$$
$$[-m\omega^2 B - s(A-B)]e^{i\omega t} = 0$$

We may, by dividing through by $me^{i\omega t}$, rewrite the above equations in matrix form as:

$$\begin{bmatrix} 2s/m-\omega^2 & -s/m \\ -s/m & s/m-\omega^2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = 0 \qquad (4.5.1)$$

which has a non-zero solution if and only if the determinant of the matrix vanishes; that is, if

$$(2s/m-\omega^2)(s/m-\omega^2) - s^2/m^2 = 0$$

i.e. $$\omega^4 - (3s/m)\,\omega^2 + s^2/m^2 = 0$$

i.e. $$\omega^2 = (3 \pm \sqrt{5})\frac{s}{2m}$$

In the slower mode, $\omega^2 = (3-\sqrt{5})\,s/2m$. By substitution of the value of frequency into equation (4.5.1), we have:

$$\frac{A}{B} = \frac{s}{2s-m\omega^2} = \frac{s-m\omega^2}{s} = \frac{\sqrt{5}-1}{2}$$

which is the ratio of the amplitude of the upper mass to that of the lower mass.

Similarly, in the slower mode, $\omega^2 = (3+\sqrt{5})\,s/2m$. By substitution of the value of frequency into equation (4.5.1), we have:

$$\frac{A}{B} = \frac{s}{2s-m\omega^2} = \frac{s-m\omega^2}{s} = -\frac{\sqrt{5}+1}{2}$$

4.6

The motions of the two pendulums in Figure 4.3 are given by:

$$x = 2a\cos\frac{(\omega_2-\omega_1)t}{2}\cos\frac{(\omega_1+\omega_2)t}{2} = 2a\cos\omega_m t\cos\omega_a t$$
$$y = 2a\sin\frac{(\omega_2-\omega_1)t}{2}\sin\frac{(\omega_1+\omega_2)t}{2} = 2a\sin\omega_m t\sin\omega_a t$$

where, the amplitude of the two masses, $2a\cos\omega_m t$ and $2a\sin\omega_m t$, are constants over one cycle at the frequency ω_a.

Supposing the spring is very weak, the stiffness of the spring is ignorable, i.e. $s \approx 0$.

Noting that $\omega_1^2 = g/l$ and $\omega_2^2 = (g/l + 2s/m)$, we have:

$$\frac{g}{l} = \omega_1^2 \approx \omega_2^2 \approx \left(\frac{\omega_1 + \omega_2}{2}\right)^2 = \omega_a^2$$

Hence, the energies of the masses are given by:

$$E_x = \frac{1}{2}s_x a_x^2 = \frac{1}{2}\frac{mg}{l}(2a\cos\omega_m t)^2 = 2ma^2\omega_a^2\cos^2\omega_m t$$
$$E_y = \frac{1}{2}s_y a_y^2 = \frac{1}{2}\frac{mg}{l}(2a\sin\omega_m t)^2 = 2ma^2\omega_a^2\sin^2\omega_m t$$

The total energy is given by:

$$E = E_x + E_y = 2ma^2\omega_a^2(\cos^2\omega_m t + \sin^2\omega_m t) = 2ma^2\omega_a^2$$

Noting that $\omega_m = (\omega_2 - \omega_1)/2$, we have:

$$E_x = 2ma^2\omega_a^2\cos^2(\omega_2 - \omega_1)t = \frac{E}{2}[1 + \cos(\omega_2 - \omega_1)t]$$
$$E_y = 2ma^2\omega_a^2\sin^2(\omega_2 - \omega_1)t = \frac{E}{2}[1 - \cos(\omega_2 - \omega_1)t]$$

which show that the constant energy E is completed exchanged between the two pendulums at the beat frequency $(\omega_2 - \omega_1)$.

4.7

By adding up the two equations of motion, we have:

$$m_1\ddot{x} + m_2\ddot{y} = -(m_1 x + m_2 y)(g/l)$$

By multiplying the equation by $1/(m_1 + m_2)$ on both sides, we have:

$$\frac{m_1\ddot{x} + m_2\ddot{y}}{m_1 + m_2} = -\frac{g}{l}\frac{m_1 x + m_2 y}{m_1 + m_2}$$

i.e. $$\frac{m_1\ddot{x} + m_2\ddot{y}}{m_1 + m_2} + \frac{g}{l}\frac{m_1 x + m_2 y}{m_1 + m_2} = 0$$

which can be written as:

$$\ddot{X} + \omega_1^2 X = 0 \qquad (4.7.1)$$

where,

$$X = \frac{m_1 x + m_2 y}{m_1 + m_2} \quad \text{and} \quad \omega_1^2 = g/l$$

On the other hand, the equations of motion can be written as:

$$\ddot{x} = -\frac{g}{l}x - \frac{s}{m_1}(x-y)$$

$$\ddot{y} = -\frac{g}{l}y + \frac{s}{m_2}(x-y)$$

By subtracting the above equations, we have:

$$\ddot{x} - \ddot{y} = -\frac{g}{l}(x-y) - \left(\frac{s}{m_1} + \frac{s}{m_2}\right)(x-y)$$

i.e.
$$\ddot{x} - \ddot{y} + \left[\frac{g}{l} + s\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\right](x-y) = 0$$

which can be written as:

$$\ddot{Y} + \omega_2^2 Y = 0 \qquad (4.7.2)$$

where,

$$Y = x - y \quad \text{and} \quad \omega_2^2 = \frac{g}{l} + s\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$$

Equations (4.7.1) and (4.7.2) take the form of linear differential equations with constant coefficients and each equation contains only one dependant variable, therefore X and Y are normal coordinates and their normal frequencies are given by ω_1 and ω_2 respectively.

4.8

Since the initial condition gives $\dot{x} = \dot{y} = 0$, we may write, in normal coordinate, the solutions to the equations of motion of Problem 4.7 as:

$$X = X_0 \cos\omega_1 t$$

$$Y = Y_0 \cos\omega_2 t$$

i.e.

$$\frac{m_1 x + m_2 y}{m_1 + m_2} = X_0 \cos\omega_1 t$$

$$x - y = Y_0 \cos\omega_2 t$$

By substitution of initial conditions: $t = 0$, $x = A$ and $y = 0$ into the above equations, we have:

$$X_0 = (m_1/M)A$$

$$Y_0 = A$$

where, $$M = m_1 + m_2$$

so the equations of motion in original coordinates x, y are given by:

$$\frac{m_1 x + m_2 y}{m_1 + m_2} = \frac{m_1}{M} A\cos\omega_1 t$$
$$x - y = A\cos\omega_2 t$$

The solutions to the above equations are given by:

$$x = \frac{A}{M}(m_1 \cos\omega_1 t + m_2 \cos\omega_2 t)$$
$$y = A\frac{m_1}{M}(\cos\omega_1 t - \cos\omega_2 t)$$

Noting that $\omega_1 = \omega_a - \omega_m$ and $\omega_2 = \omega_a + \omega_m$, where $\omega_m = (\omega_2 - \omega_1)/2$ and $\omega_a = (\omega_1 + \omega_2)/2$, the above equations can be rearranged as:

$$\begin{aligned}
x &= \frac{A}{M}[m_1 \cos(\omega_a - \omega_m)t + m_2 \cos(\omega_a + \omega_m)t] \\
&= \frac{A}{M}[m_1(\cos\omega_m t\cos\omega_a t + \sin\omega_m t\sin\omega_a t) + m_2(\cos\omega_m t\cos\omega_a t - \sin\omega_m t\sin\omega_a t)] \\
&= \frac{A}{M}[(m_1 + m_2)\cos\omega_m t\cos\omega_a t + (m_1 - m_2)\sin\omega_m t\sin\omega_a t] \\
&= A\cos\omega_m t\cos\omega_a t + \frac{A}{M}(m_1 - m_2)\sin\omega_m t\sin\omega_a t
\end{aligned}$$

and

$$\begin{aligned}
y &= A\frac{m_1}{M}[\cos(\omega_a - \omega_m)t - \cos(\omega_a + \omega_m)t] \\
&= 2A\frac{m_1}{M}\sin\omega_m t\sin\omega_a t
\end{aligned}$$

4.9

From the analysis in Problem 4.6, we know, at weak coupling conditions, $\cos\omega_m t$ and $\sin\omega_m t$ are constants over one cycle, and the relation: $\omega_a \approx g/l$, so the energy of the mass m_1, E_x, and the energy of the mass m_2, E_y, are the sums of their separate kinetic and potential energies, i.e.:

$$E_x = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}s_x x^2 = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}\frac{mg}{l}x^2 = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_1\omega_a^2 x^2$$
$$E_y = \frac{1}{2}m_{21}\dot{y}^2 + \frac{1}{2}s_y y^2 = \frac{1}{2}m_2\dot{y}^2 + \frac{1}{2}\frac{mg}{l}y^2 = \frac{1}{2}m_1\dot{y}^2 + \frac{1}{2}m_1\omega_a^2 y^2$$

By substitution of the expressions of x and y in terms of $\cos\omega_a t$ and $\sin\omega_a t$ given by Problem 4.8 into the above equations, we have:

$$E_x = \frac{1}{2}m_1\left[-A\omega_a\cos\omega_m t\sin\omega_a t + \frac{A}{M}\omega_a(m_1 - m_2)\sin\omega_m t\cos\omega_a t\right]^2 +$$

$$\frac{1}{2}m_1\omega_a^2\left[A\cos\omega_m t\cos\omega_a t + \frac{A}{M}(m_1 - m_2)\sin\omega_m t\sin\omega_a t\right]^2$$

$$= \frac{1}{2}m_1\omega_a^2\frac{A^2}{M^2}\left[(m_1+m_2)^2\cos^2\omega_m t + (m_1-m_2)^2\sin^2\omega_m t\right]$$

$$= \frac{1}{2}m_1\omega_a^2\frac{A^2}{M^2}\left[m_1^2 + m_2^2 + 2m_1m_2(\cos^2\omega_m t - \sin^2\omega_m t)\right]$$

$$= \frac{1}{2}m_1\omega_a^2\frac{A^2}{M^2}\left[m_1^2 + m_2^2 + 2m_1m_2\cos 2\omega_m t\right]$$

$$= \frac{E}{M^2}\left[m_1^2 + m_2^2 + 2m_1m_2\cos(\omega_2 - \omega_1)t\right]$$

and

$$E_y = \frac{1}{2}m_2\left[2A\frac{m_1}{M}\omega_a\sin\omega_m t\cos\omega_a t\right]^2 + \frac{1}{2}m_1\omega_a^2\left[2A\frac{m_1}{M}\sin\omega_m t\sin\omega_a t\right]^2$$

$$= 2m_1^2m_2\omega_a^2\frac{A^2}{M^2}\left[\sin^2\omega_m t(\cos^2\omega_a t + \sin^2\omega_a t)\right]$$

$$= 2m_1^2m_2\omega_a^2\frac{A^2}{M^2}\sin^2\omega_m t$$

$$= \left(\frac{1}{2}m_1\omega_a^2A^2\right)\left(\frac{2m_1m_2}{M^2}\right)\left[1 - \cos 2\omega_m t\right]$$

$$= E\left(\frac{2m_1m_2}{M^2}\right)\left[1 - \cos(\omega_2 - \omega_1)t\right]$$

where,

$$E = \frac{1}{2}m_1\omega_a^2A^2$$

4.10

Add up the two equations and we have:

$$m(\ddot{x} + \ddot{y}) + \frac{mg}{l}(x + y) + r(\dot{x} + \dot{y}) = F_0\cos\omega t$$

i.e.

$$m\ddot{X} + r\dot{X} + \frac{mg}{l}X = F_0\cos\omega t \qquad (4.10.1)$$

Subtract the two equations and we have:

$$m(\ddot{x} - \ddot{y}) + \frac{mg}{l}(x - y) + r(\dot{x} - \dot{y}) + 2s(x - y) = F_0\cos\omega t$$

i.e. $$m\ddot{Y} + r\dot{Y} + \left(\frac{mg}{l} + 2s\right)Y = F_0 \cos\omega t \qquad (4.10.2)$$

Equations (4.10.1) and (4.10.2) shows that the normal coordinates X and Y are those for damped oscillators driven by a force $F_0 \cos\omega t$.

By neglecting the effect of r, equation of (4.10.1) and (4.10.2) become:

$$m\ddot{X} + \frac{mg}{l}X \approx F_0 \cos\omega t$$

$$m\ddot{Y} + \left(\frac{mg}{l} + 2s\right)Y \approx F_0 \cos\omega t$$

Suppose the above equations have solutions: $X = X_0 \cos\omega t$ and $Y = Y_0 \cos\omega t$, by substitution of the solutions to the above equations, we have:

$$\left(-m\omega^2 + \frac{mg}{l}\right)X_0 \cos\omega t \approx F_0 \cos\omega t$$

$$\left(-m\omega^2 + \frac{mg}{l} + 2s\right)Y_0 \cos\omega t \approx F_0 \cos\omega t$$

These equations satisfy any t if

$$\left(-m\omega^2 + \frac{mg}{l}\right)X_0 \approx F_0$$

$$\left(-m\omega^2 + \frac{mg}{l} + 2s\right)Y_0 \approx F_0$$

i.e.

$$X_0 \approx \frac{F_0}{m(g/l - \omega^2)}$$

$$Y_0 \approx \frac{F_0}{m(g/l + 2s/m - \omega^2)}$$

so the expressions of X and Y are given by:

$$X = x + y \approx \frac{F_0}{m(g/l - \omega^2)}\cos\omega t$$

$$Y = x - y \approx \frac{F_0}{m(g/l + 2s/m - \omega^2)}\cos\omega t$$

By solving the above equations, the expressions of x and y are given by:

$$x \approx \frac{F_0}{2m}\cos\,\omega t\left[\frac{1}{\omega_1^2-\omega^2}+\frac{1}{\omega_2^2-\omega^2}\right]$$

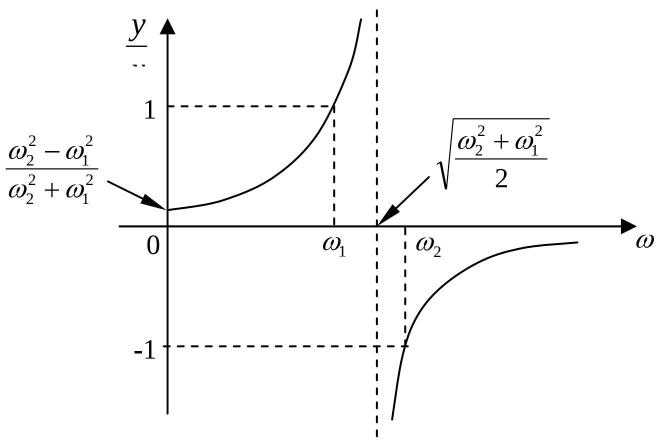
$$y \approx \frac{F_0}{2m}\cos\,\omega t\left[\frac{1}{\omega_1^2-\omega^2}-\frac{1}{\omega_2^2-\omega^2}\right]$$

where,

$$\omega_1^2=\frac{g}{l} \qquad \text{and} \qquad \omega_2^2=\frac{g}{l}+\frac{2s}{m}$$

The ratio of y/x is given by:

$$\frac{y}{x}\approx\frac{\frac{F_0}{2m}\cos\omega t\left[\frac{1}{\omega_1^2-\omega^2}+\frac{1}{\omega_2^2-\omega^2}\right]}{\frac{F_0}{2m}\cos\omega t\left[\frac{1}{\omega_1^2-\omega^2}-\frac{1}{\omega_2^2-\omega^2}\right]}=\frac{\frac{1}{\omega_1^2-\omega^2}+\frac{1}{\omega_2^2-\omega^2}}{\frac{1}{\omega_1^2-\omega^2}-\frac{1}{\omega_2^2-\omega^2}}=\frac{\omega_2^2-\omega_1^2}{\omega_2^2+\omega_1^2-2\omega^2}$$



The behaviour of y/x as a function of frequency ω is shown as the figure below:

The figure shows $|y/x|$ is less than 1 if $\omega<\omega_1$ or $\omega>\omega_2$, i.e. outside frequency range $\omega_2-\omega_1$ the motion of y is attenuated.

4.11

Suppose the displacement of mass M is x, the displacement of mass m is y, and the tension of the spring is T. Equations of motion give:

$$M\ddot{x}+kx=F_0\cos\omega t+T \qquad (4.11.1)$$

$$m\ddot{y}=-T \qquad (4.11.2)$$

$$s(y-x)=T \qquad (4.11.3)$$

Eliminating T, we have:

$$M\ddot{x} + kx = F_0 \cos \omega t + s(y - x)$$

so for $x = 0$ at all times, we have

$$F_0 \cos \omega t + sy = 0$$

that is

$$y = -\frac{F_0}{s} \cos \omega t$$

Equation (4.11.2) and (4.11.3) now give:

$$m\ddot{y} + sy = 0$$

with $\omega^2 = s/m$, so M is stationary at $\omega^2 = s/m$.

This value of ω satisfies all equations of motion for $x = 0$ including $T = -F_0 \cos \omega t$

4.12

Noting the relation: $V = q/C$, the voltage equations can be written as:

$$\frac{q_1}{C} - \frac{q_2}{C} = L\frac{dI_a}{dt}$$
$$\frac{q_2}{C} - \frac{q_3}{C} = L\frac{dI_b}{dt}$$

so we have:

$$\dot{q}_1 - \dot{q}_2 = LC\ddot{I}_a$$
$$\dot{q}_2 - \dot{q}_3 = LC\ddot{I}_b$$

i.e.

$$\dot{q}_1 - \dot{q}_2 = LC\ddot{I}_a$$
$$\dot{q}_2 - \dot{q}_3 = LC\ddot{I}_b$$

By substitution of $\dot{q}_1 = -I_a$, $\dot{q}_2 = I_a - I_b$ and $\dot{q}_3 = I_b$ into the above equations, we have:

$$-I_a - I_a + I_b = LC\ddot{I}_a$$
$$I_a - I_b - I_b = LC\ddot{I}_b$$

i.e.

$$LC\ddot{I}_a + 2I_a - I_b = 0$$
$$LC\ddot{I}_b - I_a + 2I_b = 0$$

By adding up and subtracting the above equations, we have:

$$LC(\ddot{I}_a + \ddot{I}_b) + I_a + I_b = 0$$
$$LC(\ddot{I}_a - \ddot{I}_b) + 3(I_a - I_b) = 0$$

Supposing the solutions to the above normal modes equations are given by:

$$I_a + I_b = A\cos\omega t$$

$$I_a - I_b = B\cos\omega t$$

so we have:

$$(-A\omega^2 LC + A)\cos\omega t = 0$$
$$(-B\omega^2 LC + 3B)\cos\omega t = 0$$

which are true for all t when

$$\omega^2 = \frac{1}{LC} \quad \text{and} \quad B = 0$$

or

$$\omega^2 = \frac{3}{LC} \quad \text{and} \quad A = 0$$

which show that the normal modes of oscillation are given by:

$$I_a = I_b \;\text{ at }\; \omega_1^2 = \frac{1}{LC}$$

and

$$I_a = -I_b \;\text{ at }\; \omega_2^2 = \frac{3}{LC}$$

4.13

From the given equations, we have the relation between I_1 and I_2 given by:

$$I_2 = \frac{i\omega M}{Z_2 + i\omega L_s} I_1$$

so:

$$E = i\omega L_p I_1 - i\omega M I_2 = \left(i\omega L_p + \frac{\omega^2 M}{Z_2 + i\omega L_s} \right) I_1$$

i.e.

$$\frac{E}{I_1} = i\omega L_p + \frac{\omega^2 M^2}{Z_2 + i\omega L_s}$$

which shows that E/I_1, the impedance of the whole system seen by the generator, is the sum of the primary impedance, $i\omega L_p$, and a ‘reflected impedance’ from the secondary circuit of $\omega^2 M^2 / Z_s$, where $Z_s = Z_2 + i\omega L_s$.

4.14

Problem 4.13 shows the impedance seen by the generator Z is given by:

$$Z = i\omega L_p + \frac{\omega^2 M^2}{Z_2 + i\omega L_s}$$

Noting that $M = \sqrt{L_p L_s}$ and $L_p / L_s = n_p^2 / n_s^2$, the impedance can be written as:

$$Z = \frac{i\omega L_p Z_2 - \omega^2 L_p L_s + \omega^2 M^2}{Z_2 + i\omega L_s} = \frac{i\omega L_p Z_2 - \omega^2 M^2 + \omega^2 M^2}{Z_2 + i\omega L_s} = \frac{i\omega L_p Z_2}{Z_2 + i\omega L_s}$$

so we have:

$$\frac{1}{Z} = \frac{Z_2 + i\omega L_s}{i\omega L_p Z_2} == \frac{1}{i\omega L_p} + \frac{1}{\frac{L_p}{L_s} Z_2} = \frac{1}{i\omega L_p} + \frac{1}{\frac{n_p^2}{n_s^2} Z_2}$$

which shows the impedance Z is equivalent to the primary impedance $i\omega L_p$ connected in parallel with an impedance $(n_p / n_s)^2 Z_2$.

4.15

Suppose a generator with the internal impedance of Z_1 is connected with a load with an impedance of Z_2 via an ideal transformer with a primary inductance of L_p and the ratio of the number of primary and secondary transformer coil turns given by n_p / n_s, and the whole circuit oscillate at a frequency of ω. From the analysis in Problem 4.13, the impedance of the load is given by:

$$\frac{1}{Z_L} = \frac{1}{i\omega L_p} + \frac{1}{\frac{n_p^2}{n_s^2} Z_2}$$

At the maximum output power: $Z_L = Z_1$, i.e.:

$$\frac{1}{Z_L} = \frac{1}{i\omega L_p} + \frac{1}{\frac{n_p^2}{n_s^2} Z_2} = \frac{1}{Z_1}$$

which is the relation used for matching a load to a generator.

4.16

From the second equation, we have:

$$I_1 = -\frac{Z_2}{Z_M} I_2$$

By substitution into the first equation, we have:

$$-\frac{Z_1 Z_2}{Z_M} I_2 + Z_M I_2 = E$$

i.e.

$$I_2 = \frac{E}{\left(Z_M - \dfrac{Z_1 Z_2}{Z_M} \right)} I_1$$

Noting that $Z_M = i\omega M$ and I_2 has the maximum value when $X_1 = X_2 = 0$, i.e. $Z_1 = R_1$ and $Z_2 = R_2$, we have:

$$|I_2| = \frac{E}{\left| i\omega M - \dfrac{R_1 R_2}{i\omega M} \right|} |I_1| = \frac{E}{\omega M + \dfrac{R_1 R_2}{\omega M}} |I_1| \le \frac{E}{2\sqrt{\omega M \dfrac{R_1 R_2}{\omega M}}} |I_1| = \frac{E}{2\sqrt{R_1 R_2}} |I_1|$$

which shows $|I_2|$ has the maximum value of $\dfrac{E}{2\sqrt{R_1 R_2}} |I_1|$, when $\omega M = \dfrac{R_1 R_2}{\omega M}$, i.e. $\omega M = \sqrt{R_1 R_2}$

4.17

By substitution of $j = 1$ and $n = 3$ into equation (4.15), we have:

$$\omega_1^2 = 2\omega_0^2 \left[1 - \cos\frac{\pi}{4} \right] = 2\omega_0^2 \left[1 - \frac{\sqrt{2}}{2} \right] = (2 - \sqrt{2})\omega_0^2$$

By substitution of $j = 2$ and $n = 3$ into equation (4.15), we have:

$$\omega_1^2 = 2\omega_0^2 \left[1 - \cos\frac{2\pi}{4} \right] = 2\omega_0^2$$

By substitution of $j = 3$ and $n = 3$ into equation (4.15), we have:

$$\omega_1^2 = 2\omega_0^2 \left[1 - \cos\frac{3\pi}{4} \right] = 2\omega_0^2 \left[1 + \frac{\sqrt{2}}{2} \right] = (2 + \sqrt{2})\omega_0^2$$

In equation (4.14), we have $A_0 = A_4 = 0$ when $n = 3$, and noting that $\omega_0^2 = T/ma$, equation (4.14) gives:

when $r = 1$:
$$-A_0 + \left(2 - \frac{\omega^2}{\omega_0^2}\right)A_1 - A_2 = 0$$

i.e.
$$\left(2 - \frac{\omega^2}{\omega_0^2}\right)A_1 - A_2 = 0 \qquad (4.17.1)$$

when $r = 2$:
$$-A_1 + \left(2 - \frac{\omega^2}{\omega_0^2}\right)A_2 - A_3 = 0 \qquad (4.17.2)$$

when $r = 3$:
$$-A_2 + \left(2 - \frac{\omega^2}{\omega_0^2}\right)A_3 - A_4 = 0$$

i.e.
$$-A_2 + \left(2 - \frac{\omega^2}{\omega_0^2}\right)A_3 = 0 \qquad (4.17.3)$$

Write the above equations in matrix format, we have:

$$\begin{pmatrix} 2-\omega^2/\omega_0^2 & -1 & 0 \\ -1 & 2-\omega^2/\omega_0^2 & -1 \\ 0 & -1 & 2-\omega^2/\omega_0^2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$$

which has non zero solutions provided the determinant of the matrix is zero, i.e.:

$$(2-\omega^2/\omega_0^2)^3 - 2(2-\omega^2/\omega_0^2) = 0$$

The solutions to the above equations are given by:

$$\omega_1^2 = (2-\sqrt{2})\omega_0^2 ,` \qquad \omega_2^2 = 2\omega_0^2 , \quad \text{and} \qquad \omega_1^2 = (2+\sqrt{2})\omega_0^2$$

4.18

By substitution of $\omega_1^2 = (2-\sqrt{2})\omega_0^2$ into equation (4.17.1), we have:

$$\sqrt{2}A_1 - A_2 = 0 \quad \text{i.e.} \quad A_1 : A_2 = 1 : \sqrt{2}$$

By substitution of $\omega_1^2 = (2-\sqrt{2})\omega_0^2$ into equation (4.17.3), we have:

$$-A_2 + \sqrt{2}A_3 = 0 \ \text{ i.e.} \quad A_2 : A_3 = \sqrt{2} : 1$$

Hence, when $\omega_1^2 = (2-\sqrt{2})\omega_0^2$, the relative displacements are given by:

$$A_1 : A_2 : A_3 = 1 : \sqrt{2} : 1$$

By substitution of $\omega_2^2 = 2\omega_0^2$ into equation (4.17.1), we have:

$$A_2 = 0$$

By substitution of $\omega_2^2 = 2\omega_0^2$ into equation (4.17.2), we have:

$$-A_1 + A_3 = 0 \qquad \text{i.e.} \qquad A_1 : A_3 = 1 : -1$$

Hence, when $\omega_2^2 = 2\omega_0^2$, the relative displacements are given by:

$$A_1 : A_2 : A_3 = 1 : 0 : -1$$

By substitution of $\omega_2^2 = (2+\sqrt{2})\omega_0^2$ into equation (4.17.1), we have:

$$-\sqrt{2}A_1 - A_2 = 0 \;\; \text{i.e.} \qquad A_1 : A_2 = 1 : -\sqrt{2}$$

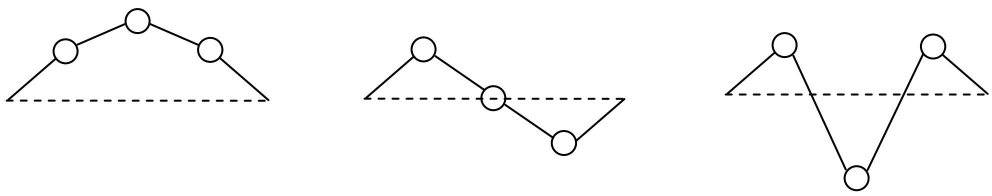
By substitution of $\omega_1^2 = (2+\sqrt{2})\omega_0^2$ into equation (4.17.3), we have:

$$-A_2 - \sqrt{2}A_3 = 0 \;\; \text{i.e.} \qquad A_2 : A_3 = -\sqrt{2} : 1$$

Hence, when $\omega_1^2 = (2+\sqrt{2})\omega_0^2$, the relative displacements are given by:

$$A_1 : A_2 : A_3 = 1 : -\sqrt{2} : 1$$

The relative displacements of the three masses at different normal frequencies are shown below:



$\omega^2 = (2-\sqrt{2})\omega_0^2$ $\qquad$ $\omega^2 = 2\omega_0^2$ $\qquad$ $\omega^2 = (2+\sqrt{2})\omega_0^2$

As we can see from the above figures that tighter coupling corresponds to higher frequency.

4.19

Suppose the displacement of the left mass m is x, and that of the central mass M is y, and that of the right mass m is z. The equations of motion are given by:

$$m\ddot{x} = s(y-x)$$
$$M\ddot{y} = -s(y-x) + s(z-y)$$
$$m\ddot{z} = -s(z-y)$$

If the system has a normal frequency of ω, and the displacements of the three masses can by written as:

$$x = \eta_1 e^{i\omega t}$$
$$y = \eta_2 e^{i\omega t}$$
$$z = \eta_3 e^{i\omega t}$$

By substitution of the expressions of displacements into the above equations of motion, we have:

$$-m\omega^2\eta_1 e^{i\omega t} = s(\eta_2 - \eta_1)e^{i\omega t}$$
$$-M\omega^2\eta_2 e^{i\omega t} = -s(\eta_2 - \eta_1)e^{i\omega t} + s(\eta_3 - \eta_2)e^{i\omega t}$$
$$-m\omega^2\eta_3 e^{i\omega t} = -s(\eta_3 - \eta_2)e^{i\omega t}$$

i.e.

$$[(s - m\omega^2)\eta_1 - s\eta_2]e^{i\omega t} = 0$$
$$[-s\eta_1 + (2s - M\omega^2)\eta_2 - s\eta_3]e^{i\omega t} = 0$$
$$[-s\eta_2 + (s - m\omega^2)\eta_3]e^{i\omega t} = 0$$

which is true for all t if

$$(s - m\omega^2)\eta_1 - s\eta_2 = 0$$
$$-s\eta_1 + (2s - M\omega^2)\eta_2 - s\eta_3 = 0$$
$$-s\eta_2 + (s - m\omega^2)\eta_3 = 0$$

The matrix format of these equations is given by:

$$\begin{pmatrix} s - m\omega^2 & -s & 0 \\ -s & 2s - M\omega^2 & -s \\ 0 & -s & s - m\omega^2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = 0$$

which has non zero solutions if and only if the determinant of the matrix is zero, i.e.:

$$(s - m\omega^2)^2(2s - M\omega^2) - 2s^2(s - m\omega^2) = 0$$

i.e. $$(s - m\omega^2)[(s - m\omega^2)(2s - M\omega^2) - 2s^2] = 0$$

i.e. $$(s - m\omega^2)[mM\omega^4 - s(M + 2m)\omega^2] = 0$$

i.e. $$\omega^2(s - m\omega^2)[mM\omega^2 - s(M + 2m)] = 0$$

The solutions to the above equation, i.e. the frequencies of the normal modes, are given by:

$$\omega_1^2 = 0\,, \quad \omega_2^2 = \frac{s}{m} \quad \text{and} \quad \omega_3^2 = \frac{s(M+2m)}{mM}$$

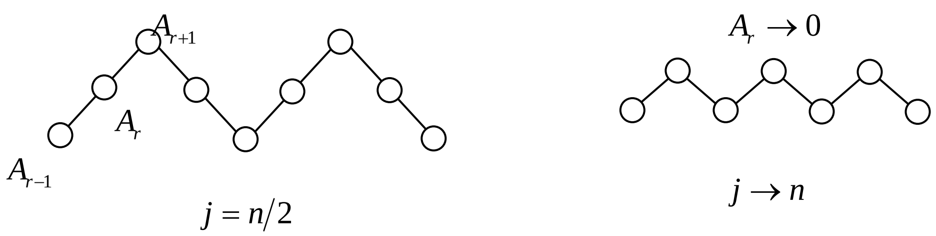
At the normal mode of $\omega^2 = 0$, all the atoms are stationary, $\eta_1 = \eta_2 = \eta_3$, i.e. all the masses has the same displacement;

At the normal mode of $\omega^2 = \frac{s}{m}$, $\eta_2 = 0$ and $\eta_1 = -\eta_3$, i.e. the mass M is stationary, and the two masses m have the same amplitude but are "anti-phase" with respect to each other;

At the normal mode of $\omega^2 = \frac{s(M+2m)}{mM}$, $\eta_1 : \eta_2 : \eta_3 = M : -2m : M$, i.e. the two mass m have the same amplitude and are "in-phase" with respect to each other. They are both "anti-phase" with respect to the mass M. The ratio of amplitude between the mass m and M is $M/2m$.

4.20

In understanding the motion of the masses it is more instructive to consider the range $n/2 \le j \le n$. For each value of the frequency ω_j the amplitude of the r^{th} mass is $A_r = C \sin \frac{rj\pi}{n+1}$ where C is a constant. For $j = n/2$ adjacent masses have a $\pi/2$ phase difference, so the ratios: $A_{r-1} : A_r : A_{r+1} = -1 : 0 : 1$, with the r^{th} masses stationary and the amplitude A_{r-1} anti-phase with respect to A_{r+1}, so that:



As $n/2 \to n$, A_r begins to move, the coupling between masses tightens and when j is close to n each mass is anti-phase with respect to its neighbour, the amplitude of each mass decreases until in the limit $j = n$ no motion is transmitted as the cut off frequency $\omega_j^2 = 4T/ma$ is reached. The end points are fixed and this restricts the motion of the masses near the end points at all frequencies except the lowest.

4.21

By expansion of the expression of ω_j^2, we have:

$$\omega_j^2 = \frac{2T}{ma}\left(1-\cos\frac{j\pi}{n+1}\right) = \frac{2T}{ma}\left[\frac{(j\pi/n+1)^2}{2!} - \frac{(j\pi/n+1)^4}{4!} + \frac{(j\pi/n+1)^6}{6!} - \cdots\right]$$

If $n >> 1$ and $j << n$, $j\pi/n+1$ has a very small value, so the high order terms of the above equation can be neglected, so the above equation become:

$$\omega_j^2 = \frac{2T}{ma}\left[\frac{(j\pi/n+1)^2}{2!}\right] = \frac{T}{ma}\left(\frac{j\pi}{n+1}\right)^2$$

i.e.
$$\omega_j = \frac{j\pi}{n+1}\sqrt{\frac{T}{ma}}$$

which can be written as:

$$\omega_j = \frac{j\pi}{l}\sqrt{\frac{T}{\rho}}$$

where, $\rho = m/a$ and $l = (n+1)a$

4.22

From the first equation, we have:

$$L\ddot{I}_{r-1} = \frac{\dot{q}_{r-1} - \dot{q}_r}{C}$$

By substitution of $\dot{q}_r = I_{r-1} - I_r$ and $\dot{q}_{r-1} = I_{r-2} - I_{r-1}$ into the above equation, we have:

$$L\ddot{I}_{r-1} = \frac{I_{r-2} - 2I_{r-1} + I_r}{C} \qquad (4.22.1)$$

If, in the normal mode, the currents oscillate at a frequency ω, we may write the displacements as:

$$I_{r-2} = A_{r-2}e^{i\omega t}, \quad I_{r-1} = A_{r-1}e^{i\omega t} \quad \text{and} \quad I_r = A_r e^{i\omega t}$$

Using these values of I in equation (4.22.1) gives:

$$-\omega^2 L A_{r-1}e^{i\omega t} = \frac{A_{r-2} - 2A_{r-1} + A_r}{C}e^{i\omega t}$$

or

$$-A_{r-2} + (2 - LC\omega^2)A_{r-1} - A_r = 0 \qquad (4.22.2)$$

By comparison of equation (4.22.2) with equation (4.14) in text book, we may find the expression of I_r is the same as that of y_r in the case of mass-loaded string, i.e.

$$I_r = A_r e^{i\omega t} = D\sin\frac{rj\pi}{n+1}e^{i\omega t}$$

Where D is constant, and the frequency ω is given by:

$$\omega_j^2 = \frac{1}{LC}\left(1 - \cos\frac{j\pi}{n+1}\right)$$

where, $j = 1,2,3...n$

4.23

By substitution of y into $\frac{\partial^2 y}{\partial t^2}$, we have:

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2}{\partial t^2}(e^{i\omega t}e^{ikx}) = -\omega^2 e^{i(\omega t + kx)}$$

By substitution of y into $\frac{\partial^2 y}{\partial x^2}$, we have:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2}{\partial x^2}(e^{i\omega t}e^{ikx}) = -k^2 e^{i(\omega t + kx)}$$

If $\omega = ck$, we have:

$$\frac{\partial^2 y}{\partial t^2} - c^2\frac{\partial^2 y}{\partial x^2} = (-\omega^2 + c^2k^2)e^{i(\omega t + kx)} = 0$$

i.e.

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

SOLUTIONS TO CHAPTER 5

5.1

Write $u = ct + x$, and try $\frac{\partial^2 y}{\partial x^2}$ with $y = f_2(ct + x)$, we have:

$$\frac{\partial y}{\partial x} = \frac{\partial f_2(u)}{\partial u}, \text{ and } \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f_2(u)}{\partial u^2}$$

Try $\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$ with $y = f_2(ct + x)$, we have:

$$\frac{\partial y}{\partial t} = c\frac{\partial f_2(u)}{\partial u}, \text{ and } \frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 f_2(u)}{\partial u^2}$$

so:

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{1}{c^2}c^2\frac{\partial^2 f_2(u)}{\partial u^2} = \frac{\partial^2 f_2(u)}{\partial u^2}$$

Therefore:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

5.2

If $y = f_1(ct - x)$, the expression for y at a time $t + \Delta t$ and a position $x + \Delta x$, where $\Delta t = \Delta x / c$, is given by:

$$\begin{aligned} y_{t+\Delta t, x+\Delta x} &= f_1[c(t + \Delta t) - (x + \Delta x)] \\ &= f_1[c(t + \Delta x / c) - (x + \Delta x)] \\ &= f_1[ct + \Delta x - x - \Delta x] \\ &= f_1[ct - x] = y_{t,x} \end{aligned}$$

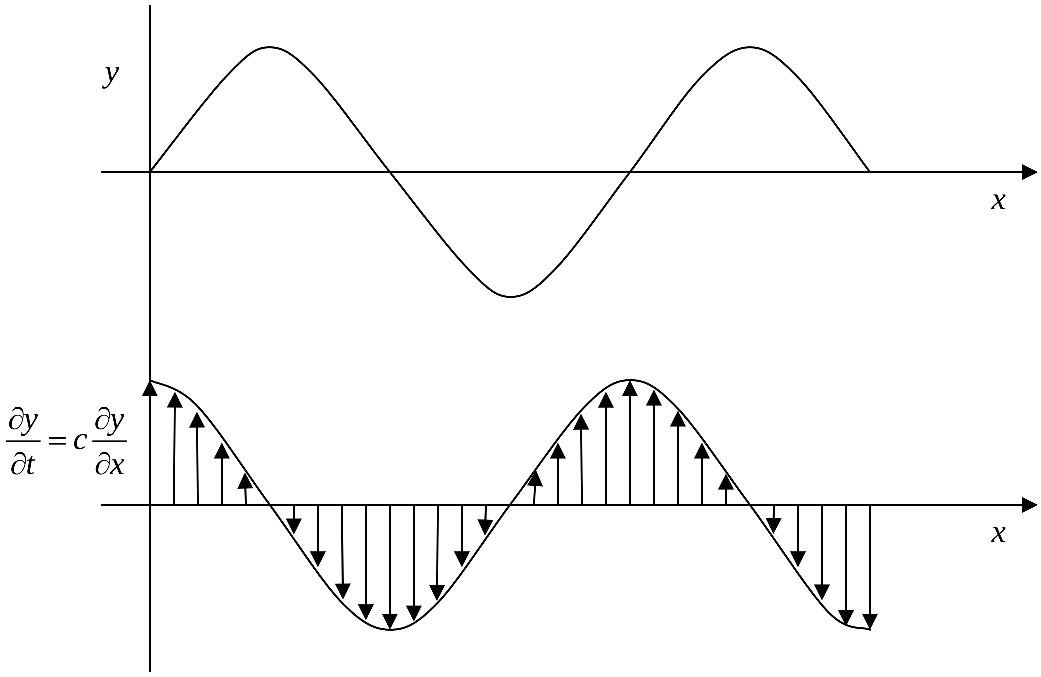
i.e. the wave profile remains unchanged.

If $y = f_2(ct + x)$, the expression for y at a time $t + \Delta t$ and a position $x + \Delta x$, where $\Delta t = -\Delta x / c$, is given by:

$$
\begin{aligned}
y_{t+\Delta t, x+\Delta x} &= f_1[c(t+\Delta t) + (x+\Delta x)] \\
&= f_1[c(t-\Delta x/c) + (x+\Delta x)] \\
&= f_1[ct - \Delta x + x + \Delta x] \\
&= f_1[ct + x] = y_{t,x}
\end{aligned}
$$

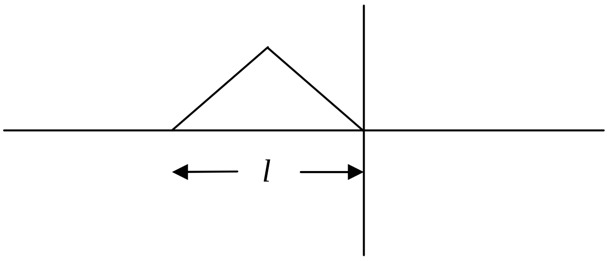
i.e. the wave profile also remains unchanged.

5.3



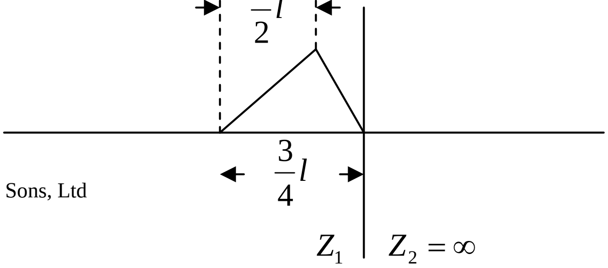
5.4

The pulse shape before reflection is given by the graph below:

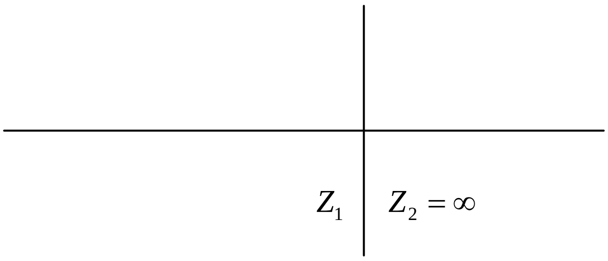


The pulse shapes after of a length of Δl of the pulse being reflected are shown below:

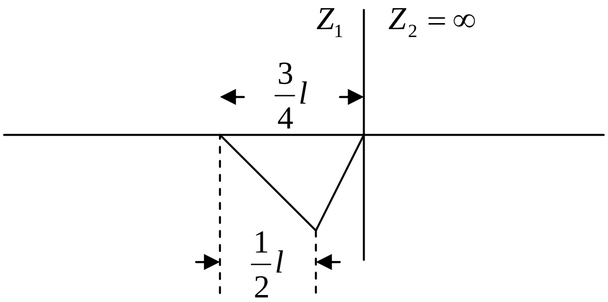
(a) $\Delta l = l/4$



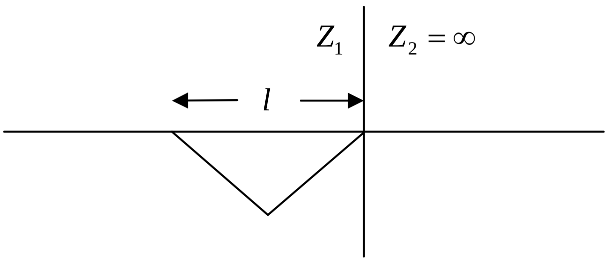
(b) $\Delta l = l/2$



(c) $\Delta l = 3l/4$



(d) $\Delta l = l$



5.5

The boundary condition $y_i + y_r = y_t$ gives:

$$A_1 e^{i(\omega t - kx)} + B_1 e^{i(\omega t + kx)} = A_2 e^{i(\omega t - kx)}$$

At $x = 0$, this equation gives:

$$A_1 + B_1 = A_2 \qquad (5.5.1)$$

The boundary condition $Ma = T\frac{\partial}{\partial x}y_t - T\frac{\partial}{\partial x}(y_i + y_r)$ gives:

$$Ma = -ikTA_2e^{i(\omega t - kx)} + ikTA_1e^{i(\omega t - kx)} - ikTB_1e^{i(\omega t + kx)}$$

At $x = 0$, $a = \ddot{y}_t = \ddot{y}_i + \ddot{y}_r$, so the above equation becomes:

$$-\omega^2 MA_2 = -i\omega\frac{T}{c}A_2 + i\omega\frac{T}{c}A_1 - i\omega\frac{T}{c}B_1$$

i.e. $$i\frac{T}{c}A_1 - i\frac{T}{c}B_1 = \left(-\omega M + i\frac{T}{c}\right)A_2$$

Noting that $T/c = \rho c$, the above equation becomes:

$$i\rho cA_1 - i\rho cB_1 = (-\omega M + i\rho c)A_2 \qquad (5.5.2)$$

By substitution of (5.5.1) into (5.5.2), we have:

$$i\rho cA_1 - i\rho cB_1 = (-\omega M + i\rho c)(A_1 + B_1)$$

i.e.

$$\frac{B_1}{A_1} = \frac{-iq}{1 + iq}$$

where $q = \omega M/2\rho c$

By substitution of the above equation into (5.5.1), we have:

$$A_1 - \frac{iq}{1 + iq}A_1 = A_2$$

i.e.

$$\frac{A_2}{A_1} = \frac{1}{1 + iq}$$

5.6

Writing $q = \tan\theta$, we have:

$$\frac{A_2}{A_1} = \frac{1}{1 + iq} = \frac{1}{1 + i\tan\theta} = \frac{\cos\theta}{\cos\theta + i\sin\theta} = \cos\theta e^{-i\theta}$$

and

$$\frac{B_1}{A_1} = \frac{-iq}{1 + iq} = \frac{-i\tan\theta}{1 + i\tan\theta} = \frac{-i\sin\theta}{\cos\theta + i\sin\theta} = \sin\theta e^{-i(\theta + \pi/2)}$$

which show that A_2 lags A_1 by θ and that B_1 lags A_1 by $(\pi/2 + \theta)$ for $0 < \theta < \pi/2$

The reflected energy coefficients are given by:

$$\left|\frac{B_1}{A_1}\right|^2 = \left|\sin\theta e^{-i(\theta+\pi/2)}\right|^2 = \sin^2\theta$$

and the transmitted energy coefficients are given by:

$$\left|\frac{A_2}{A_1}\right|^2 = \left|\cos\theta e^{-i\theta}\right|^2 = \cos^2\theta$$

5.7

Suppose T is the tension of the string, the average rate of working by the force over one period of oscillation on one-wavelength-long string is given by:

$$\overline{W} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\int_0^{1/k} -T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}dxdt$$

By substitution of $y = a\sin(\omega t - kx)$ into the above equation, we have:

$$\begin{aligned}\overline{W} &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\int_0^{1/k} -T[-ka\sin(\omega t - kx)][\omega a\sin(\omega t - kx)]dxdt \\ &= \frac{\omega^2 k^2 a^2 T}{2\pi}\int_0^{2\pi/\omega}\int_0^{1/k}\sin^2(\omega t - kx)dxdt \\ &= \frac{\omega^2 k^2 a^2 T}{2\pi}\int_0^{2\pi/\omega}\int_0^{1/k}\frac{1-\cos(2\omega t - 2kx)}{2}dxdt \\ &= \frac{\omega^2 k^2 a^2 T}{2\pi}\cdot\frac{1}{2}\cdot\frac{2\pi}{\omega}\cdot\frac{1}{k} \\ &= \frac{\omega k a^2 T}{2}\end{aligned}$$

Noting that $k = \omega/c$ and $T = \rho c^2$, the above equation becomes

$$\overline{W} = \frac{\omega^2 a^2 \rho c^2}{2c} = \frac{\omega^2 a^2 \rho c}{2}$$

which equals the rate of energy transfer along the string.

5.8

Suppose the wave equation is given by: $y = \sin(\omega t - kx)$. The maximum value of transverse harmonic force $F_{\max}$ is given by:

$$F_{\max} = T\left(\frac{\partial y}{\partial x}\right)_{\max} = T\left[\frac{\partial}{\partial x}A\sin(\omega t - kx)\right]_{\max} = TAk = \frac{TA\omega}{c}$$

i.e.

$$\frac{T}{c} = \frac{F_{\max}}{A\omega} = \frac{0.3}{0.1 \times 2\pi \times 5} = \frac{0.3}{\pi}$$

Noting that $\rho c = T/c$, the rate of energy transfer along the string is given by:

$$P = \frac{\rho c \omega^2 A^2}{2} = \frac{1}{2}\frac{T}{c}\omega^2 A^2 = \frac{1}{2} \times \frac{0.3}{\pi} \times (2\pi \times 5)^2 \times 0.1^2 = \frac{3\pi}{20}[W]$$

so the velocity of the wave c is given by:

$$c = \frac{2P}{\rho \omega^2 A^2} = \frac{2 \times 3\pi/20}{0.01 \times (2\pi \times 5)^2 \times 0.1^2} = \frac{30}{\pi}[ms^{-1}]$$

5.9

This problem is not viable in its present form and it will be revised in the next printing. The first part in the zero reflected amplitude may be solved by replacing Z_3 by Z_1, which then equates *r* with R' because each is a reflection at a Z_1Z_2 boundary. We then have the total reflected amplitude as:

$$R + tTR'(1 + R'^2 + R'^4 + \cdots) = R + \frac{tTR'}{1 - R'^2}$$

Stokes' relations show that the incident amplitude may be reconstructed by reversing the paths of the transmitted and reflected amplitudes.

T is transmitted back along the incident direction as tT in Z_1 and is reflected as TR' in Z_2.

R is reflected in Z_1 as $(R)R = R^2$ back along the incident direction and is refracted as TR in the TR' direction in Z_2.

We therefore have $tT + R^2 = 1$ in Z_1, i.e. $tT = 1 - R^2$ and $T(R + R') = 0$ in Z_2 giving $R = -R'$, $\therefore tT = 1 - R^2 = 1 - R'^2$ giving the total reflected amplitude in Z_1 as $R + R' = 0$ with $R = -R'$.

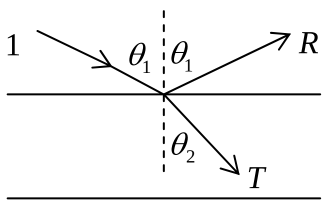


Fig Q.5.9(a)

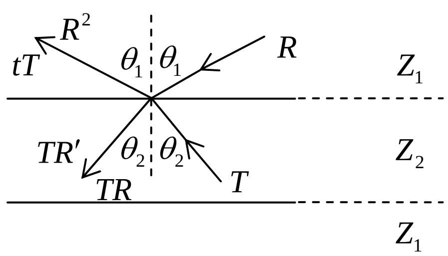


Fig Q.5.9(b)

Note that for zero total reflection in medium Z_1, the first reflection R is cancelled by the sum of all subsequent reflections.

5.10

The impedance of the anti-reflection coating Z_{coat} should have a relation to the impedance of air Z_{air} and the impedance of the lens Z_{lens} given by:

$$Z_{coat} = \sqrt{Z_{air} Z_{lens}} = \sqrt{\frac{1}{n_{air} n_{lens}}}$$

So the reflective index of the coating is given by:

$$n_{coat} = \frac{1}{Z_{coat}} = \sqrt{n_{air} n_{lens}} = \sqrt{1.5} = 1.22$$

and the thickness of the coating d should be a quarter of light wavelength in the coating, i.e.

$$d = \frac{\lambda}{4n_{coat}} = \frac{5.5\times10^{-7}}{4\times1.22} = 1.12\times10^{-7}[m]$$

5.11

By substitution of equation (5.10) into $\frac{\partial y}{\partial x}$, we have:

$$\frac{\partial y}{\partial x} = \frac{\omega_n}{c}(A_n \cos\omega_n t + B_n \sin\omega_n t)\cos\frac{\omega_n t}{c}$$

so:

$$\frac{\partial^2 y}{\partial x^2} = -\frac{\omega_n^2}{c^2}(A_n \cos\omega_n t + B_n \sin\omega_n t)\sin\frac{\omega_n t}{c} = -\frac{\omega_n^2}{c^2}y$$

Noting that $k = \frac{\omega_n}{c}$, we have:

$$\frac{\partial^2 y}{\partial x^2} + k^2 y = -\frac{\omega_n^2}{c^2}y + \frac{\omega_n^2}{c^2}y = 0$$

5.12

By substitution of the expression of $(y_n^2)_{\max}$ into the integral, we have:

$$\frac{1}{2}\rho\omega_n^2\int_0^l (y_n^2)_{\max}\,dx = \frac{1}{2}\rho\omega_n^2(A_n^2+B_n^2)\int_0^l \sin^2\frac{\omega_n x}{c}\,dx$$
$$= \frac{1}{2}\rho\omega_n^2(A_n^2+B_n^2)\int_0^l \frac{1-\cos(2\omega_n x/c)}{2}\,dx$$
$$= \frac{1}{4}\rho\omega_n^2(A_n^2+B_n^2)\left(l-\frac{c}{2\omega_n}\sin\frac{2\omega_n l}{c}\right)$$

Noting that $\omega_n = \frac{n\pi c}{l}$, i.e. $\sin\frac{2\omega_n l}{c} = \sin 2n\pi = 0$, the above equation becomes:

$$\frac{1}{2}\rho\omega_n^2\int_0^l (y_n^2)_{\max}\,dx = \frac{1}{4}\rho l\omega_n^2(A_n^2+B_n^2)$$

which gives the expected result.

5.13

Expand the expression of $y(x,t)$, we have:

$$\begin{aligned} y(x,t) &= A\cos(\omega t - kx) + rA\cos(\omega t + kx) \\ &= A\cos\omega t\cos kx + A\sin\omega t\sin kx + rA\cos\omega t\cos kx - rA\sin\omega t\sin kx \\ &= A(1+r)\cos\omega t\cos kx + A(1-r)\sin\omega t\sin kx \end{aligned}$$

which is the superposition of standing waves.

5.14

The wave group has a modulation envelope of:

$$A = A_0\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right)$$

where $\Delta\omega = \omega_1 - \omega_2$ is the frequency difference and $\Delta k = k_1 - k_2$ is the wave number difference. At a certain time t, the distance between two successive zeros of the modulation envelope Δx satisfies:

$$\frac{\Delta k}{2}\Delta x = \pi$$

Noting that $k = 2\pi/\lambda$, for a small value of $\Delta\lambda/\lambda$, we have: $\Delta k \approx (2\pi/\lambda^2)\Delta\lambda$, so the above equation becomes:

$$\frac{2\pi\Delta\lambda}{2\lambda^2}\Delta x \approx \pi$$

i.e.

$$\Delta x \approx \frac{\lambda}{\Delta\lambda}\lambda$$

which shows that the number of wavelengths λ contained between two successive zeros of the modulating envelop is $\approx \lambda/\Delta\lambda$

5.15

The expression for group velocity is given by:

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}(kv) = v + k\frac{dv}{dk}$$

By substitution of the expression of v into the above equation, we have:

$$\begin{aligned} v_g &= c\frac{\sin(ka/2)}{ka/2} + k\frac{d}{dk}\left[c\frac{\sin(ka/2)}{ka/2}\right] \\ &= c\frac{\sin(ka/2)}{ka/2} + ck\frac{(ka^2/4)\cos(ka/2) - (a/2)\sin(ka/2)}{(ka/2)^2} \\ &= c\frac{\sin(ka/2)}{ka/2} + c\cos\frac{ka}{2} - c\frac{\sin(ka/2)}{ka/2} \\ &= c\cos\frac{ka}{2} \end{aligned}$$

At long wavelengths, i.e. $k \to 0$, the limiting value of group velocity is the phase velocity c.

5.16

Noting that the group velocity of light in gas is given on page 131 as:

$$V_g = v\left(1 + \frac{\lambda}{2\varepsilon_r}\frac{\partial \varepsilon_r}{\partial \lambda}\right)$$

we have:

$$\begin{aligned} V_g\varepsilon_r &= v\left(1 + \frac{\lambda}{2\varepsilon_r}\frac{\partial \varepsilon_r}{\partial \lambda}\right)\varepsilon_r = v\left(\varepsilon_r + \frac{\lambda}{2}\frac{\partial \varepsilon_r}{\partial \lambda}\right) \\ &= v\left[\left(A + \frac{B}{\lambda^2} - D\lambda^2\right) + \frac{\lambda}{2}\frac{\partial}{\partial \lambda}\left(A + \frac{B}{\lambda^2} - D\lambda^2\right)\right] \\ &= v\left[\left(A + \frac{B}{\lambda^2} - D\lambda^2\right) + \frac{\lambda}{2}\left(-\frac{2B}{\lambda^3} - 2D\lambda\right)\right] \\ &= v\left[\left(A + \frac{B}{\lambda^2} - D\lambda^2\right) + \left(-\frac{B}{\lambda^2} - D\lambda^2\right)\right] \\ &= v(A - 2D\lambda^2) \end{aligned}$$

5.17

The relation $\varepsilon_r = \frac{c^2}{v^2} = 1 - \left(\frac{\omega_e}{\omega}\right)^2$ gives:

$$\frac{\omega^2 c^2}{v^2} = \omega^2 - \omega_e^2$$

By substitution of $v = \omega/k$, the above equation becomes:

$$\omega^2 = \omega_e^2 + c^2k^2 \qquad (5.17.1)$$

As $\omega \to \omega_e$, we have:

$$\frac{c^2}{v^2} = 1 - \left(\frac{\omega_e}{\omega}\right)^2 < 1$$

i.e. $v > c$, which means the phase velocity exceeds that of light c.
From equation (5.17.1), we have:

$$d(\omega^2) = d(\omega_e^2 + c^2k^2)$$

i.e. $$2\omega d\omega = 2kc^2 dk$$

which shows the group velocity v_g is given by:

$$v_g = \frac{d\omega}{dk} = c^2\frac{k}{\omega} = \frac{c^2}{v} = \frac{c}{v}c < c$$

i.e. the group velocity is always less than c.

5.18

From equation (5.17.1), we know that only electromagnetic waves of $\omega > \omega_e$ can propagate through the electron plasma media.

For an electron number density $n_e \sim 10^{20}$, the electron plasma frequency is given by:

$$\omega_e = e\sqrt{\frac{n_e}{m_e\varepsilon_0}} = 1.6\times10^{-19}\times\sqrt{\frac{10^{20}}{9.1\times10^{-31}\times8.8\times10^{-12}}} = 5.65\times10^{11}[rad\cdot s^{-1}]$$

Now consider the wavelength of the wave in the media given by:

$$\lambda = \frac{v}{f} = \frac{2\pi v}{\omega} < \frac{2\pi v}{\omega_e} < \frac{2\pi c}{\omega_e} = \frac{2\pi\times3\times10^8}{5.65\times10^{11}} = 3\times10^{-3}[m]$$

which shows the wavelength has an upper limit of $3\times10^{-3}m$.

5.19

The dispersion relation $\omega^2/c^2 = k^2 + m^2c^2/\hbar^2$ gives

$$d(\omega^2/c^2) = d(k^2 + m^2c^2/\hbar^2)$$

i.e. $$\frac{2\omega}{c^2}d\omega = 2kdk$$

i.e. $$\frac{\omega}{k}\frac{d\omega}{dk} = c^2$$

Noting that the group velocity is $d\omega/dk$ and the particle (phase) velocity is ω/k, the above equation shows their product is c^2.

5.20

The series in the problem is that at the bottom of page 132. The frequency components can be expressed as:

$$R = na\frac{\sin(\Delta\omega \cdot t/2)}{\Delta\omega \cdot t/2}\cos\overline{\omega}t$$

which is a symmetric function to the average frequency ω_0. It shows that at $\Delta t = \frac{2\pi}{\Delta\omega}$, $R = 0$,

$\therefore \Delta t \cdot \Delta\omega = 2\pi$

In k space, we may write the series as:

$$y(k) = a\cos k_1 x + a\cos(k_1 + \delta k)x + \cdots + a\cos[k_1 + (n-1)\delta k]x$$

As an analogy to the above analysis, we may replace ω by k and t by x, and R is zero at $\Delta x = \frac{2\pi}{\Delta k}$, i.e. $\Delta k \Delta x = 2\pi$

5.21

The frequency of infrared absorption of NaCl is given by:

$$\omega = \sqrt{\frac{2T}{a}\left(\frac{1}{m_{Na}} + \frac{1}{m_{Cl}}\right)} = \sqrt{2\times 15\times\left(\frac{1}{23\times 1.66\times 10^{-27}} + \frac{1}{35\times 1.66\times 10^{-27}}\right)} = 3.608\times 10^{13}[rad\cdot s^{-1}]$$

The corresponding wavelength is given by:

$$\lambda = \frac{2\pi c}{\omega} = \frac{2\pi\times 3\times 10^8}{3.608\times 10^{13}} \approx 52[\mu m]$$

which is close to the experimental value: $61\mu m$

The frequency of infrared absorption of KCl is given by:

$$\omega = \sqrt{\frac{2T}{a}\left(\frac{1}{m_K} + \frac{1}{m_{Cl}}\right)} = \sqrt{2\times 15\times\left(\frac{1}{39\times 1.66\times 10^{-27}} + \frac{1}{35\times 1.66\times 10^{-27}}\right)} = 3.13\times 10^{13}[rad\cdot s^{-1}]$$

The corresponding wavelength is given by:

$$\lambda = \frac{2\pi c}{\omega} = \frac{2\pi\times 3\times 10^8}{3.13\times 10^{13}} \approx 60[\mu m]$$

which is close to the experimental value: $71\mu m$

5.22

Before the source passes by the observer, the source has a velocity of u, the frequency noted by the observer is given by:

$$\nu_1 = \frac{c}{c-u}\nu$$

After the source passes by the observer, the source has a velocity of $-u$, the frequency noted by the observer is given by:

$$\nu_2 = \frac{c}{c+u}\nu$$

So the change of frequency noted by the observer is given by:

$$\Delta\nu = \nu_2 - \nu_1 = \left(\frac{c}{c-u} - \frac{c}{c+u}\right)\nu = \frac{2\nu cu}{(c^2-u^2)}$$

5.23

By superimposing a velocity of $-v$ on the system, the observer becomes stationary and the source has a velocity of $u-v$ and the wave has a velocity of $c-v$. So the frequency registered by the observer is given by:

$$\nu''' = \frac{c-v}{c-v-(u-v)} = \frac{c-v}{c-u}\nu$$

5.24

The relation between wavelength λ and frequency ν of light is given by:

$$\nu = \frac{c}{\lambda}$$

So the Doppler Effect $\nu' = \dfrac{\nu c}{c-u}$ can be written in the format of wavelength as:

$$\frac{c}{\lambda'} = \frac{c^2}{\lambda(c-u)}$$

i.e.

$$\lambda' = \frac{c-u}{c}\lambda$$

Noting that wavelength shift is towards red, i.e. $\lambda' > \lambda$, so we have:

$$\Delta\lambda = \lambda' - \lambda = -\frac{u}{c}\lambda$$

i.e.

$$u = -\frac{c\Delta\lambda}{\lambda} = -\frac{3\times10^8\times10^{-11}}{6\times10^{-7}} = -5[Kms^{-1}]$$

which shows the earth and the star are separating at a velocity of $5Kms^{-1}$.

5.25

Suppose the aircraft is flying at a speed of u, and the signal is being transmitted from the aircraft at a frequency of ν and registered at the distant point at a frequency of ν'. Then, the Doppler Effect gives:

$$\nu' = \nu \frac{c}{c-u}$$

Now, let the distant point be the source, reflecting a frequency of ν' and the flying aircraft be the receiver, registering a frequency of ν''. By superimposing a velocity of $-u$ on the flying aircraft, the distant point and signal waves, we bring the aircraft to rest; the distant point now has a velocity of $-u$ and signal waves a velocity of $-c-u$. Then, the Doppler Effect gives:

$$\nu'' = \nu' \frac{-c-u}{-c-u-(-u)} = \nu' \frac{c+u}{c} = \nu \frac{c+u}{c-u}$$

which gives:

$$u = \frac{\nu''-\nu}{\nu''+\nu} c = \frac{\Delta\nu}{2\nu+\Delta\nu} c = \frac{15\times10^3}{2\times3\times10^9} \times 3\times10^8 = 750[ms^{-1}]$$

i.e. the aircraft is flying at a speed of $750\, m/s$

5.26

Problem 5.24 shows the Doppler Effect in the format of wavelength is given by:

$$\lambda' = \frac{c-u}{c} \lambda$$

where u is the velocity of gas atom. So we have:

$$|\Delta\lambda| = |\lambda' - \lambda| = \frac{|u|}{c} \lambda$$

i.e.

$$|u| = |\lambda' - \lambda| = \frac{|\Delta\lambda|}{\lambda} c = \frac{2\times10^{-12}}{6\times10^{-7}} \times 3\times10^8 = 1\times10^3[ms^{-1}]$$

The thermal energy of sodium gas is given by:

$$\frac{1}{2} m_{Na} u^2 = \frac{3}{2} kT$$

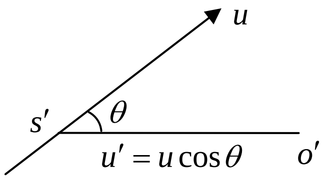
where $k = 1.38\times10^{-23}[JK^{-1}]$ is Boltzmann's constant, so the gas temperature is given by:

$$T = \frac{m_{Na} u^2}{3k} = \frac{23\times1.66\times10^{-27} \times 1000^2}{3\times1.38\times10^{-23}} \approx 900[K]$$

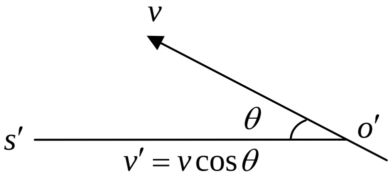
5.27

A point source radiates spherical waves equally in all directions.

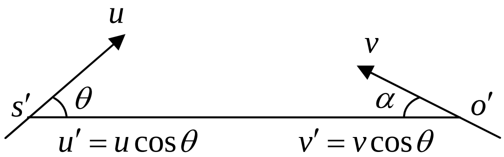
$\nu' = \left(\frac{\nu c}{c-u'} \right)$: Observer is at rest with a moving source.



$v'' = \left(\dfrac{c - v'}{c}\right)$: Source at rest with a moving observer.



$v''' = \left(\dfrac{c - v'}{c - u'}\right)$: Source and observer both moving.



5.28

By substitution of equation (2) into (3) and eliminating x', we can find the expression of t' given by:

$$t' = \frac{1}{v}\left[\frac{x}{k'} - k(x - vt)\right]$$

Now we can eliminate x' and t' by substituting the above equation and the equation (2) into equation (1), i.e.

$$x^2 - c^2t^2 = k^2(x - vt)^2 - \frac{c^2}{v^2}\left[\frac{x}{k'} - k(x - vt)\right]^2$$

i.e.

$$\left[1 - k^2 + \frac{c^2}{v^2}\left(\frac{1}{k'} - k\right)^2\right]x^2 + 2kv\left[k + \frac{c^2}{v^2}\left(\frac{1}{k'} - k\right)\right]xt + \left[k^2v^2\left(\frac{c^2}{v^2} - 1\right) - c^2\right]t^2 = 0$$

which is true for all x and t if and only if the coefficients of all terms are zeros, so we have:

$$1 - k^2 = \frac{c^2}{v^2}\left(k - \frac{1}{k'}\right)^2$$

$$\left(\frac{c^2}{v^2} - 1\right)kk' = \frac{c^2}{v^2}$$

$$k^2(c^2 - v^2) = c^2$$

The solution to the above equations gives:

$$k = k' = \frac{1}{\sqrt{1-\beta^2}}$$

where, $\beta = v/c$

5.29

Source at rest at x_1 in O frame gives signals at intervals measured by O as $\Delta t = t_2 - t_1$ where t_2 is later than t_1. O' moving with velocity v with respect to O measures these intervals as:

$$t_2' - t_1' = \Delta t' = k(\Delta t - \frac{v}{c^2}\Delta x) \text{ with } \Delta x = 0$$
$$\therefore \Delta t' = k\Delta t$$

$l = (x_2 - x_1)$ as seen by O, O' sees it as $(x_2' - x_1') = k[(x_2 - x_1) - v(t_2 - t_1)]$.

Measuring l' puts $t_2' = t_1'$ or $\Delta t' = 0$

$$\therefore \Delta t' = k\left[\Delta t - \frac{v}{c^2}(x_2 - x_1)\right] = 0 \text{ i.e. } \Delta t = \frac{v}{c^2}(x_2 - x_1) = t_2 - t_1$$

$$\therefore l' = x_2' - x_1' = k[(x_2 - x_1) - v(\Delta t)] = k\left[(x_2 - x_1) - \frac{v^2}{c^2}(x_2 - x_1)\right] = \frac{x_2 - x_1}{k}$$

$$\therefore l' = l/k$$

5.30

Two events are simultaneous $(t_1 = t_2)$ at x_1 and x_2 in O frame. They are not simultaneous in O' frame because:

$$t_1' = k\left(t_1 - \frac{v}{c^2}x_1\right) \neq t_2' = k\left(t_2 - \frac{v}{c^2}x_2\right) \text{ i.e. } x_1 \neq x_2$$

5.31

The order of cause followed by effect can never be reversed.

2 events x_1, t_1 and x_2, t_2 in O frame with $t_2 > t_1$ i.e. $t_2 - t_1 > 0$ (t_2 is later).

$$t_2' - t_1' = k\left[(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)\right] \text{ i.e. } \Delta t' = k\left[\Delta t - \frac{v}{c^2}\Delta x\right] \text{ in } O' \text{ frame.}$$

$\Delta t'$ real requires k real that is $v < c$, $\Delta t'$ is $+ve$ if $\Delta t > \frac{v}{c}\left(\frac{\Delta x}{c}\right)$ where $\frac{v}{c}$ is $+ve$ but < 1 and $\frac{\Delta x}{c}$ is shortest possible time for signal to traverse Δx.

SOLUTIONS TO CHAPTER 6

6.1

Elementary kinetic theory shows that, for particles of mass m in a gas at temperature T, the energy of each particle is given by:

$$\frac{1}{2}mv^2 = \frac{3}{2}kT$$

where v is the root mean square velocity and k is Boltzmann's constant.
Page 154 of the text shows that the velocity of sound c is a gas at pressure P is given by:

$$c^2 = \frac{\gamma P}{\rho} = \frac{\gamma PV}{M} = \frac{\gamma RT}{M} = \frac{\gamma NkT}{M}$$

where V is the molar volume, M is the molar mass and N is Avogadro's number, so:

$$Mc^2 = \gamma NkT = \alpha kT \approx \frac{5}{3}kT$$

6.2

The intensity of sound wave can be written as:

$$I = P^2/\rho_0 c$$

where P is acoustic pressure, ρ_0 is air density, and c is sound velocity, so we have:

$$P = \sqrt{I\rho_0 c} = \sqrt{10 \times 1.29 \times 330} \approx 65[Pa]$$

which is 6.5×10^{-4} of the pressure of an atmosphere.

6.3

The intensity of sound wave can be written as:

$$I = \frac{1}{2}\rho_0 c\omega^2\eta^2$$

where η is the displacement amplitude of an air molecule, so we have:

$$\eta = \frac{1}{2\pi\nu}\sqrt{\frac{2I}{\rho_0 c}} = \frac{1}{2\pi\times 500}\times\sqrt{\frac{2\times 10}{1.29\times 330}} = 6.9\times 10^{-5}[m]$$

6.4

The expression of displacement amplitude is given by Problem 6.3, i.e.:

$$\eta = \frac{1}{2\pi\nu}\sqrt{\frac{2\times 10^{-10} I_0}{\rho_0 c}} = \frac{1}{2\pi\times 500}\times\sqrt{\frac{2\times 10\times 10^{-10}\times 10^{-2}}{1.29\times 330}} \approx 10^{-10}[m]$$

6.5

The audio output is the product of sound intensity and the cross section area of the room, i.e.:

$$P = IA = 100 I_0 A = 100\times 10^{-2}\times 3\times 3 \approx 10[W]$$

6.6

The expression of acoustic pressure amplitude is given by Problem 6.2, so the ratio of the pressure amplitude in water and in air, at the same sound intensity, are given by:

$$\frac{p_{water}}{p_{air}} = \frac{\sqrt{I(\rho_0 c)_{water}}}{\sqrt{I(\rho_0 c)_{air}}} = \sqrt{\frac{(\rho_0 c)_{water}}{(\rho_0 c)_{air}}} = \sqrt{\frac{1.45\times 10^6}{400}} \approx 60$$

And at the same pressure amplitudes, we have:

$$\frac{I_{water}}{I_{air}} = \frac{(\rho_0 c)_{air}}{(\rho_0 c)_{water}} = \frac{400}{1.45\times 10^6} \approx 3\times 10^{-4}$$

6.7

If η is the displacement of a section of a stretched spring by a disturbance, which travels along it in the x direction, the force at that section is given by: $F = Y\frac{\partial \eta}{\partial x}$, where Y is young's modulus.

The relation between Y and s, the stiffness of the spring, is found by considering the force required to increase the length L of the spring slowly by a small amount $l << L$, the force F being the same at all points of the spring in equilibrium. Thus

$$\frac{\partial \eta}{\partial x} = \frac{l}{L} \quad \text{and} \quad F = \left(\frac{Y}{L}\right) l$$

If $l = x$ in the stretched spring, we have:

$$F = sx = \left(\frac{Y}{L}\right) x \quad \text{and} \quad Y = sL\,.$$

If the spring has mass m per unit length, the equation of motion of a section of length dx is given by:

$$m\frac{\partial^2\eta}{\partial t^2}dx = \frac{\partial F}{\partial x}dx = Y\frac{\partial^2\eta}{\partial x^2}dx$$

or
$$\frac{\partial^2\eta}{\partial t^2} = \frac{Y}{m}\frac{\partial^2\eta}{\partial x^2} = \frac{sL}{m}\frac{\partial^2\eta}{\partial x^2}$$

a wave equation with a phase velocity $\sqrt{\frac{sL}{m}}$

6.8

At $x = 0$,

$$\eta = B\sin kx \sin\omega t$$

At $x = L$,

$$M\frac{\partial^2\eta}{\partial t^2} = -sL\frac{\partial\eta}{\partial x}$$

i.e.
$$-M\omega^2\sin kL = -sLk\cos kL$$

(which for $k = \omega/v$, $\rho = m/L$ and $v = \sqrt{sL/\rho}$ from problem 7 when $l << L$)

becomes:

$$\frac{\omega L}{v}\tan\frac{\omega L}{v} = \frac{sL^2}{Mv^2} = \frac{\rho L}{M} = \frac{m}{M} \qquad (6.8.1)$$

For $M >> m$, $v >> \omega L$ and writing $\omega L/v = \theta$ where θ is small, we have:

$$\tan\theta = \theta + \theta^3/3 + ...$$

and the left hand side of equation 6.8.1 becomes

$$\theta^2[1+\theta^2/3+...] = (\omega L/v)^2[1+(\omega L/v)^2/3+...]$$

Now $v = (sL/\rho)^{1/2} = (sL^2/m)^{1/2} = L(s/m)^{1/2}$ and $\omega L/v = \omega\sqrt{m/s}$

So eq. 6.8.1 becomes:

$$\omega^2 m/s\,(1+\omega^2 m/3s+...) = m/M$$

or

$$\omega^2(1+\omega^2 m/3s) = s/M \qquad (6.8.2)$$

Using $\omega^2 = s/M$ as a second approximation in the bracket of eq. 6.8.2, we have:

$$\omega^2\left(1+\frac{1}{3}\frac{m}{M}\right)=\frac{s}{M}$$

i.e. $$\omega^2=\frac{s}{M+\frac{1}{3}m}$$

6.9

The Poissons ratio $\sigma = 0.25$ gives:

$$\frac{\lambda}{2(\lambda+\mu)}=0.25$$

i.e. $$\lambda=\mu$$

So the ratio of the longitudinal wave velocity to the transverse wave velocity is given by:

$$\frac{v_l}{v_t}=\frac{\sqrt{\lambda+2\mu}}{\sqrt{\mu}}=\sqrt{\frac{\mu+2\mu}{\mu}}=\sqrt{3}$$

In the text, the longitudinal wave velocity of the earth is $8kms^{-1}$ and the transverse wave velocity is $4.45kms^{-1}$, so we have:

$$\frac{\sqrt{\lambda+2\mu}}{\sqrt{\mu}}=\frac{8}{4.45}$$

i.e. $$\lambda=1.23\mu$$

so the Poissons ratio for the earth is given by:

$$\sigma=\frac{\lambda}{2(\lambda+\mu)}=\frac{1.23\mu}{2\times(1.23\mu+\mu)}\approx 0.276$$

6.10

At a plane steel water interface, the energy ratio of reflected wave is given by:

$$\frac{I_r}{I_i}=\left(\frac{Z_{steel}-Z_{water}}{Z_{steel}+Z_{water}}\right)^2=\left(\frac{3.9\times10^7-1.43\times10^6}{3.9\times10^7+1.43\times10^6}\right)^2\approx 86\%$$

At a plane steel water interface, the energy ratio of transmitted wave is given by:

$$\frac{I_t}{I_i}=\frac{4Z_{ice}Z_{water}}{(Z_{ice}+Z_{water})^2}=\frac{4\times3.49\times10^6\times1.43\times10^6}{(3.49\times10^6+1.43\times10^6)^2}\approx 82.3\%$$

6.11

Solution follow directly from the coefficients at top of page 165.

Closed end is zero displacement with $\frac{n_r}{n_i} = -1$ (node).

Open end: $\frac{n_r}{n_i} = 1$ (antinode, η is a max)

Pressure: closed end: $\frac{p_r}{p_i} = 1$. Pressure doubles at antinode

Open end : $\frac{p_r}{p_i} = -1$ (out of phase – cancels to give zero pressure, i.e. node)

6.12

(a) The boundary condition $\frac{\partial \eta}{\partial x} = 0$ at $x = 0$ gives:

$$(-Ak \sin kx + Bk \cos kx) \sin \omega t\big|_{x=0} = 0$$

i.e. $B = 0$, so we have: $\eta = A \cos kx \sin \omega t$

The boundary condition $\frac{\partial \eta}{\partial x} = 0$ at $x = L$ gives:

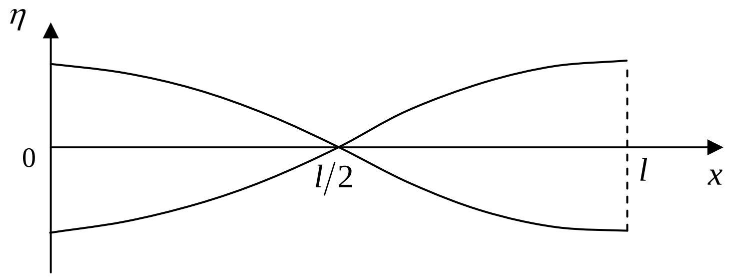
$$-kA \sin kx \sin \omega t\big|_{x=l} = 0$$

i.e. $kA \sin kL \sin \omega t = 0$

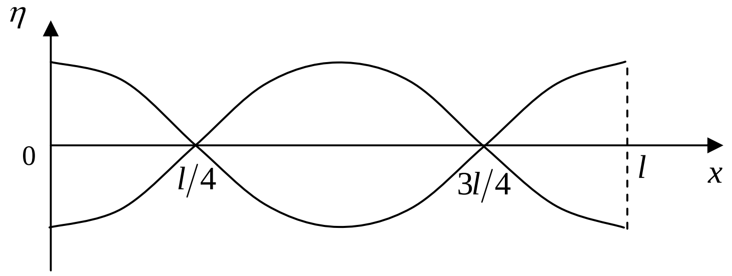
which is true for all t if $kl = n\pi$, i.e. $\frac{2\pi}{\lambda} l = n\pi$ or $\lambda = \frac{2l}{n}$

The first three harmonics are shown below:

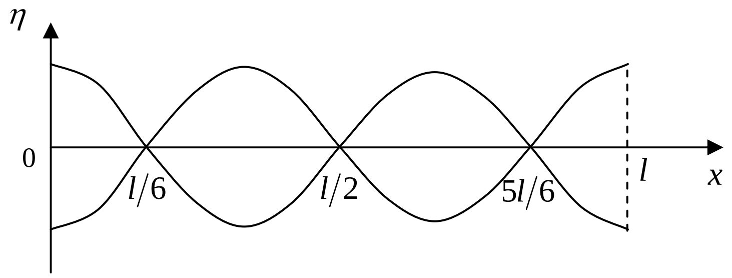
$n=1$:



$n=2$:



$n=3$:



(b) The boundary condition $\frac{\partial \eta}{\partial x} = 0$ at $x = 0$ gives:

$$(-Ak\sin kx + Bk\cos kx)\sin \omega t\Big|_{x=0} = 0$$

i.e. $B = 0$, so we have: $\eta = A\cos kx \sin \omega t$

The boundary condition $\eta = 0$ at $x = L$ gives:

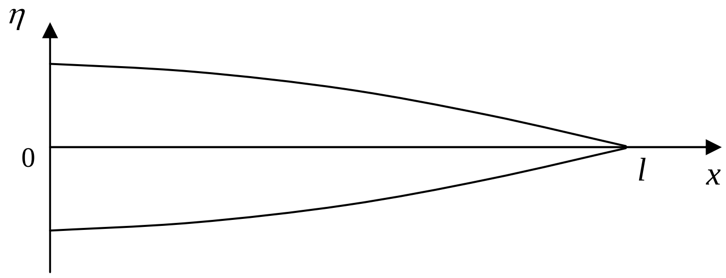
$$A\sin kx \sin \omega t\Big|_{x=l} = 0$$

i.e. $$A\cos kl \sin \omega t = 0$$

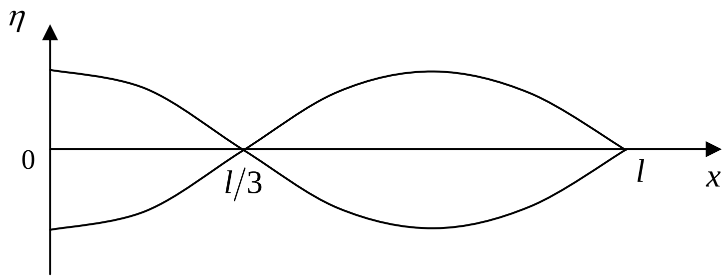
which is true for all t if $kl = \left(n + \frac{1}{2}\right)\pi$, i.e. $\frac{2\pi}{\lambda} l = \left(n + \frac{1}{2}\right)\pi$ or $\lambda = \frac{4l}{2n+1}$

The first three harmonics are shown below:

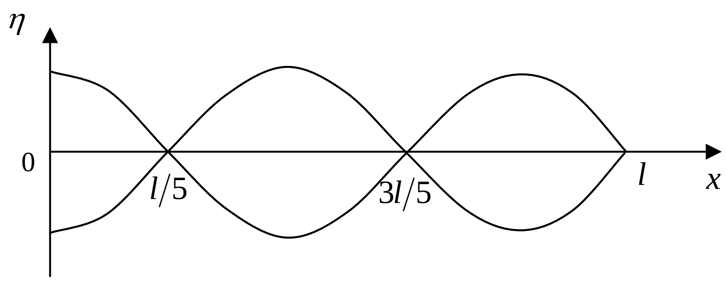
$n = 0$:



$n = 1$:



$n = 2$:



6.13

The boundary condition for pressure continuity at $x = 0$ gives:

$$[A_1 e^{i(\omega t - k_1 x)} + B_1 e^{i(\omega t - k_1 x)}]_{x=0} = [A_2 e^{i(\omega t - k_2 x)} + B_2 e^{i(\omega t - k_2 x)}]_{x=0}$$

i.e. $$A_1 + B_1 = A_2 + B_2 \qquad (6.13.1)$$

In acoustic wave, the pressure is given by: $p = Z\dot{\eta}$, so the continuity of particle velocity $\dot{\eta}$ at $x = 0$ gives:

$$\left.\frac{A_1 e^{i(\omega t - k_1 x)} + B_1 e^{i(\omega t - k_1 x)}}{Z_1}\right|_{x=0} = \left.\frac{A_2 e^{i(\omega t - k_2 x)} + B_2 e^{i(\omega t - k_2 x)}}{Z_2}\right|_{x=0}$$

i.e. $$Z_2(A_1 - B_1) = Z_1(A_2 - B_2) \qquad (6.13.2)$$

At $x = l$, the continuity of pressure gives:

$$[A_2 e^{i(\omega t - k_2 x)} + B_2 e^{i(\omega t - k_2 x)}]_{x=l} = A_3 e^{i(\omega t - k_3 x)}\Big|_{x=l}$$

i.e. $$A_2 e^{-ik_2 l} + B_2 e^{ik_2 l} = A_3 \qquad (6.13.3)$$

The continuity of particle velocity gives:

$$\left.\frac{A_2 e^{i(\omega t - k_2 x)} + B_2 e^{i(\omega t - k_2 x)}}{Z_2}\right|_{x=l} = \left.\frac{A_3 e^{i(\omega t - k_3 x)}}{Z_3}\right|_{x=l}$$

i.e.
$$Z_3(A_2 e^{-ik_2 l} - B_2 e^{ik_2 l}) = Z_2 A_3 \qquad (6.13.4)$$

By comparison of the boundary conditions derived above with the derivation in page 121-124, we can easily find:

$$\frac{Z_1}{Z_3}\frac{A_3^2}{A_1^2} = \frac{4r_{31}}{(r_{31}+1)^2\cos^2 k_2 l + (r_{21}+r_{32})^2 \sin^2 k_2 l}$$

where $r_{31} = \frac{Z_3}{Z_1}$, $r_{21} = \frac{Z_2}{Z_1}$, and $r_{32} = \frac{Z_3}{Z_2}$.

If we choose $l = \lambda_2/4$, $\cos k_2 l = 0$ and $\sin k_2 l = 1$, we have:

$$\frac{Z_1}{Z_3}\frac{A_3^2}{A_1^2} = \frac{4r_{31}}{(r_{21}+r_{32})^2} = 1$$

when $r_{21} = r_{32}$, i.e. $\frac{Z_2}{Z_1} = \frac{Z_3}{Z_2}$ or $Z_2^2 = Z_1 Z_3$.

6.14

The differentiation of the adiabatic condition:

$$\frac{P}{P_0} = \left[\frac{V_0}{V_0(1+\delta)}\right]^{\gamma}$$

gives:

$$\frac{\partial P}{\partial x} = \frac{\partial p}{\partial x} = -\gamma P_0 (1+\delta)^{-(\gamma+1)} \frac{\partial^2 \eta}{\partial x^2}$$

since $\delta = \partial\eta/\partial x$.

Since $(1+\delta)(1+s) = 1$, we may write:

$$\frac{\partial p}{\partial x} = -\gamma P_0 (1+s)^{\gamma+1} \frac{\partial^2 \eta}{\partial x^2}$$

and from Newton’s second law we have:

$$\frac{\partial p}{\partial x} = -\rho_0 \frac{\partial^2 \eta}{\partial t^2}$$

so that

$$\frac{\partial^2 \eta}{\partial t^2} = c_0^2 (1+s)^{\gamma+1} \frac{\partial^2 \eta}{\partial x^2}, \text{ where } c_0^2 = \frac{\gamma P_0}{\rho_0}$$

which shows the sound velocity of high amplitude wave is given by $c_0(1+s)^{(\gamma+1)/2}$

6.15

The differentiation of equation $\omega^2 = \omega_e^2 + 3aTk^2$ gives:

$$2\omega\frac{d\omega}{dk} = 6aTk$$

i.e.

$$\frac{\omega}{k}\frac{d\omega}{dk} = 3aT$$

where a represents Boltzmann constant, ω/k is the phase velocity, $d\omega/dk$ is the group velocity.

6.16

The fluid is incompressible so that during the wave motion there is no change in the volume of the fluid element of height h, horizontal length Δx and unit width. The distortion η in the element Δx is therefore directly translated to a change in its height h and its constant volume requires that:

$$h\Delta x = (h+\alpha)(\Delta x + \Delta\eta) = h\Delta x + h\Delta\eta + \alpha\Delta x + \alpha\Delta\eta$$

Because $\alpha << h$ and $\Delta\eta << \Delta x$, the second order term $\alpha\Delta\eta$ is ignorable, we then have

$\alpha = -h\Delta\eta/\Delta x = -h\partial\eta/\partial x$, and from now on we replace Δx by dx.

We see that for $\alpha + ve$ (or increase in height), we have $\partial\eta/\partial x - ve$, that is, a compression.

On page 153 of the text, the horizontal motion of the element is shown to be due to the difference in forces acting on the opposing faces of the element $h\Delta x$, that is:

$$-\frac{\partial F}{\partial x}\Delta x = \rho h\frac{\partial^2\eta}{\partial t^2}\Delta x$$

where the force difference, dF is $-ve$ when measured in the $+ve$ x direction for a compression.

Thus:

$$-\frac{\partial F}{\partial x}dx = \rho h\frac{\partial^2\eta}{\partial t^2}dx = -h\frac{\partial P_{av}}{\partial x}dx \qquad (6.16.1)$$

where the pressure must be averaged over the height of the element because it varies with the liquid depth. This average value is found from the pressure difference (to unit depth) at the liquid surface to between the two values of α on the opposing faces of the element. This gives:

$$dP_{av} = \rho g d\alpha$$

so $$\frac{dP_{av}}{dx}dx = \rho g \frac{d\alpha}{dx}dx$$

and we have from eq. 6.16.1:

$$-\frac{\partial F}{\partial x}dx = -h\frac{\partial P_{av}}{\partial x}dx = -h\rho g\frac{d\alpha}{dx}dx = h^2\rho g\frac{\partial^2\eta}{\partial x^2}dx = \rho h\frac{\partial^2\eta}{\partial t^2}dx$$

The last two terms equate to give the wave equation. For horizontal motion as:

$$\frac{\partial^2\eta}{\partial t^2} = gh\frac{\partial^2\eta}{\partial x^2}$$

with phase velocity $v = \sqrt{gh}$.

The horizontal motion translates directly to the vertical displacement α to give an equation of wave motion:

$$\frac{\partial^2\alpha}{\partial t^2} = gh\frac{\partial^2\alpha}{\partial x^2}$$

with a similar phase velocity $v = \sqrt{gh}$

6.17

(a) Since $h >> \lambda$, i.e. $kh >> 1$, we have: $\tanh kh \approx 1$, therefore:

$$v^2 = \left[\frac{g}{k} + \frac{Tk}{\rho}\right]\tanh kh \approx \frac{g}{k} + \frac{Tk}{\rho} \geq 2\sqrt{\frac{g}{k}\cdot\frac{Tk}{\rho}} = 2\sqrt{\frac{gT}{\rho}}$$

i.e. the velocity has a minimum value given by:

$$v^4 = \frac{4gT}{\rho}$$

when $\frac{g}{k} = \frac{Tk}{\rho}$, i.e. $k^2 = \frac{g\rho}{T}$ or $\lambda_c = 2\pi\sqrt{\frac{T}{\rho g}}$

(b) If T is negligible, we have:

$$v^2 \approx \frac{g}{k}\tanh kh$$

and when $\lambda >> \lambda_c$, $k \to 0$, and for a shallow liquid, $h \to 0$. Noting that when $hk \to 0$, $\tanh kh \to kh$, we have:

$$v = \sqrt{\frac{g}{k}\tanh kh} \approx \sqrt{\frac{g}{k}kh} = \sqrt{gh}$$

(c) For a deep liquid, $h \to +\infty$ i.e. $\tanh kh \to 1$, the phase velocity is given by:

$$v_p^2 = \frac{g}{k}\tanh kh \approx \frac{g}{k} \quad \text{i.e.} \quad v_p = \sqrt{\frac{g}{k}}$$

and the group velocity is given by:

$$v_g = v_p + \frac{kdv_p}{dk} = v_p - \frac{1}{2}k\sqrt{\frac{g}{k^3}} = \sqrt{\frac{g}{k}} - \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}\sqrt{\frac{g}{k}}$$

(d) For the case of short ripples dominated by surface tension in a deep liquid, i.e. $\frac{g}{k} << \frac{Tk}{\rho}$ and $h \to +\infty$, we have:

$$v_p^2 = \lim_{h\to+\infty} \frac{Tk}{\rho}\tanh kh = \frac{Tk}{\rho} \quad \text{i.e.} \quad v_p = \sqrt{\frac{Tk}{\rho}}$$

and the group velocity is given by:

$$v_g = v_p + \frac{kdv_p}{dk} = \sqrt{\frac{Tk}{\rho}} + \frac{k}{2}\sqrt{\frac{T}{\rho k}} = \frac{3}{2}\sqrt{\frac{Tk}{\rho}} = \frac{3}{2}v_p$$

SOLUTIONS TO CHAPTER 7

7.1

The equation

$$I_{r-1} - I_r = \frac{d}{dt} q_r = C_0 dx \frac{d}{dt} V_r$$

at the limit of $dx \to 0$ becomes:

$$\frac{dI}{dx} = C_0 \frac{dV}{dt}$$

(7.1.1)

The equation

$$L_0 dx \frac{d}{dt} I_r = V_r - V_{r+1}$$

at the limit of $dx \to 0$ becomes:

$$\frac{\partial V}{\partial x} = L_0 \frac{\partial I}{\partial t}$$

(7.1.2)

The derivative of equation (7.1.1) on t gives:

$$\frac{\partial^2 I}{\partial x \partial t} = C_0 \frac{\partial^2 V}{\partial t^2}$$

(7.1.3)

The derivative of equation (7.1.2) on x gives:

$$\frac{\partial^2 V}{\partial x^2} = L_0 \frac{\partial^2 I}{\partial x \partial t}$$

(7.1.4)

Equation (7.1.3) and (7.1.4) give:

$$\frac{\partial^2 V}{\partial x^2} = L_0 C_0 \frac{\partial^2 V}{\partial t^2}$$

The derivative of equation (7.1.1) on x gives:

$$\frac{\partial^2 I}{\partial x^2} = C_0 \frac{\partial^2 V}{\partial x \partial t}$$

(7.1.5)

The derivative of equation (7.1.2) on t gives:

$$\frac{\partial^2 V}{\partial x \partial t} = L_0 \frac{\partial^2 I}{\partial t^2}$$

(7.1.6)

Equation (7.1.5) and (7.1.6) give:

$$\frac{\partial^2 I}{\partial x^2} = L_0 C_0 \frac{\partial^2 I}{\partial t^2}$$

7.2

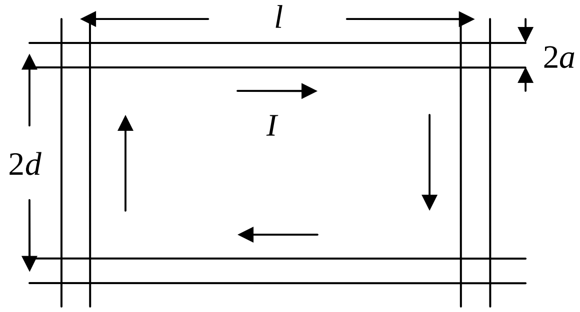


Fig Q.7.2.1

A pair of parallel wires of circular cross section and radius a are separated at a distance $2d$ between their centres.

To find the inductance per unit length we close the circuit by joining the sides of a section of length l.

The self inductance of this circuit is the magnetic flux through the circuit when a current of 1 amp flows around it.

If the current is 1 amp the field outside the wire at a distance r from the centre is $\mu_0 I/2\pi r$,

where μ_0 is the permeability of free space. For a clockwise current in the circuit (Fig Q.7.2.1)

both wires contribute to the magnetic flux B which points downwards into the page and the total flux through the circuit is given by:

$$2l\int_a^{2d-2a} \frac{\mu_0 I dr}{2\pi r} = \frac{\mu_0 l I}{\pi} \ln\left(\frac{2d}{a}\right) \quad \text{for} \quad d >> a$$

Hence the self inductance per unit length is:

$$L = \frac{\mu_0}{\pi} \ln\left(\frac{2d}{a}\right)$$

To find the capacitance per unit length of such a pair of wires we first find the electrostatic potential at a distance r from a single wire and proceed to find the potential from a pair of wires via the principle of electrostatic images.

If the radius of the wire is a and it carries a charge of λ per unit length then the electrostatic flux E per unit length of the cylindrical surface is: $2\pi r E(r) = \lambda/\varepsilon_0$, where ε_0 is the

permittivity of free space. Thus $E(r) = \lambda/2\pi\varepsilon_0 r$ for $r > a$ and we have the potential:

$$\phi(r) = -\frac{\lambda}{2\pi\varepsilon_0}\ln(r) + \text{constant} = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{r}\right) \quad \text{for} \quad \phi = 0 \quad \text{at} \quad r = b\,.$$

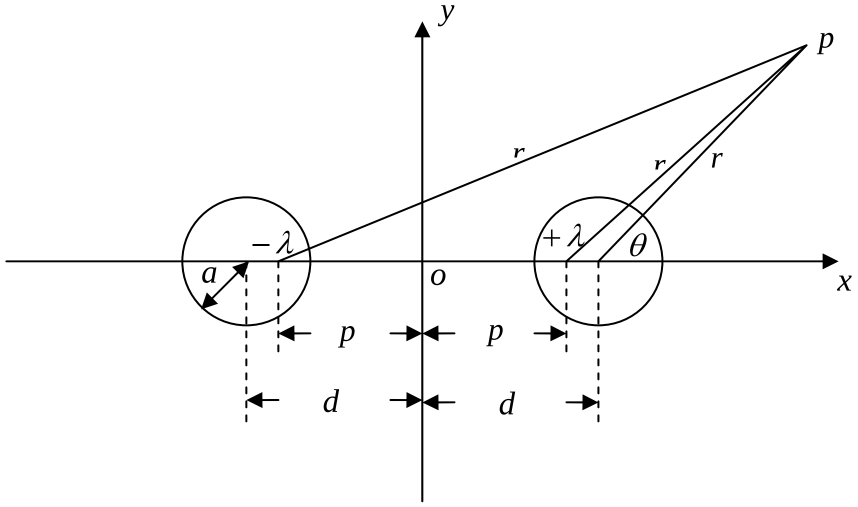


Fig Q.7.2.2

The conducting wires are now represented in the image system of Fig Q.7.2.2. The equipotential surfaces will be seen to be cylindrical but not coaxial with the wires. Neither the electric field nor the charge density is uniform on the conducting surface.

The surface charge is collapsed onto two line carrying charges $\pm\lambda$ per unit length. The y axis represents an equipotential plane.

The conducting wires, of radius a, are centred a distance d from the origin $o(x = y = 0)$. The distances $\pm p$ can be chosen of the line charges so that the conducting surfaces lie on the equipotentials of the image charge. Choosing the potential to be zero at ∞ on the y axis, the potential at point p in the xy plane is given by:

$$\phi_p = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{1}{r_1}\right) - \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{1}{r_2}\right) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{r_2^2}{r_1^2}\right)$$

In Fig Q.7.2.2:

$$r_2^2 = 2\alpha\gamma + \delta$$
$$r_1^2 = 2\beta\gamma + \delta$$

where $\alpha = (d + p)$; $\beta = (d - p)$; $\gamma = (d + r\cos\theta)$ and $\delta = (r^2 + p^2 - d^2)$.

If the position of the image charge is such that $p^2 = (d^2 - a^2)$, then, at $r = a$:

$\phi(a) = \dfrac{\lambda}{4\pi\varepsilon_0}\ln\left(\dfrac{d+p}{d-p}\right)$, independent of θ, and the right-hand conductor is an equipotential of the image charge. Symmetry requires that the potential at the surface of the other conductor is

$-\phi(a)$ and the potential difference between the conductors is:

$$V = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{d+p}{d-p}\right)$$

Gauss's theorem applied to one of the equipotentials surrounding each conductor proves that the surface charge on each conductor is equal to the image charge.
The capacitance per unit length is now given by:

$$C = \frac{\lambda}{V} = \frac{2\pi\varepsilon_0}{\ln\left(\dfrac{d+p}{d-p}\right)} \approx \frac{2\pi\varepsilon_0}{\ln\left(\dfrac{2d}{a}\right)} \quad \text{for} \quad d >> a$$

and

$$Z_0 \quad \text{for the parallel wires} \quad = \sqrt{\frac{L}{C}} = \left(\frac{\dfrac{\mu_0}{\pi}\ln\left(\dfrac{2d}{a}\right)}{\dfrac{2\pi\varepsilon_0}{\ln(2d/a)}}\right)^{1/2} \approx \left(\frac{\mu_0}{\pi^2\varepsilon_0}\right)^{1/2}\ln\left(\frac{2d}{a}\right)$$

7.3

The integral of magnetic energy over the last quarter wavelength is given by:

$$\int_{-\lambda/4}^{0}\frac{1}{2}L_0 I^2 dx = \int_{-\lambda/4}^{0}\frac{1}{2}L_0\left(\frac{2V_{0+}}{Z_0}\cos kx\right)^2 dx = \int_{-\lambda/4}^{0}2L_0\frac{V_{0+}^2}{Z_0^2}\frac{1+\cos 4\pi x/\lambda}{2}dx = -\frac{\lambda L_0 V_{0+}^2}{4Z_0^2}$$

The integral of electric energy over the last quarter wavelength is given by:

$$\int_{-\lambda/4}^{0}\frac{1}{2}C_0 V^2 dx = \int_{-\lambda/4}^{0}\frac{1}{2}C_0(2V_{0+}\sin kx)^2 dx = \int_{-\lambda/4}^{0}2C_0 V_{0+}^2\frac{1-\cos 4\pi x/\lambda}{2}dx = -\frac{\lambda C_0 V_{0+}^2}{4}$$

Noting that $Z_0 = \sqrt{L_0/C_0}$, we have:

$$\int_{-\lambda/4}^{0}\frac{1}{2}L_0 I^2 dx = \left|\frac{\lambda L_0 V_{0+}^2}{4Z_0^2}\right| = \left|\frac{\lambda C_0 V_{0+}^2}{4}\right| = \int_{-\lambda/4}^{0}\frac{1}{2}C_0 V^2 dx$$

7.4

The maximum of the magnetic energy is given by:

$$(E_m)_{\max} = \left(\frac{1}{2}L_0 I^2\right)_{\max} = \left[\frac{1}{2}L_0\left(\frac{2V_{0+}}{Z_0}\cos kx\right)^2\right]_{\max} = \frac{2L_0 V_{0+}^2}{Z_0^2} = 2C_0 V_{0+}^2$$

The maximum of the electric energy is given by:

$$(E_e)_{\max} = \left(\frac{1}{2}C_0 V^2\right)_{\max} = \left[\frac{1}{2}C_0(2V_{0+}\sin kx)^2\right]_{\max} = 2C_0 V_{0+}^2$$

The instantaneous value of the two energies over the last quarter wavelength is given by:

$$
\begin{aligned}
(E_m + E_e)_i &= \frac{1}{2} L_0 \left(\frac{2V_{0+}}{Z_0} \cos kx \right)^2 + \frac{1}{2} C_0 (2V_{0+}^2 \sin kx)^2 \\
&= 2C_0 V_{0+}^2 \cos^2 kx + 2C_0 V_{0+}^2 \sin^2 kx \\
&= 2C_0 V_{0+}^2
\end{aligned}
$$

So we have:

$$
(E_m)_{\max} = (E_e)_{\max} = (E_m + E_e)_i = 2C_0 V_{0+}^2
$$

7.5

For a real transmission line with a propagation constant γ, the forward current wave I_{x+} at position x is given by:

$$
I_{x+} = I_{0+} e^{-\gamma x} = A e^{-\gamma x}
$$

where $I_{0+} = A$ is the forward current wave at position $x = 0$. So the forward voltage wave at position x is given by:

$$
V_{x+} = Z_0 I_{x+} = Z_0 A e^{-\gamma x}
$$

The backward current wave I_{x-} at position x is given by:

$$
I_{x-} = I_{0-} e^{+\gamma x} = B e^{+\gamma x}
$$

where $I_{0-} = B$ is the backward current wave at position $x = 0$. So the backward voltage wave at position x is given by:

$$
V_{x-} = -Z_0 I_{x-} = -Z_0 B e^{+\gamma x}
$$

Therefore the impedance seen from position x is given by:

$$
Z_x = \frac{V_{x+} + V_{x-}}{I_{x+} + I_{x-}} = \frac{Z_0 A e^{-\gamma x} - Z_0 B e^{+\gamma x}}{A e^{-\gamma x} + B e^{+\gamma x}} = Z_0 \frac{A e^{-\gamma x} - B e^{+\gamma x}}{A e^{-\gamma x} + B e^{+\gamma x}}
$$

If the line has a length l and is terminated by a load Z_L, the value of Z_L is given by:

$$
Z_L = \frac{V_L}{I_L} = \frac{V_{l+} + V_{l-}}{I_{l+} + I_{l-}} = Z_0 \frac{A e^{-\gamma l} - B e^{+\gamma l}}{A e^{-\gamma l} + B e^{+\gamma l}}
$$

7.6

The impedance of the line at $x = 0$ is given by:

$$
Z_i = \left(Z_0 \frac{A^{-\gamma x} - B e^{+\gamma x}}{A e^{-\gamma x} + B e^{+\gamma x}} \right)_{x=0} = Z_0 \frac{A - B}{A + B}
$$

Noting that:

$$Z_L = Z_0 \frac{Ae^{-\gamma l} - Be^{\gamma l}}{Ae^{-\gamma l} + Be^{\gamma l}}$$

we have:

$$(Z_0 - Z_L)Ae^{-\gamma l} = (Z_0 + Z_L)Be^{\gamma l}$$

i.e.
$$\frac{A}{B} = \frac{(Z_0 + Z_L)}{(Z_0 - Z_L)} e^{2\gamma l}$$

so we have:

$$Z_i = Z_0 \frac{A/B - 1}{A/B + 1} = Z_0 \frac{Z_0(e^{\gamma l} - e^{-\gamma l}) + Z_L(e^{\gamma l} + e^{-\gamma l})}{Z_0(e^{\gamma l} + e^{-\gamma l}) + Z_L(e^{\gamma l} - e^{-\gamma l})} = Z_0 \frac{Z_0 \sinh \gamma l + Z_L \cosh \gamma l}{Z_0 \cosh \gamma l + Z_L \sinh \gamma l}$$

7.7

If the transmission line of Problem 7.6 is short-circuited, i.e. $Z_L = 0$, The expression of input impedance in Problem 7.6 gives:

$$Z_{sc} = Z_0 \frac{Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l} = Z_0 \tanh \gamma l$$

If the transmission line of Problem 7.6 is open-circuited, i.e. $Z_L = \infty$, The expression of input impedance in Problem 7.6 gives:

$$Z_{oc} = Z_0 \frac{Z_L \cosh \gamma l}{Z_L \sinh \gamma l} = Z_0 \coth \gamma l$$

By taking the product of these two impedances we have:

$$Z_{sc} Z_{oc} = Z_0^2, \text{ i.e. } Z_0 = \sqrt{Z_{sc} Z_{oc}}$$

which shows the characteristic impedance of the line can be obtained by measuring the impedances of short-circuited line and open-circuited line separately and then taking the square root of the product of the two values.

7.8

The forward and reflected voltage waves at the end of the line are given by:

$$V_{l+} = -V_{l-} = V_{0+} e^{-ikl}$$

where V_{0+} is the forward voltage at the beginning of the line. So the reflected voltage wave at the beginning of the line is given by:

$$V_{0-} = V_{l-} e^{-ikl} = -V_{0+} e^{-i2kl}$$

The forward and reflected current waves at the end of the line are given by:

$$I_{l+} = V_{l+} / Z_0 = V_{0+} e^{-ikl} / Z_0 = I_{0+} e^{-ikl}$$
$$I_{l-} = -V_{l-} / Z_0 = V_{l+} / Z_0 = I_{0+} e^{-ikl}$$

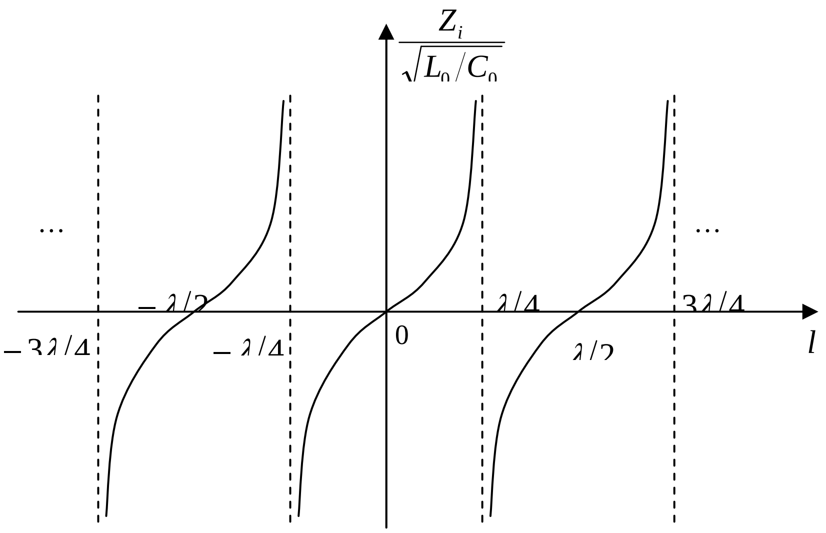
where I_{0+} is the forward current at the beginning of the line. So the reflected current wave at the beginning of the line is given by:

$$I_{0-} = I_{l-} e^{-ikl} = I_{0+} e^{-i2kl}$$

Therefore the input impedance of the line is given by:

$$Z_i = \frac{V_{0+} + V_{0-}}{I_{0+} + I_{0-}} = \frac{V_{0+}(1 - e^{-i2kl})}{I_{0+}(1 + e^{-i2kl})} = \frac{V_{0+}(e^{ikl} - e^{-ikl})}{I_{0+}(e^{ikl} + e^{-ikl})} = iZ_0 \frac{\sin kl}{\cos kl} = i\sqrt{\frac{L_0}{C_0}} \tan \frac{2\pi l}{\lambda}$$

The variation of the ratio $Z_i / \sqrt{L_0 / C_0}$ with l is shown in the figure below:



7.9

The boundary condition at $Z_0 Z_m$ junction gives:

$$V_{0+} + V_{0-} = V_{m0+} + V_{m0-}$$

$$I_{0+} + I_{0-} = I_{m0+} + I_{m0-}$$

where V_{0+}, V_{0-} are the voltages of forward and backward waves on Z_0 side of $Z_0 Z_m$ junction; I_{0+}, I_{0-} are the currents of forward and backward waves on Z_0 side of $Z_0 Z_m$ junction; V_{m0+}, V_{m0-} are the voltages of forward and backward waves on Z_m side of $Z_0 Z_m$

junction; I_{m0+}, I_{m0-} are the currents of forward and backward waves on Z_m side of $Z_0 Z_m$ junction;

The boundary condition at $Z_m Z_L$ junction gives:

$$V_{mL+} + V_{mL-} = V_L$$

$$I_{mL+} + I_{mL-} = I_L$$

where V_{mL+}, V_{mL-} are the voltages of forward and backward waves on Z_m side of $Z_m Z_L$ junction; I_{mL+}, I_{mL-} are the currents of forward and backward waves on Z_m side of $Z_m Z_L$ junction; V_L, I_L are the voltage and current across the load.

If the length of the matching line is l, we have:

$$V_{m0+} = V_{mL+} e^{ikl}$$

$$I_{m0+} = I_{mL+} e^{ikl}$$

$$V_{m0-} = V_{mL-} e^{-ikl}$$

$$I_{m0-} = I_{mL-} e^{-ikl}$$

In addition, we have the relations:

$$\frac{V_L}{I_L} = Z_L$$

$$\frac{V_0}{I_0} = Z_0$$

$$\frac{V_{m0+}}{I_{m0+}} = -\frac{V_{m0-}}{I_{m0-}} = \frac{V_{mL+}}{I_{mL+}} = -\frac{V_{mL-}}{I_{mL-}} = Z_m$$

The above conditions yield:

$$V_{mL+} = V_{m0+} e^{-ikl}$$

$$I_{mL+} = I_{m0+} e^{-ikl}$$

$$V_{mL-} = \frac{Z_L - Z_m}{Z_L + Z_m} V_{mL+}$$

$$I_{mL-} = \frac{Z_m - Z_L}{Z_m + Z_L} I_{mL+}$$

$$V_{m0-} = V_{mL-}e^{-ikl} = \frac{Z_L - Z_m}{Z_L + Z_m} V_{mL+}e^{-ikl} = V_{m0+}\frac{Z_L - Z_m}{Z_L + Z_m} e^{-i2kl}$$

$$I_{m0-} = I_{mL-}e^{-ikl} = \frac{Z_m - Z_L}{Z_L + Z_m} I_{mL+}e^{-ikl} = I_{m0+}\frac{Z_m - Z_L}{Z_L + Z_m} e^{-i2kl}$$

Impedance mating requires $V_{0-} = 0$ and $I_{0-} = 0$, i.e.:

$$V_{0+} = V_{m0+} + V_{m0-}$$

$$I_{0+} = I_{m0+} + I_{m0-}$$

i.e.

$$V_{0+} = V_{m0+}\left(1 + \frac{Z_L - Z_m}{Z_L + Z_m} e^{-i2kl}\right)$$

$$I_{0+} = I_{m0+}\left(1 + \frac{Z_m - Z_L}{Z_L + Z_m} e^{-i2kl}\right)$$

By dividing the above equations we have:

$$Z_0 = Z_m \frac{(Z_L + Z_m)e^{ikl} + (Z_L - Z_m)e^{-ikl}}{(Z_L + Z_m)e^{ikl} + (Z_m - Z_L)e^{-ikl}} = Z_m \frac{Z_L \cos kl + iZ_m \sin kl}{Z_L \sin kl + iZ_m \cos kl}$$

which is true if $kl = \pi/2$, or $l = \lambda/4$ and yields:

$$Z_m^2 = Z_0 Z_L$$

7.10

Analysis in Problem 7.8 shows the impedance of a short-circuited loss-free line has an impedance given by:

$$Z_i = iZ_0 \tan \frac{2\pi l}{\lambda}$$

so, if the length of the line is a quarter of one wavelength, we have:

$$Z_i = iZ_0 \tan \frac{2\pi}{\lambda}\frac{\lambda}{4} = iZ_0 \tan \frac{\pi}{2} = \infty$$

If this line is bridged across another transmission line, due to the infinite impedance, the transmission of fundamental wavelength λ will not be affected. However for the second harmonic wavelength $\lambda/2$, the impedance of the bridge line is given by:

$$Z_i = iZ_0 \tan \frac{2\pi}{\lambda/2} \frac{\lambda}{4} = iZ_0 \tan \pi = 0$$

which shows the bridge line short circuits the second harmonic waves.

7.11

For Z_0 to act as a high pass filter with zero attenuation, the frequency $\omega^2 > \frac{1}{2LC}$, where $Z_0 = \sqrt{L/C}$.

The exact physical length of Z_0 is determined by ω. Choosing the frequency ω_1 determines $k_1 = 2\pi/\lambda_1$.

For a high frequency load Z_L and a loss- free line, we have, for the input impedance:

$$Z_{in} = Z_0 \left(\frac{Z_L \cos kl + iZ_0 \sin kl}{Z_0 \cos kl + iZ_L \sin kl} \right)$$

For n even, we have:

$$\cos k_1 l = \cos \frac{2\pi}{\lambda_1} \frac{n\lambda_1}{2} = \cos n\pi = 1$$

For n odd, we have:

$$\cos k_1 l = \cos \frac{2\pi}{\lambda_1} \frac{n\lambda_1}{2} = \cos n\pi = -1$$

The sine terms are zero.

So $Z_{in} = Z_L$ for n odd or even, and the high frequency circuits, input and load, are uniquely matched at ω_1 when the circuits are tuned to ω_1.

7.12

The phase shift per section β should satisfy:

$$\cos \beta = 1 + \frac{Z_1}{2Z_2} = 1 + \frac{i\omega L}{2/i\omega C} = 1 - \frac{\omega^2 LC}{2}$$

i.e. $$1 - \cos \beta = \frac{\omega^2 LC}{2}$$

i.e. $$2 \sin^2 \frac{\beta}{2} = \frac{\omega^2 LC}{2}$$

For a small β, $\beta \approx \sin\beta$, so the above equation becomes:

$$2\left(\frac{\beta}{2}\right)^2 = \frac{\omega^2 LC}{2}$$

i.e. $$\beta = \omega\sqrt{LC} = \omega/v = k$$

where the phase velocity is given by $v = 1/\sqrt{LC}$ and is independent of the frequency.

7.13

The propagation constant γ can be expanded as:

$$\begin{aligned}\gamma &= \sqrt{(R_0 + i\omega L_0)(G_0 + i\omega C_0)}\\ &= \omega\sqrt{L_0C_0}\sqrt{\left(\frac{R_0}{\omega L_0}+i\right)\left(\frac{G_0}{\omega C_0}+i\right)}\\ &= \omega\sqrt{L_0C_0}\sqrt{i^2+\left(\frac{R_0}{\omega L_0}+\frac{G_0}{\omega C_0}\right)i+\frac{R_0}{\omega L_0}\frac{G_0}{\omega C_0}}\\ &= \omega\sqrt{L_0C_0}\sqrt{\left(i+\frac{R_0}{2\omega L_0}+\frac{G_0}{2\omega C_0}\right)^2-\left(\frac{R_0^2}{4\omega^2L_0^2}+\frac{G_0^2}{4\omega C_0^2}+\frac{R_0}{\omega L_0}\frac{G_0}{\omega C_0}\right)}\end{aligned}$$

Since $R_0/\omega L_0$ and $G_0/\omega C_0$ are both small quantities, the above equation becomes:

$$\gamma = \omega\sqrt{L_0C_0}\left(i+\frac{R_0}{2\omega L_0}+\frac{G_0}{2\omega C_0}\right) = \alpha + ik$$

where $\alpha = \frac{R_0}{2}\sqrt{\frac{C_0}{L_0}}+\frac{G_0}{2}\sqrt{\frac{L_0}{C_0}}$, and $k = \omega\sqrt{L_0C_0} = \omega/v$

If $G = 0$, we have:

$$\frac{k}{2\alpha} = \frac{\omega\sqrt{L_0C_0}}{R_0\sqrt{C_0/L_0}+G_0\sqrt{L_0/C_0}} = \frac{\omega\sqrt{L_0C_0}}{R_0\sqrt{C_0/L_0}} = \frac{\omega L_0}{R_0}$$

which is the Q value of this transmission line.

7.14

Suppose $\frac{R_0}{L_0} = \frac{G_0}{C_0} = K$, where K is constant, the characteristic impedance of a lossless line is given by:

$$Z_0 = \sqrt{\frac{R_0 + i\omega L_0}{G_0 + i\omega C_0}} = \sqrt{\frac{KL_0 + i\omega L_0}{KC_0 + i\omega C_0}} = \sqrt{\frac{L_0}{C_0}}$$

which is a real value.

7.15

Try solution $\psi = \psi_m e^{\gamma x}$ in wave equation:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{8\pi^2 m}{h^2}(E - V)\psi = 0$$

we have:

$$\gamma^2 = \frac{8\pi^2 m}{h^2}(V - E)$$

For $E > V$ (inside the potential well), the value of γ is given by:

$$\gamma_{in} = \pm i \frac{2\pi}{h}\sqrt{2m(E - V)}$$

So the ψ has a standing wave expression given by:

$$\psi = Ae^{i\frac{2\pi}{h}\sqrt{m(V-E)}x} + Be^{-i\frac{2\pi}{h}\sqrt{m(V-E)}x}$$

where A, B are constants.

For $E < V$ (outside the potential well), the value of γ is given by:

$$\gamma_{out} = \pm \frac{2\pi}{h}\sqrt{2m(V - E)}$$

So the expression of ψ is given by:

$$\psi = Ae^{\frac{2\pi}{h}\sqrt{m(V-E)}x} + Be^{-\frac{2\pi}{h}\sqrt{m(V-E)}x}$$

where A, B are constants. i.e. the x dependence of ψ is $e^{\pm\gamma x}$, where

$$\gamma = \frac{2\pi}{h}\sqrt{2m(V - E)}$$

7.16

Form the diffusion equation:

$$\frac{\partial H}{\partial t} = \frac{1}{\mu\sigma}\frac{\partial^2 H}{\partial x^2}$$

we know the diffusivity is given by: $d = 1/\mu\sigma$. The time of decay of the field is approximately given by Einstein's diffusivity relation:

$$t = \frac{L^2}{d} = \frac{L^2}{1/\mu\sigma} = L^2\mu\sigma$$

where L is the extent of the medium.

For a copper sphere of radius $1m$, the time of decay of the field is approximately given by:

$$t = L^2 \mu\sigma = 1^2 \times 1.26 \times 10^{-6} \times 5.8 \times 10^7 \approx 73[s] < 100[s]$$

7.17

Try solution $f(\alpha,t) = \frac{r}{\sqrt{\pi}} e^{-(r\alpha)^2}$ in $\frac{\partial f(\alpha,t)}{\partial t}$, we have:

$$\begin{aligned}\frac{\partial f(\alpha,t)}{\partial t} &= \frac{\partial}{\partial t}\left[\frac{r}{\sqrt{\pi}} e^{-(r\alpha)^2}\right] \\ &= \frac{r}{\sqrt{\pi}} e^{-(r\alpha)^2}(-2r\alpha)\alpha\frac{dr}{dt} + \frac{1}{\sqrt{\pi}}\frac{dr}{dt} e^{-(r\alpha)^2} \\ &= \frac{1-2r^2\alpha^2}{\sqrt{\pi}} e^{-(r\alpha)^2}\frac{dr}{dt} \\ &= \frac{2r^2\alpha^2 - 1}{4t\sqrt{\pi dt}} e^{-(r\alpha)^2}\end{aligned} \qquad \text{(A.7.17.1)}$$

Try solution $f(\alpha,t) = \frac{r}{\sqrt{\pi}} e^{-(r\alpha)^2}$ in $\frac{\partial f(\alpha,t)}{\partial x}$, we have:

$$\frac{\partial f(\alpha,t)}{\partial x} = -\frac{2r^3\alpha}{\sqrt{\pi}} e^{-(r\alpha)^2}$$

so:

$$\begin{aligned}\frac{\partial^2 f(\alpha,t)}{\partial x^2} &= -\frac{2r^3}{\sqrt{\pi}} e^{-(r\alpha)^2} - \frac{2r^3\alpha}{\sqrt{\pi}} e^{-(r\alpha)^2}(-2r\alpha)r \\ &= -\frac{2r^3}{\sqrt{\pi}}(1-2r^2\alpha^2)e^{-(r\alpha)^2} \\ &= \frac{2r^2\alpha^2 - 1}{4td\sqrt{\pi dt}} e^{-(r\alpha)^2}\end{aligned}$$

(A.7.17.2)

By comparing the above derivatives, A.7.17.1 and 2, we can find the solution

$$f(\alpha,t) = \frac{r}{\sqrt{\pi}} e^{-(r\alpha)^2}$$

satisfies the equation:

$$\frac{\partial f}{\partial t} = d\frac{\partial^2 f}{\partial x^2}$$

SOLUTIONS TO CHAPTER 8

8.1

Write the expressions of E_x and H_y as:

$$E_x = E_0 \sin \frac{2\pi}{\lambda_E}(v_E t - z)$$
$$H_y = H_0 \sin \frac{2\pi}{\lambda_H}(v_H t - z)$$

where λ_E and λ_H are the wavelengths of electric and magnetic waves respectively,

and v_E and v_H are the velocities of electric and magnetic waves respectively.

By substitution of the these expressions into equation (8.1a), we have:

$$-\mu \frac{2\pi}{\lambda_H} v_H H_0 \cos \frac{2\pi}{\lambda_H}(v_H t - z) = -\frac{2\pi}{\lambda_E} E_0 \cos \frac{2\pi}{\lambda_E}(v_E t - z)$$

i.e.

$$\mu \frac{v_H H_0}{\lambda_H} \cos \frac{2\pi}{\lambda_H}(v_H t - z) = \frac{E_0}{\lambda_E} \cos \frac{2\pi}{\lambda_E}(v_E t - z)$$

which is true for all t and z, provided:

$$v_H = v_E = \frac{E_0}{\mu H_0}$$

and

$$\lambda_H = \lambda_E$$

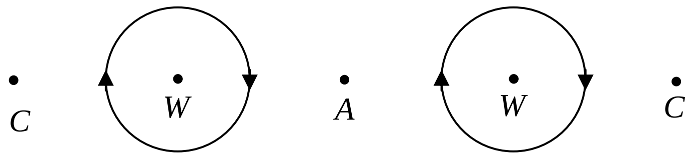
so, at any t and z, we have:

$$\phi_E = \phi_H = \frac{E_0}{\mu H_0} t - z$$

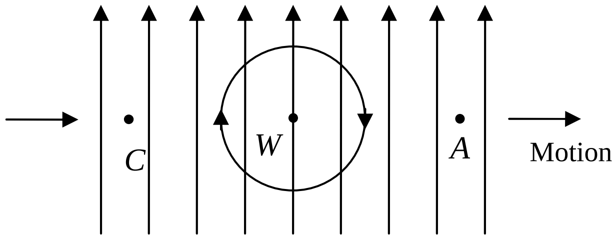
Therefore E_x and H_y have the same wavelength and phase.

8.2

$$\frac{\text{Energy}}{\text{Volume}} = \frac{\text{Force} \cdot \text{Distance}}{L^3} = \frac{\text{Force}}{\text{Area}} = \text{pressure}$$



Currents in W into page. Field lines at A cancel. Those at C force wires together.
Reverse current in one wire. Field lines at A in same direction, force wires apart.
Fig Q.8.2.a



Field lines at C in same direction as those from current in wire –
in opposite direction at A. Motion to the right
Fig Q.8.2.b

8.3

The volume of a thin shell of thickness dr is given by: $4\pi r^2 dr$, so the electrostatic energy over the spherical volume from radius a to infinity is given by: $\int_a^{+\infty} \frac{1}{2}\varepsilon_0 E^2 (4\pi r^2) dr$, which equals mc^2, i.e.:

$$\int_a^{+\infty} \frac{1}{2}\varepsilon_0 E^2 (4\pi r^2) dr = mc^2$$

By substitution of $E = e/4\pi\varepsilon_0 r^2$ into the above equation, we have:

$$\int_a^{+\infty} \frac{1}{2}\varepsilon_0 \frac{e^2}{(4\pi\varepsilon_0 r^2)^2} (4\pi r^2) dr = mc^2$$

i.e. $$\frac{e^2}{8\pi\varepsilon_0} \int_a^{+\infty} \frac{1}{r^2} dr = mc^2$$

i.e. $$\frac{e^2}{8\pi\varepsilon_0 a} = mc^2$$

Then, the value of radius a is given by:

$$a = \frac{e^2}{8\pi\varepsilon_0 mc^2} = \frac{(1.6\times10^{-19})^2}{8\pi\times8.8\times10^{-12}\times9.1\times10^{-31}\times(3\times10^8)^2} \approx 1.41\times10^{-15}[m]$$

Another approach to the problem yields the value:

$$a = 2.82\times10^{-15}[m]$$

8.4

The magnitude of Poynting vector on the surface of the wire can be calculated by deriving the electric and magnetic fields respectively.

The vector of magnetic field on the surface of the cylindrical wire points towards the azimuthal direction, and its magnitude is given by Ampere's Law:

$$\mathbf{H} = H_\theta \mathbf{e}_\theta = \frac{I}{2\pi r}\mathbf{e}_\theta$$

where r is the radius of the wire's cross circular section, and I is the current in the wire.

Ohm's Law, $\mathbf{J} = \sigma\mathbf{E}$, shows the vector of electric field on the surface of the cylindrical wire points towards the current's direction, and its magnitude equals the voltage drop per unit length, i.e.:

$$\mathbf{E} = E_z \mathbf{e}_z = \frac{V}{l}\mathbf{e}_z = \frac{IR}{l}\mathbf{e}_z$$

where, l is the length of the wire, and the V is the voltage drop along the whole length of the wire and is given by Ohm's Law: $V = IR$, where R is the resistance of the wire.

Hence, the Poynting vector on the surface of the wire points towards the axis of the wire is given by:

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = E_z\mathbf{e}_z \times H_\theta \mathbf{e}_\theta = -E_z H_\theta \mathbf{e}_r$$

which shows the Poynting vector on the surface of the wire points towards the axis of the wire, which corresponds to the flow of energy into the wire from surrounding space. The product of its magnitude and the surface area of the wire is given by:

$$S\times 2\pi rl = E_z H_\theta \times 2\pi rl = \frac{IR}{l}\frac{I}{2\pi r}2\pi rl = I^2 R$$

which is the rate of generation of heat in the wire.

8.5

By relating Poynting vector to magnetic energy, we first need to derive the magnitude of Poynting vector in terms of magnetic field.

The electric field on the inner surface of the solenoid can be derived from the integral format of Faraday's Law:

$$\oint Edl = -\mu\oint \frac{\partial H}{\partial t}dS$$

where S is the area of the solenoid's cross section. E is the electric field on the inner surface of the solenoid, and H is the magnetic field inside the solenoid.

For a long uniformly wound solenoid the electric field uniformly points towards azimuthal direction, i.e. $\mathbf{E} = E_\theta \mathbf{e}_\theta$, and the magnetic field inside the solenoid uniformly points along the axis direction, i.e. $\mathbf{H} = H_z\mathbf{e}_z$. So the above equation becomes

$$E_\theta \times 2\pi r = -\mu \frac{\partial H_z}{\partial t}\pi r^2$$

i.e.
$$E_\theta = -\frac{\mu r}{2}\frac{\partial H_z}{\partial t}$$

where r is the radius of the cross section of the solenoid.
Hence, the Poynting vector on the inner surface of the solenoid is given by:

$$\mathbf{E}\times\mathbf{H} = E_\theta\mathbf{e}_\theta \times H_z\mathbf{e}_z = -\frac{\mu r}{2}H_z\frac{\partial H_z}{\partial t}\mathbf{e}_r$$

which points towards the axis of the solenoid and corresponds to the inward energy flow. The product of its magnitude and the surface area of the solenoid is given by:

$$S\times 2\pi r l = \frac{\mu r}{2}H_z\frac{\partial H_z}{\partial t}\times 2\pi r l = \pi r^2 l H_z\frac{\partial H_z}{\partial t}$$

where l is the length of the solenoid.
On the other hand, the time rate of change of magnetic energy stored in the solenoid of a length l is given by:

$$\frac{d}{dt}\left(\frac{1}{2}\mu H^2\times\pi r^2 l\right) = \pi r^2 l H_z\frac{\partial H_z}{\partial t}$$

which equals $S\times 2\pi r l$

8.6

For plane polarized electromagnetic wave (E_x, H_y) in free space, we have the relation:

$$\frac{E_x}{H_y} = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

Its Poynting vector is given by:

$$S = E_x H_y = E_x\frac{E_x}{\sqrt{\mu_0/\varepsilon_0}} = \sqrt{\frac{\varepsilon_0}{\mu_0}}E_x^2 = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\varepsilon_0 E_x^2 = c\varepsilon_0 E_x^2$$

where $c = 1\big/\sqrt{\mu_0\varepsilon_0}$ is the velocity of light.

Noting that:

$$\frac{1}{2}\varepsilon_0 E_x^2 = \frac{1}{2}\varepsilon_0\left(\sqrt{\frac{\mu_0}{\varepsilon_0}}H_y\right)^2 = \frac{1}{2}\mu_0 H_y^2$$

we have:

$$S = E_x H_y = c\left(\frac{1}{2}\varepsilon_0 E_x^2 + \frac{1}{2}\mu_0 H_y^2\right) = c\varepsilon_0 E_x^2$$

Since the intensity in such a wave is given by:

$$I = \bar{S}_{av} = c\varepsilon_0 \overline{E^2} = \frac{1}{2} c\varepsilon_0 E_{\max}^2$$

we have:

$$\bar{S} = \frac{1}{2} \times 3\times 10^8 \times 8.8\times 10^{-12} \times E_{\max}^2 \approx 1.327\times 10^{-3} E_{\max}^2$$

$$E_{\max} = \sqrt{\frac{2}{c\varepsilon_0}} \bar{S}^{1/2} = \sqrt{\frac{2}{3\times 10^8 \times 8.8\times 10^{-12}}} \bar{S}^{1/2} \approx 27.45 \bar{S}^{1/2} [Vm^{-1}]$$

$$H_{\max} = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_{\max} = \sqrt{\frac{2}{c\mu_0}} \bar{S}^{1/2} = \sqrt{\frac{2}{3\times 10^8 \times 4\pi \times 10^{-7}}} \bar{S}^{1/2} \approx 7.3\times 10^{-2} \bar{S}^{1/2} [Am^{-1}]$$

8.7

The average intensity of the beam and is given by:

$$\bar{I} = \frac{\text{Power}}{\text{area}} = \frac{\text{Energy}}{\text{area} \times \text{pulse duration}} = \frac{0.3}{\pi \times (2.5\times 10^{-3})^2 \times 10^{-4}} = 1.53\times 10^8 [Wm^{-2}]$$

Using the result in Problem 8.6, the root mean square value of the electric field in the wave is given by:

$$\sqrt{\overline{E^2}} = \sqrt{\frac{\bar{I}}{c\varepsilon_0}} = \sqrt{\frac{1.53\times 10^8}{3\times 10^8 \times 8.8\times 10^{-12}}} \approx 2.4\times 10^5 [Vm^{-1}]$$

8.8

Using the result of Problem 8.6, the amplitude of the electric field at the earth's surface is given by:

$$E_0 = 27.45 \bar{S}^{1/2} = 27.45 \times \sqrt{1350} \approx 1010 [Vm^{-1}]$$

and the amplitude of the associated magnetic field in the wave is given by:

$$H_0 = 7.3\times 10^{-2} \times \sqrt{1350} \approx 2.7 [Am^{-1}]$$

The radiation pressure of the sunlight upon the earth equals the sum of the electric field energy density and the magnetic field energy density, i.e.

$$p_{rad} = \frac{1}{2}\varepsilon_0 E_0^2 + \frac{1}{2}\mu_0 H_0^2 = \varepsilon_0 E_0^2 = 8.8\times 10^{-12} \times 1010^2 = 8.98\times 10^{-6} [Pa]$$

8.9

The total radiant energy loss per second of the sun is given by:

$$E_{loss} = S \times 4\pi r^2 = 1350 \times 4\pi \times (15\times 10^{10})^2 = 3.82\times 10^{26} [J]$$

which is associated with a mass of:

$$m = E_{loss} / c^2 = \frac{3.82\times 10^{26}}{(3\times 10^8)^2} = 4.2\times 10^9 [kg]$$

8.10

At a point $10km$ from the station, the Poynting vector is given by:

$$S = \frac{P}{2\pi r^2} = \frac{10^5}{2\times\pi\times(10\times10^3)^2} = 1.6\times10^{-4}[W/m^2]$$

Using the result in Problem 8.6, the amplitude of electric field is given by:

$$E_0 = 27.45\times S^{1/2} = 27.45\times\sqrt{1.6\times10^{-4}} = 0.346[V/m]$$

The amplitude of magnetic field is given by:

$$H_0 = 7.3\times10^{-2}S^{1/2} = 7.3\times10^{-2}\times\sqrt{1.6\times10^{-4}} = 9.2\times10^{-4}[A/m]$$

8.11

The surface current in the strip is given by:

$$I = Qv$$

where Q is surface charge per unit area on the strip and is given by: $Q = \varepsilon E_x$, and v is the velocity of surface charges along the transmission line.

Since the surface charges change along the transmission line at the same speed as the electromagnetic wave travels, i.e. $v = c = \frac{1}{\sqrt{\mu\varepsilon}}$, the surface current becomes:

$$I = Qv = \varepsilon E_x \frac{1}{\sqrt{\mu\varepsilon}} = \sqrt{\frac{\varepsilon}{\mu}}E_x$$

Analysis in page 207 shows, for plane electromagnetic wave, $\sqrt{\mu}H_y = \sqrt{\varepsilon}E_x$, so the surface current is now given by:

$$I = \sqrt{\frac{\varepsilon}{\mu}}\sqrt{\frac{\mu}{\varepsilon}}H_y = H_y$$

On the other hand, the voltage across the two strips is given by:

$$V = E_x L = E_x$$

where $L = 1$ is the distance between the two strips.

Therefore the characteristic impedance of the transmission line is given by:

$$Z = \frac{V}{I} = \frac{E_x}{H_y} = \sqrt{\frac{\mu}{\varepsilon}}$$

8.12

Write equation (8.6) in form:

$$\frac{\partial^2}{\partial z^2}E_x = \mu\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\varepsilon E_x\right) + \mu\frac{\partial}{\partial t}(\sigma E_x)$$

which can be dimensionally expressed as:

$$\frac{\text{voltage/length}}{\text{length}\times\text{length}} = \frac{\text{inductance}}{\text{length}}\times\frac{\text{displacement current}}{\text{time}\times\text{area}} + \frac{\text{inductance}}{\text{length}}\times\frac{\text{current/area}}{\text{time}}$$

Multiplied by a dimension term, length, the above equation has the dimension:

$$\frac{\text{voltage}}{\text{area}} = \text{inductance}\times\frac{\text{current}}{\text{time}\times\text{area}}$$

which is the dimensional form (per unit area) of the equation:

$$V = L\frac{dI}{dt}$$

where V is a voltage, L is a inductance and I is a current.

8.13

Analysis in page 210 and 211 shows, in a conducting medium, the wave number of electromagnetic wave is given by:

$$k = \sqrt{\frac{\omega\mu\sigma}{2}}$$

where ω is angular frequency of the electromagnetic wave, μ and σ are the permeability and conductivity of the conducting medium.

Differentiation of the above equation gives:

$$dk = \frac{1}{2}\sqrt{\frac{2}{\omega\mu\sigma}}\frac{\mu\sigma}{2}d\omega = \frac{1}{2}\sqrt{\frac{\mu\sigma}{2\omega}}d\omega$$

i.e.

$$\frac{d\omega}{dk} = 2\sqrt{\frac{2\omega}{\mu\sigma}} = 2\sqrt{\frac{2}{\omega\mu\sigma}}\omega = 2\frac{\omega}{k}$$

which shows, when a group of electromagnetic waves of nearly equal frequencies propagates in a conducting medium, where the group velocity and the phase velocity can be treated as fixed values, the group velocity, $v_g = d\omega/dk$, is twice the wave velocity, $v_p = \omega/k$.

8.14

(a) $\frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{2\pi\nu\varepsilon_r\varepsilon_0} = \frac{0.1}{2\pi\times 50\times 10^3\times 50}\times 36\pi\times 10^9 = 720 > 100$

which shows, at a frequency of $50kHz$, the medium is a conductor

(b) $\frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{2\pi\nu\varepsilon_r\varepsilon_0} = \frac{0.1}{2\pi\times10^4\times10^6\times50}\times36\pi\times10^9 = 3.6\times10^{-3} < 10^{-2}$

which shows, at a frequency of $10^4 MHz$, the medium is a dielectric.

8.15

The Atlantic Ocean is a conductor when:

$$\frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{2\pi\nu\varepsilon_r\varepsilon_0} > 100$$

i.e. $$\nu < \frac{\sigma}{2\pi\times100\times\varepsilon_r\varepsilon_0} = \frac{4.3}{2\pi\times100\times81}\times36\pi\times10^9 \approx 10[MHz]$$

Therefore the longest wavelength that could propagate under water is given by:

$$\nu = \frac{v}{\lambda_{\max}} = \frac{c}{\sqrt{\varepsilon_r}\lambda_{\max}} = 10\times10^6$$

i.e. $$\lambda_{\max} = \frac{c}{\sqrt{\varepsilon_r}\times10\times10^6} = \frac{3\times10^8}{\sqrt{81}\times10\times10^6} \approx 3[m]$$

8.16

When a plane electromagnetic wave travelling in air with an impedance of Z_{air} is reflected normally from a plane conducting surface with an impedance of Z_c, the transmission coefficient of magnetic field is given by:

$$T_H = \frac{H_t}{H_i}$$

Using the relations: $E_t = Z_c H_t$, $E_i = Z_{air} H_i$, and $\frac{E_t}{E_i} = \frac{2Z_c}{Z_c + Z_{air}}$, the above equation becomes:

$$T_H = \frac{H_t}{H_i} = \frac{E_t Z_{air}}{E_i Z_c} = \frac{Z_{air}}{Z_c}\frac{2Z_c}{Z_c + Z_{air}} = \frac{2Z_{air}}{Z_c + Z_{air}}$$

The impedance of a good conductor tends to zero, i.e. $Z_c \to 0$, so we have:

$$T_H \approx \frac{2Z_{air}}{Z_{air}} = 2 \quad \text{or} \quad H_t \approx 2H_i$$

After reflection from the air-conductor interface, standing waves are formed in the air with a magnitude of $H_i + H_r$ in magnetic field and a magnitude of $E_i + E_r$ in electric field.

Using the relations: $H_i + H_r = H_t$, $E_i + E_r = E_t$, the standing wave ratio of magnetic field to electric field in air is given by:

$$\frac{H_i + H_r}{E_i + E_r} = \frac{H_t}{E_t} = \frac{1}{Z_c}$$

which is a large quantity due to $Z_c \to 0$.

As an analogy, for a short-circuited transmission line, the relation between forward (incident) and backward (reflected) voltages is given by: $V_+ + V_- = 0$ or $V_i + V_r = 0$, the forward (incident) current is given by: $I_i = I_+ = V_+ / Z_0$, and the backward (reflected) current is given by: $I_r = I_- = -V_- / Z_0$, so the transmitted current is given by:

$$I_t = I_i + I_r = \frac{V_+}{Z_0} - \frac{V_-}{Z_0} = \frac{V_+}{Z_0} - \frac{-V_+}{Z_0} = 2\frac{V_+}{Z_0} = 2I_i$$

When a plane electromagnetic wave travelling in a conductor with an impedance of Z_c is reflected normally from a plane conductor-air interface, the transmission coefficient of electric field is given by:

$$T_E = \frac{E_t}{E_i} = \frac{2Z_{air}}{Z_c + Z_{air}}$$

The impedance of a good conductor tends to zero, i.e. $Z_c \to 0$, so we have:

$$T_E \approx \frac{2Z_{air}}{Z_{air}} = 2 \quad \text{or} \quad E_t \approx 2E_i$$

As an analogy, for a open-circuited transmission line, the forward (incident) voltage equals the backward (reflected) voltages, i.e. $V_i = V_+ = V_r = V_-$, the transmitted voltage is given by:

$$V_t = V_i + V_r = V_+ + V_+ = 2V_+$$

8.17

Analysis in page 215 and 216 shows, in a conductor, magnetic field H_y lags electric field E_x by a phase angle of $\phi = 45^\circ$, so we can write the electric field and magnetic field in a conductor as:

$$E_x = E_0 \cos \omega t \quad \text{and} \quad H_y = H_0 \cos(\omega t - \phi)$$

so the average value of the Poynting vector is the integral of the Poynting vector $E_x H_y$ over one time period T divided by the time period, i.e.:

$$\begin{aligned} S_{av} &= \frac{1}{T}\int_0^T E_x H_y \\ &= \frac{1}{T}\int_0^T E_0 \cos \omega t H_0 \cos(\omega t - \phi) dt \\ &= \frac{E_0 H_0}{T}\int_0^T \frac{1}{2}[\cos(2\omega t - \phi) + \cos\phi] dt \\ &= \frac{1}{2}\frac{E_0 H_0}{T} T \cos\phi = \frac{1}{2} E_0 H_0 \cos 45^\circ [Wm^2] \end{aligned}$$

Noting that the real part of impedance of the conductor is given by:

$$(\text{real part of } Z_c) = \frac{E_0}{H_0}\cos\phi = \frac{E_0}{H_0}\cos 45^\circ$$

i.e.

$$E_0 = \frac{H_0}{\cos 45^\circ} \times (\text{real part of } Z_c)$$

so we have:

$$\begin{aligned} S_{av} &= \frac{1}{2} E_0 H_0 \cos 45^\circ \\ &= \frac{1}{2}\frac{H_0^2}{\cos 45^\circ} \times \text{real part of } Z_c \times \cos 45^\circ \\ &= \frac{1}{2} H_0^2 \times (\text{real part of } Z_c)[Wm^2] \end{aligned}$$

We know from analysis in page 216 that, at a frequency $\nu = 3000MHz$, the value of $\omega\varepsilon/\sigma$ for copper is 2.9×10^{-9}, hence, at of frequency of $1000MHz$, the value of $\omega\varepsilon/\sigma$ for copper is given by $2.9 \times 10^{-9}/3 = 9.7 \times 10^{-10}$, and $\mu_r \approx \varepsilon_r \approx 1$. So, the real part of impedance of the large copper sheet is given by:

$$\begin{aligned} &(\text{real part of } Z_{copper}) = \frac{\sqrt{2}}{2}\left|Z_{copper}\right| \\ &= \frac{\sqrt{2}}{2} \times 376.6 \times \sqrt{\frac{\mu_r}{\varepsilon_r}}\sqrt{\frac{\omega\varepsilon}{\sigma}} = 376.6 \times \frac{\sqrt{2}}{2} \times \sqrt{9.7 \times 10^{-10}} = 8.2 \times 10^{-3}[\Omega] \end{aligned}$$

Noting that, at an air-conductor interface, the transmitted magnetic field in copper H_{copper} doubles the incident magnetic field H_0, i.e. $H_{copper} = 2H_0$, the average power absorbed by the copper per square metre is the average value of transmitted Poynting vector, which is given by:

$$\overline{S_{copper}} = \frac{1}{2} H_{copper}^2 \times (\text{real part of } Z_{copper})$$
$$= \frac{1}{2}(2H_{copper})^2 \times (\text{real part of } Z_{copper})$$
$$= 2H_{copper}^2 \times (\text{real part of } Z_{copper})$$
$$= 2 \times \left(\frac{E_0}{376.6} \right)^2 \times (\text{real part of } Z_{copper})$$
$$= 2 \times \left(\frac{1}{376.6} \right)^2 \times 8.2 \times 10^{-3} = 1.16 \times 10^{-7} [W]$$

8.18

Analysis in page 222 and 223 shows that when an electromagnetic wave is reflected normally from a conducting surface its reflection coefficient I_r is given by:

$$I_r = 1 - 2\sqrt{\frac{2\omega\varepsilon_0}{\sigma}}$$

Noting that $\varepsilon_r = 1$, the fractional loss of energy is given by:

$$1 - I_r = 1 - \left(1 - 2\sqrt{\frac{2\omega\varepsilon_0}{\sigma}} \right) = \sqrt{\frac{8\omega\varepsilon_0}{\sigma}} = \sqrt{\frac{8\omega\,\varepsilon/\varepsilon_r}{\sigma}} = \sqrt{\frac{8\omega\varepsilon}{\sigma}}$$

8.19

Following the discussion of solution to problem 8.17, we can also find the average value of Poynting vector in air.

The electric and magnetic field of plane wave in air have the same phase, so the Poynting vector in air is given by:

$$S_{air} = E_x H_y = E_0 \cos\omega t \times H_0 \cos\omega t = E_0 H_0 \cos^2 \omega t$$

and its average value is given by:

$$\overline{S_{air}} = \frac{1}{T}\int_0^T E_x H_y$$
$$= \frac{1}{T}\int_0^T E_0 \cos\omega t H_0 \cos\omega t dt$$
$$= \frac{E_0 H_0}{T}\int_0^T \frac{1}{2}[1 + \cos(2\omega t)]dt$$
$$= \frac{1}{2}\frac{E_0 H_0}{T} T = \frac{1}{2} E_0 H_0 = \frac{1}{2} \times \frac{E_0^2}{376.6} = \frac{1}{2 \times 376.6} = 1.33 \times 10^{-3} [Wm^{-1}]$$

So, the ratio of transmitted Poynting vector in copper to the incident Poynting vector in air is given by:

$$\frac{\overline{S_{copper}}}{\overline{S_{air}}} = \frac{1.16\times10^{-7}}{1.33\times10^{-3}} = 8.81\times10^{-5}$$

which equals the fractional loss of energy I_r given by:

$$I_r = \sqrt{\frac{8\omega\varepsilon}{\sigma}} = \sqrt{8\times9.7\times10^{-10}} = 8.81\times10^{-5}$$

8.20

E_x and H_y are in complex expression, we have:

$$\frac{1}{2}E_x H_y^* = \frac{1}{2}Ae^{-kz}e^{i(\omega t-kz)}A\left(\frac{\sigma}{\omega\mu}\right)^{1/2}e^{-kz}e^{-i(\omega t-kz)}e^{i\pi/4}$$
$$= \frac{1}{2}A^2\left(\frac{\sigma}{\omega\mu}\right)^{1/2}e^{-2kz}e^{i\pi/4}$$

So, the average value of the Poynting vector in the conductor is given by:

$$S_{av} = \text{real part of } \left(\frac{1}{2}E_x H_y^*\right) = \frac{1}{2}A^2\left(\frac{\sigma}{2\omega\mu}\right)^{1/2}e^{-2kz}[Wm^{-2}]$$

The mean value of the electric field vector, $\overline{E_x}$, is a constant value, which contributes to the same electric energy density at the same amount of time, i.e.:

$$\text{(average electric energy density)} = \frac{1}{2}\varepsilon\overline{E_x}^2 = \frac{1}{T}\int_0^T \frac{1}{2}\varepsilon E_x^2 dt$$

i.e.

$$\overline{E_x}^2 = A^2e^{-2kz}\frac{1}{T}\int_0^T \cos^2\omega t dt = A^2e^{-2kz}\frac{1}{T}\int_0^T \frac{1+\cos 2\omega t}{2}dt = \frac{A^2e^{-2kz}}{2}$$

or: $$\overline{E_x} = \frac{\sqrt{2}}{2}Ae^{-kz}$$

Noting that:

$$\frac{\partial S_{av}}{\partial z} = -2k\times\frac{1}{2}A^2\left(\frac{\sigma}{2\omega\mu}\right)^{1/2}e^{-2kz}$$
$$= -A^2\left(\frac{\omega\mu\sigma}{2}\right)^{1/2}\left(\frac{\sigma}{2\omega\mu}\right)^{1/2}e^{-2kz}$$
$$= -\frac{A^2\sigma}{2}e^{-2kz} = -\sigma\overline{E_x}^2$$

we find the value of $\partial S_{av}/\partial z$ is the product of the conductivity σ and the square of the mean value of the electric field vector $\overline{E_x}$. The negative sign in the above equation shows the energy is decreasing with distance.

8.21

Noting that the relation between refractive index n of a dielectric and its impedance Z_d is given by: $n = \dfrac{Z_0}{Z_d}$, where Z_0 is the impedance in free space, so, when light travelling in free space is normally incident on the surface of a dielectric, the reflected intensity is given by:

$$I_r = \left(\frac{E_r}{E_i}\right)^2 = \left(\frac{Z_d - Z_0}{Z_d + Z_0}\right)^2 = \left(\frac{1 - Z_0/Z_d}{1 + Z_0/Z_d}\right)^2 = \left(\frac{1-n}{1+n}\right)^2$$

and the transmitted intensity is given by:

$$I_t = \frac{Z_0 E_t^2}{Z_d E_i^2} = \frac{Z_0}{Z_d}\left(\frac{2Z_d}{Z_d + Z_0}\right)^2 = \frac{Z_0}{Z_d}\left(\frac{2}{1 + Z_0/Z_d}\right)^2 = n\left(\frac{2}{1+n}\right)^2 = \frac{4n}{(1+n)^2}$$

8.22

If the dielectric is a glass $(n_{glass} = 1.5)$, we have:

$$I_{r\_glass} = \left(\frac{1 - n_{glass}}{1 + n_{glass}}\right)^2 = \left(\frac{1 - 1.5}{1 + 1.5}\right)^2 = 4\%$$

$$I_{t\_glass} = \frac{4n_{glass}}{(1 + n_{glass})^2} = \frac{4 \times 1.5}{(1 + 1.5)^2} = 96\%$$

Problem 8.15 shows water is a conductor up to a frequency of 10MHz, i.e. water is a dielectric at a frequency of 100MHz and has a refractive index of:

$$n_{water} = \sqrt{\varepsilon_r} = \sqrt{81} = 9$$

So, the reflectivity is given by:

$$I_{r\_water} = \left(\frac{1 - n_{water}}{1 + n_{water}}\right)^2 = \left(\frac{1 - 9}{1 + 9}\right)^2 = 64\%$$

and transmittivity given by:

$$I_{t\_water} = \frac{4n_{water}}{(1 + n_{water})^2} = \frac{4 \times 9}{(1 + 9)^2} = 36\%$$

8.23

The loss of intensity is given by:

$$I_{loss} = 1 - I_{t1} I_{t2}$$

where I_{t1} is the transmittivity from air to glass and I_{t2} is the transmittivity from glass to air. Following the discussion in problem 8.21, we have:

$$I_{t2} = \frac{Z_i E_t^2}{Z_t E_i^2} = \frac{Z_d}{Z_0}\left(\frac{2Z_0}{Z_0 + Z_d}\right)^2 = \frac{Z_d}{Z_0}\left(\frac{2}{1 + Z_d / Z_0}\right)^2 = \frac{1}{n}\left(\frac{2}{1 + 1/n}\right)^2 = \frac{4n}{(1+n)^2} = I_{t1}$$

So we have:

$$I_{loss} = 1 - I_{t1}^2 = 1 - 0.96^2 = 7.84\%$$

8.24

Noting that $c = 1\big/\sqrt{\mu_0 \varepsilon_0} = \lambda\omega/2\pi$, the radiating power can be written as:

$$\begin{aligned} P = \frac{dE}{dt} &= \frac{q^2 \omega^4 x_0^2}{12\pi\varepsilon_0 c^3} \\ &= \frac{\omega^2 x_0^2}{12\pi\varepsilon_0 c \cdot c^2}\omega^2 q^2 \\ &= \frac{4\pi^2 \omega^2 x_0^2}{12\pi\varepsilon_0 \lambda^2 \omega^2}\sqrt{\mu_0 \varepsilon_0} I_0^2 \\ &= \frac{1}{2} \times \frac{2\pi}{3}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left(\frac{x_0}{\lambda}\right)^2 I_0^2 \end{aligned}$$

i.e.

$$R = \frac{2\pi}{3}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left(\frac{x_0}{\lambda}\right)^2 = 787\left(\frac{x_0}{\lambda}\right)^2 [\Omega]$$

By substitution of given parameters, the wavelength is given by:

$$\lambda = \frac{v}{c} = \frac{3\times 10^8}{5\times 10^5} = 600[m] >> x_0 = 30[m]$$

So the radiation resistance and the radiated power are given by:

$$R = 787\times\left(\frac{x_0}{\lambda}\right)^2 = 787\times\left(\frac{30}{600}\right)^2 = 1.97[\Omega]$$

$$P = \frac{1}{2} R I_0^2 = \frac{1}{2}\times 1.97 \times 20^2 \approx 400[W]$$

SOLUTIONS TO CHAPTER 9

9.1

Substituting the expression of z into $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}$, we have:

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=-(k_1^2+k_2^2)Ae^{i[\omega t-(k_1x+k_2y)]}=-(k_1^2+k_2^2)z$$

Noting that $k^2=\omega^2/c^2=k_1^2+k_2^2$, we have:

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=-\frac{\omega^2}{c^2}z$$

Substituting the expression of z into $\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$, we have:

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}=-\frac{\omega^2}{c^2}Ae^{[\omega t-(k_1x+k_2y)]}=-\frac{\omega^2}{c^2}z$$

So we have:

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$

9.2

Boundary condition $z=0$ at $y=0$ gives:

$$A_1e^{i(\omega t-k_1x)}+A_2e^{i(\omega t-k_1x)}=0 \quad \text{i.e.} \quad A_1=-A_2$$

so the expression of z can be written as:

$$z=A_1\{e^{i[\omega t-(k_1x+k_2y)]}-e^{i[\omega t-(k_1x-k_2y)]}\}=A_1[e^{i(\omega t-k_1x)}(e^{-ik_2y}-e^{ik_2y})]=-2iA_1\sin(k_2y)e^{i(\omega t-k_1x)}$$

Therefore, the real part of z is given by:

$$z_{real}=+2A_1\sin k_2y\sin(\omega t-k_1x)$$

Using the above expression, boundary condition $z=0$ at $y=b$ gives:

$$z=-2iA_1\sin k_2be^{i(\omega t-k_1x)}=0$$

which is true for any t and x, provided: $\sin k_2b=0$, i.e. $k_2=\frac{n\pi}{b}$.

9.3

As an analogy to discussion in text page 242, electric field E_z between these two planes is the superposition of the incident and reflected waves, which can be written as:

$$E_z = E_1 e^{i[(k_x x + k_y y) - \omega t]} + E_2 e^{i[(-k_x x + k_y y) - \omega t]}$$

where $k_x = k\cos\theta$ and $k_y = k\sin\theta$

Boundary condition $E_z = 0$ at $x = 0$ gives:

$$(E_1 + E_2)e^{i(k_y y - \omega t)} = 0$$

which is true for any t and y if $E_1 = -E_2 = E_0$, so we have:

$$E_z = E_0 e^{i[(k_x x + k_y y) - \omega t]} - E_0 e^{i[(-k_x x + k_y y) - \omega t]} = E_0(e^{ik_x x} - e^{-ik_x x})e^{i(k_y y - \omega t)}$$

Using the above equation, boundary condition $E_z = 0$ at $x = a$ gives:

$$E_z = E_0(e^{ik_x a} - e^{-ik_x a})e^{i(k_y y - \omega t)} = 0$$

i.e. $$\sin k_x a e^{i(k_y y - \omega t)} = 0$$

which is true for any t and y if $\sin k_x a = 0$, i.e. $k_x = n\pi/a$.

By substitution of the expressions for λ_c and λ_g into $\frac{1}{\lambda_c^2} + \frac{1}{\lambda_g^2}$, we have:

$$\frac{1}{\lambda_c^2} + \frac{1}{\lambda_g^2} = \left(\frac{k_x}{2\pi}\right)^2 + \left(\frac{k_y}{2\pi}\right)^2 = \frac{k_x^2 + k_y^2}{(2\pi)^2} = \frac{k^2}{(2\pi)^2} = \left(\frac{\omega}{2\pi c}\right)^2 = \frac{1}{\lambda_0^2}$$

9.4

Electric field components in x, y, z directions in problem 9.3 are given by:

$$E_x = E_y = 0 \quad \text{and} \quad E_z = E_0(e^{ik_x x} - e^{-ik_x x})e^{i(k_y y - \omega t)}$$

By substitution of these values into equation 8.1, we have:

$$-\mu\frac{\partial}{\partial t}H_x = \frac{\partial}{\partial y}E_z = ik_y E_0(e^{ik_x x} - e^{-ik_x x})e^{i(k_y y - \omega t)}$$

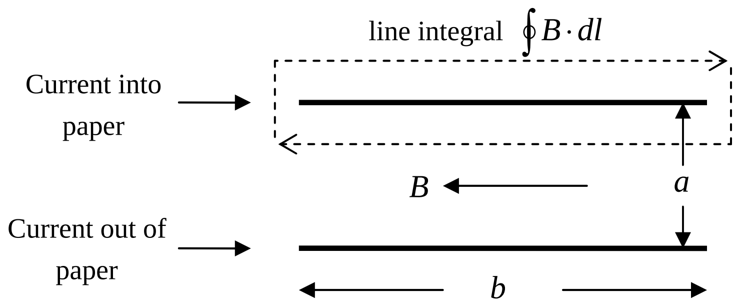
$$-\mu\frac{\partial}{\partial t}H_y = -\frac{\partial}{\partial x}E_z = -ik_x E_0(e^{ik_x x} + e^{-ik_x x})e^{i(k_y y - \omega t)}$$

which yields:

$$H_x = \frac{k_y E_0}{\mu\omega}(e^{ik_x x} - e^{-ik_x x})e^{i(k_y y - \omega t)} + C$$
$$H_y = -\frac{k_x E_0}{\mu\omega}(e^{ik_x x} + e^{-ik_x x})e^{i(k_y y - \omega t)} + D$$

where C and D are constants, which shows the magnetic fields in both x and y directions have non-zero values.

9.5



$$\oint B \cdot dl = \mu I \quad \therefore B = \mu I / b$$

Closed circuit formed by connecting ends of line length l threaded by flux:

$$B = \frac{\mu I l a}{b}$$

$$\therefore \text{inductance } L \text{ per unit length} = \mu \frac{a}{b}$$

$$\text{capacitance } C \text{ per unit length} = \varepsilon \frac{b}{a}$$

$$\therefore Z_0 = \sqrt{\frac{L}{C}} = \frac{a}{b}\sqrt{\frac{\mu}{\varepsilon}}\Omega$$

9.6

Text in page 208 shows the time averaged value of Poynting vector for an electromagnetic wave in a media with permeability of μ and permittivity of ε is given by:

$$I = \frac{1}{2} c\varepsilon E_0^2$$

Noting that, in the waveguide of Problem 9.5, the area of cross section is given by: $A = ab$, and the velocity of the electromagnetic wave is given by: $c = 1/\sqrt{\mu\varepsilon}$, the power transmitted by a single positive travelling wave is given by:

$$P = IA = \frac{1}{2} c\varepsilon E_0^2 ab = \frac{1}{2} ab \frac{1}{\sqrt{\mu\varepsilon}} \varepsilon E_0^2 = \frac{1}{2} ab E_0^2 \sqrt{\frac{\varepsilon}{\mu}}$$

9.7

The wave equation of such an electromagnetic wave is given by:

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}(y,z)}{\partial t^2}$$

i.e.
$$\frac{\partial^2 \mathbf{E}}{\partial x^2} + \frac{\partial^2 \mathbf{E}}{\partial y^2} + \frac{\partial^2 \mathbf{E}}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

By substitution of the solution $\mathbf{E} = E(y,z)\mathbf{n}\cos(\omega t - k_x x)$ into the above equation, we have:

$$\left[-k_x^2 E(y,z) + \frac{\partial^2 E(y,z)}{\partial y^2} + \frac{\partial^2 E(y,z)}{\partial z^2}\right]\cos(\omega t - k_x x) = -\frac{\omega^2}{c^2}\frac{\partial^2}{\partial t^2}E(y,z)\cos(\omega t - k_x x)$$

which is true for any t and x if:

$$\frac{\partial^2 E(y,z)}{\partial y^2} + \frac{\partial^2 E(y,z)}{\partial z^2} = \left(k_x^2 - \frac{\omega^2}{c^2}\right)E(y,z)$$

or:
$$\frac{\partial^2 E(y,z)}{\partial y^2} + \frac{\partial^2 E(y,z)}{\partial z^2} = -k^2 E(y,z)$$

where $k^2 = \dfrac{\omega^2}{c^2} - k_x^2$

9.8

Using the result of Problem 9.7, the electric field in x direction can be written as:

$$E_x = F(y,z)\cos(\omega t - k_x z)$$

and equation:

$$\frac{\partial^2 F(y,z)}{\partial y^2} + \frac{\partial^2 F(y,z)}{\partial z^2} = -k^2 F(y,z)$$

is satisfied.

Write $F(y,z)$ in form: $F(y,z) = G(y)H(z)$ and substitute to the above equation, we have:

$$\frac{1}{G(y)}\frac{\partial^2 G(y)}{\partial y^2} + \frac{1}{H(z)}\frac{\partial^2 H(z)}{\partial z^2} = -k^2$$

The solution to the above equation is given by:

$$G(y) = C_1 e^{ik_y y} + C_2 e^{-ik_y y} \quad \text{and} \quad H(z) = D_1 e^{ik_z z} + D_2 e^{-ik_z z}$$

where C_1, C_2, D_1, D_2 are constants and $k_y^2 + k_z^2 = k^2$

So the electric field in x direction is given by:

$$\begin{aligned} E_x &= F(y,z)\cos(\omega t - k_x x) = G(y)H(z)\cos(\omega t - k_x x) \\ &= (C_1 e^{ik_y y} + C_2 e^{-ik_y y})(D_1 e^{ik_z z} + D_2 e^{-ik_z z})\cos(\omega t - k_x x) \end{aligned} \qquad (9.8.1)$$

Using boundary condition $E_x = 0$ at $y = 0$ in equation (9.8.1) gives: $C_1 = -C_2$.

Using boundary condition $E_x = 0$ at $z = 0$ in equation (9.8.1) gives: $D_1 = -D_2$.

So equation (9.8.1) becomes:

$$E_x = C_1 D_1 (e^{ik_y y} - e^{-ik_y y})(e^{ik_z z} - e^{-ik_z z})\cos(\omega t - k_x x)$$

or
$$E_x = A \sin k_y y \sin k_z z \cos(\omega t - k_x x) \qquad (9.8.2)$$

where A is constant.

Using boundary condition $E_x = 0$ at $y = a$ in equation (9.8.2) gives: $\sin k_y a = 0$, i.e.

$k_y = m\pi/a$, where $m = 1,2,3,\cdots$.

Using boundary condition $E_x = 0$ at $z = b$ in equation (9.8.2) gives: $\sin k_z b = 0$, i.e.

$k_z = n\pi/b$, where $n = 1,2,3,\cdots$.

Finally, we have:

$$E_x = A \sin\frac{m\pi y}{a} \sin\frac{n\pi z}{b} \cos(\omega t - k_x x)$$

where

$$k^2 = k_y^2 + k_z^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

9.9

From problem 9.7 and 9.8, we know:

$$k_x^2 = \omega^2/c^2 - k^2 = \omega^2/c^2 - \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

For k_x to be real, we have:

$$k_x^2 = \omega^2/c^2 - \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) > 0$$

i.e.
$$\omega \geq \pi c\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

Therefore, when $m = n = 1$, ω has the lowest possible value (the cut-off frequency) given by:

$$\omega_{\min} = \pi c\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}$$

9.10

The dispersion relation of the waves of Problem 9.7 – 9.9 is given by:

$$k_x^2 = \omega^2/c^2 - \pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$$

The differentiation of this equation gives:

$$2k_x dk_x = \frac{2}{c^2}\omega d\omega$$

i.e. $$\frac{\omega}{k_x}\frac{d\omega}{dk_x} = c^2 \quad \text{or} \quad v_p v_g = c^2$$

9.11

Using boundary condition $z = 0$ at $x = 0$ in the displacement equation gives:

$$(A_1 + A_4)e^{i(\omega t - k_2 y)} + (A_2 + A_3)e^{i(\omega t + k_2 y)} = 0$$

which is true for any t and y if:

$$A_1 = -A_4 \quad \text{and} \quad A_2 = -A_3$$

so we have:

$$\begin{aligned} z &= A_1\{e^{i[\omega t-(k_1x+k_2y)]} - e^{i[\omega t-(-k_1x+k_2y)]}\} + A_2\{e^{i[\omega t-(k_1x-k_2y)]} - e^{i[\omega t-(-k_1x-k_2y)]} \\ &= -2A_1 i\sin k_1 x e^{i(\omega t-k_2y)} + 2A_2 i\sin k_1 x e^{i(\omega t+k_2y)} \qquad (9.11.1) \\ &= -2i\sin k_1 x[A_1 e^{i(\omega t-k_2y)} - A_2 e^{i(\omega t+k_2y)}] \end{aligned}$$

Using boundary condition $z = 0$ at $y = 0$ in equation (9.11.1) gives:

$$-2i\sin k_1 x(A_1 - A_2)e^{i\omega t} = 0$$

which is true for any t and x if:

$$A_1 = A_2$$

Therefore, equation (9.11.1) becomes:

$$z = -4A_1 \sin k_1 x \sin k_2 y e^{i\omega t}$$

and the real part of z is given by:

$$z_{real} = -4A_1 \sin k_1 x \sin k_2 y \cos\omega t \qquad (9.11.2)$$

Using boundary condition $z = 0$ at $x = a$ in equation (9.11.2) gives:

$\sin k_1 a = 0$, i.e. $k_1 = \dfrac{n_1\pi}{a}$, where $n_1 = 1,2,3,\cdots$.

Using boundary condition $z = 0$ at $y = b$ in equation (9.11.2) gives:

$\sin k_2 b = 0$, i.e. $k_2 = \dfrac{n_2 \pi}{b}$, where $n_2 = 1,2,3,\cdots$.

9.12

Multiplying the equation of geometric progression series by $e^{-h\nu/kT}$ on both sides gives:

$$e^{-h\nu/kT} N = e^{-h\nu/kT} \sum_n N_n = N_0 [e^{-h\nu/kT} + e^{-2h\nu/kT} + e^{-3h\nu/kT} + \cdots + e^{-(n+1)h\nu/kT}]$$

so we have:

$$N - e^{-h\nu/kT} N = N_0 [1 - \lim_{n\to\infty} e^{-(n+1)h\nu/kT}] = N_0$$

i.e.

$$N = \frac{N_0}{1 - e^{-h\nu/kT}}$$

The total energy over all the n energy states is given by:

$$E = \sum_n E_n = \sum_n N_n nh\nu = h\nu \sum_n N_n n$$
$$= h\nu N_0 (e^{-h\nu/kT} + 2e^{-2h\nu/kT} + 3e^{-3h\nu/kT} + \cdots + ne^{-nh\nu/kT})$$

Multiplying the above equation by $e^{-h\nu/kT}$ on both sides gives:

$$Ee^{-h\nu/kT} = h\nu N_0 [e^{-2h\nu/kT} + 2e^{-3h\nu/kT} + 3e^{-4h\nu/kT} + \cdots + ne^{-(n+1)h\nu/kT}]$$

so we have:

$$E - Ee^{-h\nu/kT} = h\nu N_0 \lim_{n\to\infty} [e^{-h\nu/kT} + e^{-2h\nu/kT} + e^{-3h\nu/kT} + \cdots + e^{-nh\nu/kT} - ne^{-(n+1)h\nu/kT}]$$
$$= h\nu N_0 e^{-h\nu/kT} \lim_{n\to\infty} [1 + e^{-h\nu/kT} + e^{-2h\nu/kT} + \cdots + e^{-(n-1)h\nu/kT} - ne^{-nh\nu/kT}]$$
$$= h\nu N_0 e^{-h\nu/kT} \lim_{n\to\infty} \left(\frac{1 - e^{-nh\nu/kT}}{1 - e^{-h\nu/kT}} - \frac{n}{e^{nh\nu/kT}} \right)$$
$$= h\nu N_0 e^{-h\nu/kT} \frac{1}{1 - e^{-h\nu/kT}}$$

i.e.
$$E = N_0 \frac{h\nu e^{-h\nu/kT}}{(1 - e^{-h\nu/kT})^2}$$

Hence, the average energy per oscillator is given by:

$$\bar{\varepsilon} = \frac{E}{N} = \frac{N_0 \dfrac{h\nu e^{-h\nu/kT}}{(1 - e^{-h\nu/kT})^2}}{\dfrac{N_0}{1 - e^{-h\nu/kT}}} = h\nu \frac{e^{-h\nu/kT}}{1 - e^{-h\nu/kT}} = \frac{h\nu}{e^{h\nu/kT} - 1}$$

By expanding the denominator of the above equation for $h\nu << kT$, we have:

$$\bar{\varepsilon} = \frac{h\nu}{[1 + h\nu/kT + (h\nu/kT)^2/2 + \cdots] - 1} \approx \frac{h\nu}{h\nu/kT} = kT$$

which is the classical expression of Rayleigh-Jeans for an oscillator with two degrees of freedom.

Alternative derivation for E and $\bar{\varepsilon}$:

$$E = N\bar{\varepsilon} = N\overline{nh\nu} \quad \text{where} \quad \overline{nh\nu} = \frac{\sum_{n=0}^{\infty}(nh\nu e^{-nh\nu/kT})}{\sum_{n=0}^{\infty} e^{-nh\nu/kT}}$$

$$\begin{aligned}
\therefore \overline{nh\nu} &= -\frac{\partial}{\partial (kT)^{-1}} \log \sum_{n=0}^{\infty} e^{-nh\nu/kT} \\
&= -\frac{\partial}{\partial (kT)^{-1}} \log \frac{1}{1 - e^{-h\nu/kT}} \\
&= \frac{h\nu e^{-h\nu/kT}}{1 - e^{-h\nu/kT}}
\end{aligned}$$

$$\therefore E = N\overline{nh\nu} = \frac{N_0 h\nu e^{-h\nu/kT}}{(1 - e^{-h\nu/kT})^2}$$

$$\text{and} \quad \bar{\varepsilon} = \overline{nh\nu} = \frac{h\nu}{e^{-h\nu/kT} - 1}$$

9.13

One solution of this Schrodinger’s time-independent equation can be written as:

$$\psi = X(x)Y(y)Z(z)$$

Substituting this expression into the Schrodinger’s equation and dividing ψ on both sides of the equation, we have:

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} = -\frac{8\pi^2 m}{h^2}E$$

which yields:

$$\frac{\partial^2 X(x)}{\partial x^2} + E_x X(x) = 0$$

$$\frac{\partial^2 Y(y)}{\partial y^2} + E_y Y(y) = 0$$

$$\frac{\partial^2 Z(z)}{\partial z^2} + E_z Z(z) = 0$$

where E_x, E_y, E_z are constants and satisfy: $E_x + E_y + E_z = \frac{8\pi^2 m}{h^2}E$

By solving the above three equations, we have:

$$X(x) = C_x e^{i\sqrt{E_x}x} + D_x e^{-i\sqrt{E_x}x}$$
$$Y(y) = C_y e^{i\sqrt{E_y}y} + D_y e^{-i\sqrt{E_y}y}$$
$$Z(z) = C_z e^{i\sqrt{E_z}z} + D_z e^{-i\sqrt{E_z}z}$$

and

$$\psi = (C_x e^{i\sqrt{E_x}x} + D_x e^{-i\sqrt{E_x}x})(C_y e^{i\sqrt{E_y}y} + D_y e^{-i\sqrt{E_y}y})(C_z e^{i\sqrt{E_z}z} + D_z e^{-i\sqrt{E_z}z})$$

where $C_x, D_x, C_y, D_y, C_z, D_z$ are constants.

Boundary condition $\psi = 0$ at $x = 0$ gives $C_x = -D_x$, boundary condition $\psi = 0$ at $y = 0$ gives $C_y = -D_y$, boundary condition $\psi = 0$ at $z = 0$ gives $C_z = -D_z$, so we have:

$$\psi = C_x C_y C_z (e^{i\sqrt{E_x}x} - e^{-i\sqrt{E_x}x})(e^{i\sqrt{E_y}y} - e^{-i\sqrt{E_y}y})(e^{i\sqrt{E_z}z} - e^{-i\sqrt{E_z}z})$$
$$= A\sin\sqrt{E_x}x\sin\sqrt{E_y}y\sin\sqrt{E_z}z$$

Using the above expression ψ, boundary condition $\psi = 0$ at $x = L_x$ gives: $E_x = \left(\frac{l\pi}{L_x}\right)^2$, boundary condition $\psi = 0$ at $x = L_y$ gives: $E_y = \left(\frac{r\pi}{L_y}\right)^2$, boundary condition $\psi = 0$ at $x = L_z$ gives: $E_z = \left(\frac{n\pi}{L_z}\right)^2$, where $l, r, n = 0,1,2,\cdots$, so we have:

$$\left(\frac{l\pi}{L_x}\right)^2 + \left(\frac{r\pi}{L_y}\right)^2 + \left(\frac{n\pi}{L_z}\right)^2 = \frac{8\pi^2 m}{h^2}E$$

i.e.
$$E = \frac{h^2}{8m}\left(\frac{l^2}{L_x^2} + \frac{r^2}{L_y^2} + \frac{n^2}{L_z^2}\right)$$

When $L_x = L_y = L_z = L$, $E = \frac{h^2}{8mL^2}\left(l^2 + r^2 + n^2\right)$. If $E = E_0$ for $l = 1$, $r = n = 0$, the next energy levels are given by:

$E = 3E_0$ for $l = r = n = 1$.

$E = 6E_0$ for $l = r = 1, n = 2$; $l = n = 1, r = 2$ and $n = r = 1, l = 2$, which is a three-fold degenerate state.

$E = 9E_0$ for $l = r = 2, n = 1$; $l = n = 2, r = 1$ and $n = r = 2, l = 1$ which is a three-fold degenerate state.

$E = 11E_0$ for $l = r = 1, n = 3$; $l = n = 1, r = 3$ and $n = r = 1, l = 3$ which is a three-fold degenerate state.

$E = 12E_0$ for $l = r = n = 2$.

$E = 14E_0$ for $l = 1, r = 2, n = 3$; $l = 1, n = 3, r = 2$; $l = 2, n = 1, r = 3$; $l = 2, n = 3, r = 1$; $l = 3, n = 1, r = 2$ and $l = 3, n = 2, r = 1$ which is a six-fold degenerate state.

9.14

Planck's Radiation Law is given by:

$$E_\nu d\nu = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{h\nu/kT} - 1}d\nu$$

At low energy levels $h\nu << kT$, by expansion of $e^{h\nu/kT}$ in series, the above equation becomes:

$$E_\nu d\nu = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{1 + \frac{h\nu}{kT} + \frac{1}{2}\left(\frac{h\nu}{kT}\right)^2 + \cdots - 1}d\nu$$

$$\approx \frac{8\pi\nu^2}{c^3}\frac{h\nu}{h\nu/kT}d\nu = \frac{8\pi\nu^2 kT}{c^3}d\nu$$

which is Rayleigh-Jeans expression

9.15

Using the variable $x = ch/\lambda kT$, energy per unit range of wavelength can be written as:

$$E_\lambda = \frac{8\pi ch(kTx)^5}{(ch)^5(e^x - 1)} = \frac{8\pi(kTx)^5}{(ch)^4(e^x - 1)}$$

Substitute the expression of E_λ into integral $\int_0^\infty E_\lambda d\lambda$ and, we have:

$$\int_0^\infty E_\lambda d\lambda = \int_0^\infty \frac{8\pi(kTx)^5}{(ch)^4(e^x-1)} d\,\frac{ch}{xkT}$$
$$= \int_0^\infty \frac{8\pi(kT)^4 x^3}{(ch)^3(e^x-1)}dx$$
$$= \frac{8\pi(kT)^4}{(ch)^3} \times \frac{\pi^4}{15}$$
$$= \frac{8\pi^5 k^4}{15c^3h^3}T^4$$

i.e.

$$\int_0^\infty E_\lambda d\lambda = aT^4$$

where $a = \dfrac{8\pi^5 k^4}{15c^3h^3}$

9.16

Using the expression of E_λ in Problem 9.15, the wavelength λ_m at which E_λ is maximum should satisfy the equation:

$$\frac{d}{dx}E_\lambda = 0$$

which yields:

$$\frac{d}{dx}\frac{x^5}{(e^x-1)} = 0$$

i.e.

$$\frac{5x^4(e^x-1) - e^x x^5}{(e^x-1)^2} = 0$$

i.e.

$$\left(1-\frac{x}{5}\right)e^x = 1$$

where $x = \dfrac{ch}{\lambda kT}$

9.17

The most sensitive wavelength to the human eye can be given by substituting the sun's temperature $T = 6000[K]$ into equation $ch/\lambda_m = 5kT$, i.e.:

$$\lambda_m = \frac{ch}{5kT} = \frac{3\times10^8 \times 6.63\times10^{-34}}{5\times1.38\times10^{-23}\times6000} \approx 4.7\times10^{-7}[m]$$

which is in the green region of the visible spectrum.

9.18

Substituting the tungsten's temperature $T = 2000[K]$ into equation $ch/\lambda_m = 5kT$, i.e.:

$$\lambda_m = \frac{ch}{5kT} = \frac{3\times10^8 \times 6.63\times10^{-34}}{5\times1.38\times10^{-23}\times 2000} \approx 14\times10^{-7}[m]$$

which is well into infrared.

9.19

As an analogy to the derivation of number of points in ν state shown in text page 250, 251, the number of points in k space between k and $k+dk$ is given by:

$$\frac{1}{8}\frac{\text{(volume of spherical shell)}}{\text{volume of cell}}$$
$$= \frac{4\pi k^2 dk}{8}\cdot\left(\frac{L}{\pi}\right)^3$$

Noting that for each value of k there are two allowed states, the total number of states in k space between k and $k+dk$ is given by:

$$P(k) = 2\cdot\frac{4\pi k^2 dk}{8}\cdot\left(\frac{L}{\pi}\right)^3$$

From $E = (\hbar^2/2m^*)k^2$, we have $k = \sqrt{(2m^*/\hbar^2)E}$. By substitution into the above equation, we get the number of states $S(E)dE$ in the energy interval dE given by:

$$S(E)dE = 2\cdot\frac{4\pi(2m^*/\hbar^2)^{3/2}Ed\sqrt{E}}{8}\left(\frac{L}{\pi}\right)^3$$
$$= \frac{L^3(2m^*/\hbar^2)^{3/2}E}{2\pi^2\sqrt{E}}dE = \frac{L^3(2m^*/\hbar^2)^{3/2}\sqrt{E}}{2\pi^2}dE$$

Provided $m \approx m^*$ and $A = L^3$, we have:

$$S(E) = \frac{A}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{E}$$

Since Fermi energy level satisfies the equation:

$$\int_0^{E_f} S(E)dE = N$$

By substitution of the expression of $S(E)$, we have:

$$\int_0^{E_f}\frac{A}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{E}dE = N$$

i.e.

$$\frac{A}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{E_f^{3/2}}{3/2} = N$$

which gives:

$$E_f = \frac{\hbar^2}{2m^*}\left(\frac{3\pi^2 N}{A}\right)^{2/3}$$

provided $m \approx m^*$

SOLUTIONS TO CHAPTER 10

10.1

The wave form in the upper figure has an average value of zero and is an odd function of time, so its Fourier series has a constant of zero and only sine terms. Since the wave form is constant over its half period, the Fourier coefficient b_n will be zero if n is even, i.e. there are only odd harmonics and the harmonics range from 1,3,5 to infinity.

The wave form in the lower figure has a positive average value and is a even function of time, so its Fourier series has a constant of positive value and only cosine terms. Since $\tau/T \neq 1/2$, there are both odd and even harmonics. The harmonics range from 1,2,3 to infinity.

10.2

Such a periodic waveform should satisfy: $f(x) = -f(x - T/2)$, where T is the period of the waveform. Its Fourier coefficient of cosine terms can be written as:

$$a_n = \frac{2}{T}\int_0^T f(x)\cos\frac{2\pi nx}{T}dx$$
$$= \frac{2}{T}\left[\int_0^{T/2} f(x)\cos\frac{2\pi nx}{T}dx + \int_{T/2}^{T} f(x)\cos\frac{2\pi nx}{T}dx\right]$$
$$= \frac{2}{T}\left[\int_0^{T/2} f(x)\cos\frac{2\pi nx}{T}dx + \int_{T/2}^{T} - f(x - T/2)\cos\frac{2\pi nx}{T}d(x - T/2)\right]$$

If n is even, we have

$$\cos\frac{2\pi n(x - T/2)}{T} = \cos\left(\frac{2\pi nx}{T} - n\pi\right) = \cos\frac{2\pi nx}{T}$$

Hence, by substituting into a_n and using $u = x - T/2$, we have:

$$a_n = \frac{2}{T}\left[\int_0^{T/2} f(x)\cos\frac{2\pi nx}{T}dx - \int_0^{T/2} f(u)\cos\frac{2\pi nu}{T}du\right] = 0$$

Similarly, the coefficient of sine terms is given by:

$$b_n = \frac{2}{T}\int_0^T f(x)\sin\frac{2\pi nx}{T}dx$$
$$= \frac{2}{T}\left[\int_0^{T/2} f(x)\sin\frac{2\pi nx}{T}dx + \int_{T/2}^{T} f(x)\sin\frac{2\pi nx}{T}dx\right]$$
$$= \frac{2}{T}\left[\int_0^{T/2} f(x)\sin\frac{2\pi nx}{T}dx + \int_{T/2}^{T} - f(x - T/2)\sin\frac{2\pi nx}{T}d(x - T/2)\right]$$

If n is even, using $u = x - T/2$, we have:

$$b_n = \frac{2}{T}\left[\int_0^{T/2} f(x)\sin\frac{2\pi nx}{T}dx - \int_0^{T/2} f(u)\sin\frac{2\pi nu}{T}du\right] = 0$$

Therefore, if n is even, the Fourier coefficients of both cosine and sine terms are zero, i.e. there are no even order frequency components.

10.3

The constant term of the Fourier series is given by:

$$\frac{1}{2}a_0 = \frac{1}{2\pi}\int_0^{2\pi} ydx = \frac{1}{2\pi}\int_0^{\pi} h\sin xdx = \frac{h}{\pi}$$

The Fourier coefficient of cosine term is given by:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} y\cos nxdx = \frac{h}{\pi}\int_0^{\pi}\sin x\cos nxdx$$

when $n = 1$, we have:

$$a_1 = \frac{h}{\pi}\int_0^{\pi}\sin x\cos xdx = \frac{h}{2\pi}\int_0^{\pi}\sin 2xdx = 0$$

when $n > 1$, we have:

$$\begin{aligned}
a_n &= \frac{h}{\pi}\int_0^{\pi}\sin x\cos nxdx \\
&= \frac{h}{2\pi}\int_0^{\pi}\sin(1+n)x + \sin(1-n)xdx \\
&= -\frac{h}{2\pi}\left[\frac{1}{1+n}\cos(1+n)x + \frac{1}{1-n}\cos(1-n)x\right]_0^{\pi}
\end{aligned}$$

which gives:

$$a_2 = -\frac{h}{\pi}\frac{2}{1\cdot 3},\quad a_3 = 0,\quad a_4 = -\frac{h}{\pi}\frac{2}{3\cdot 5},\quad a_5 = 0,\quad a_6 = -\frac{h}{\pi}\frac{2}{5\cdot 7},\ \ldots$$

The Fourier coefficient of sine term is given by:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} y\sin nxdx = \frac{h}{\pi}\int_0^{\pi}\sin x\sin nxdx$$

when $n = 1$, we have:

$$b_1 = \frac{h}{\pi}\int_0^{\pi}\sin x\sin xdx = \frac{h}{\pi}\int_0^{\pi}\sin^2 xdx = \frac{h}{2}$$

when $n > 1$, we have:

$$\begin{aligned}
b_n &= \frac{h}{\pi}\int_0^{\pi}\sin x\sin nxdx \\
&= \frac{h}{2\pi}\int_0^{\pi}\cos(1-n)x - \cos(1+n)xdx \\
&= \frac{h}{2\pi}\left[\frac{1}{1-n}\sin(1-n)x + \frac{1}{1+n}\sin(1+n)x\right]_0^{\pi} \\
&= 0
\end{aligned}$$

Overall, the Fourier series is given by:

$$y = \frac{1}{2}a_0 + \sum_1^\infty a_n \cos nx + \sum_1^\infty b_n \sin nx$$
$$= \frac{h}{\pi}\left(1 + \frac{\pi}{1\cdot 2}\sin x - \frac{2}{1\cdot 3}\cos 2x - \frac{2}{3\cdot 5}\cos 4x - \frac{2}{5\cdot 7}\cos 6x \cdots\right)$$

10.4

Such a wave form is a even function with a period of π. Hence, there are only constant term and cosine terms.

The constant term is given by:

$$\frac{1}{2}a_0 = \frac{1}{\pi}\int_0^\pi h\sin x dx = \frac{2h}{\pi}$$

which doubles the constant shown in Problem 10.3

The coefficient of cosine term is given by:

$$a_n = \frac{4h}{\pi}\int_0^{\pi/2} \sin x \cos\frac{2\pi nx}{\pi} dx$$
$$= \frac{4h}{\pi}\int_0^{\pi/2} \sin x \cos 2nx dx$$
$$= \frac{2h}{\pi}\int_0^{\pi/2} [\sin(1+2n)x + \sin(1-2n)x] dx$$
$$= -\frac{2h}{\pi}\left[\frac{1}{1+2n}\cos(1+2n)x + \frac{1}{1-2n}\cos(1-2n)x\right]_0^{\pi/2}$$

which gives:

$$a_1 = -\frac{h}{\pi}\frac{2}{1\cdot 3}, \quad a_2 = -\frac{h}{\pi}\frac{2}{3\cdot 5}, \quad a_3 = -\frac{h}{\pi}\frac{2}{5\cdot 7}, \ \ldots$$

Therefore the Fourier series is given by:

$$y = \frac{1}{2}a_0 + \sum_1^\infty a_n \cos\frac{2\pi nx}{\pi}$$
$$= \frac{h}{\pi}\left(1 - \frac{2}{1\cdot 3}\cos 2x - \frac{2}{3\cdot 5}\cos 4x - \frac{2}{5\cdot 7}\cos 6x \cdots\right)$$

Compared with Problem 10.3, the modulating ripple of the first harmonic $\frac{h}{2}\sin x$ disappears.

10.5

$f(x)$ is even function in the interval $\pm\pi$, so its Fourier series has a constant term given by:

$$\frac{1}{2}a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{3}$$

The coefficient of cosine term is given by:

$$
\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos\frac{2\pi nx}{2\pi}dx \\
&= \frac{2}{\pi}\int_0^{\pi} x^2\cos nxdx = \frac{2}{n\pi}\int_0^{\pi} x^2 d\sin nx \\
&= \frac{2}{n\pi}\left[x^2\sin nx\Big|_0^{\pi} - \int_0^{\pi}\sin nxdx^2\right] = \frac{4}{n^2\pi}\int_0^{\pi} xd\cos nx \\
&= \frac{4}{n^2\pi}\left[x\cos nx\Big|_0^{\pi} - \int_0^{\pi}\cos nxdx\right] = \frac{4}{n^2}\cos n\pi = (-1)^n\frac{4}{n^2}
\end{aligned}
$$

Therefore the Fourier series is given by:

$$f(x) = \frac{1}{2}a_0 + \sum_1^{\infty} a_n\cos\frac{2\pi nx}{2\pi} = \frac{1}{3}\pi^2 + \sum_1^{\infty}(-1)^n\frac{4}{n^2}\cos nx$$

10.6

The square wave function of unit height $f(x)$ has a constant value of 1 over its first half period $[0,\pi]$, so we have:

$$f(\pi/2) = 1$$

By substitution into its Fourier series, we have:

$$f(\pi/2) \approx \frac{4}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2}\right) = 1$$

i.e.

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} = \frac{\pi}{4}$$

10.7

It is obvious that the pulse train satisfies $f(t) = f(-t)$, i.e. it is an even function. The cosine coefficients of its Fourier series are given by:

$$a_n = \frac{4}{T}\int_0^{\tau}\cos\frac{2\pi nx}{T}dx = \frac{4}{T}\cdot\frac{T}{2\pi n}\sin\frac{2\pi nx}{T}\Big|_0^{\tau} = \frac{2}{n\pi}\sin\frac{2\pi}{T}n\tau$$

10.8

As τ becomes very small, $\sin\frac{2\pi}{T}n\tau \to \frac{2\pi}{T}n\tau$, so we have:

$$a_n = \frac{2}{n\pi}\sin\frac{2\pi}{T}n\tau \approx \frac{2}{n\pi}\cdot\frac{2\pi}{T}n\tau = \frac{4\tau}{T}$$

We can see as $\tau \to 0$, $a_n \to 0$, which shows as the energy representation in time domain $\to 0$, the energy representation in frequency domain $\to 0$ as well.

10.9

The constant term of the Fourier series is given by:

$$\frac{1}{2}a_0 = \frac{1}{T}\int_{-T/2}^{T/2}\frac{1}{2\tau}dt = \frac{1}{T}\int_{-\tau}^{\tau}\frac{1}{2\tau}dt = \frac{1}{T}$$

The coefficient of cosine term is given by:

$$a_n = \frac{4}{T}\int_0^{\tau}\frac{1}{2\tau}\cos\frac{2\pi nt}{T} = \frac{4}{T}\cdot\frac{1}{2\tau}\cdot\frac{T}{2\pi n}\sin\frac{2\pi nt}{T}\bigg|_0^{\tau} = \frac{1}{n\pi\tau}\sin\frac{2\pi n\tau}{T}$$

As $\tau \to 0$, we have:

$$a_n = \frac{1}{n\pi\tau}\sin\frac{2\pi n\tau}{T} \approx \frac{1}{n\pi\tau}\cdot\frac{2\pi n\tau}{T} = \frac{2}{T}$$

Now we have the Fourier series given by:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}a_n\cos\frac{2\pi nt}{T} = \frac{1}{T} + \frac{2}{T}\sum_{n=1}^{\infty}\cos\frac{2\pi nt}{T}$$

10.10

Following the derivation in the problem, we have:

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\omega)e^{i\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{i\omega}(1-e^{i\omega T})e^{i\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{i\omega}(1-e^{i\omega T})e^{i\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\frac{1}{i\omega}e^{i\omega t} - \frac{1}{i\omega}e^{i\omega(t-T)}\right]d\omega \end{aligned}$$

Using the fact that for T very large:

$$\int_{-\infty}^{+\infty}\frac{1}{i\omega}e^{i\omega(t-T)}d\omega = \int_{-\infty}^{+\infty}\frac{1}{i\omega}e^{-i\omega T}d\omega = -\pi$$

we have:

$$f(t) = \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{i\omega}e^{i\omega t}d\omega$$

10.11

Following the derivation in the problem, we have:

$$F(\nu) = \int_{-\infty}^{+\infty} f(t')e^{-i2\pi\nu t'}dt'$$
$$= \int_{-\tau/2}^{+\tau/2} f_0 e^{-i2\pi(\nu-\nu_0)t'}dt'$$
$$= \frac{f_0}{-i2\pi(\nu-\nu_0)}\int_{-\tau/2}^{+\tau/2} e^{-i2\pi(\nu-\nu_0)t'}d[-i2\pi(\nu-\nu_0)t']$$
$$= \frac{f_0}{-i2\pi(\nu-\nu_0)}(e^{-i2\pi(\nu-\nu_0)\tau/2} - e^{i2\pi(\nu-\nu_0)\tau/2})$$
$$= f_0\tau \frac{\sin[\pi(\nu-\nu_0)\tau]}{\pi(\nu-\nu_0)\tau}$$

which shows the relative energy distribution in the spectrum given by:

$$|F(\nu)|^2 = (f_0\tau)^2 \frac{\sin^2[\pi(\nu-\nu_0)\tau]}{[\pi(\nu-\nu_0)\tau]^2}$$

follows the intensity distribution curve in a single slit diffraction pattern given by:

$$I = I_0 \frac{\sin^2(\pi d \sin\theta/\lambda)}{(\pi d \sin\theta/\lambda)^2}$$

10.12

The energy spectrum has a maximum when:

$$\frac{\sin^2[\pi(\nu_{\max}-\nu_0)\tau]}{[\pi(\nu_{\max}-\nu_0)\tau]^2} = 1$$

i.e. $\pi(\nu_{\min}-\nu_0)\tau = 0$ or $\nu_{\min} = \nu_0$

The frequencies for the minima of the energy spectrum are given by:

$$|F(\nu_{\min})| = (f_0\tau)^2 \frac{\sin^2[\pi(\nu_{\min}-\nu_0)\tau]}{[\pi(\nu_{\min}-\nu_0)\tau]^2} = 0$$

i.e. $\pi(\nu_{\min}^n-\nu_0)\tau = n\pi$ or $\nu_{\min}^n - \nu_0 = \dfrac{n}{\tau}$

where $n = -\infty,\cdots,-3,-2,-1,1,2,3,\cdots,+\infty$

Hence, the total width of the first maximum of the energy spectrum is given by:

$$2\Delta\nu = \nu_{\min}^{+1} - \nu_{\min}^{-1} = \frac{2}{\tau} \quad \text{or} \quad \Delta\nu = \frac{1}{\tau}$$

Using the differentiation of the relation $\nu = \dfrac{c}{\lambda}$:

$$\Delta\nu = \frac{c}{\lambda^2}\Delta\lambda$$

we have

$$\frac{c}{\lambda^2}\Delta\lambda = \frac{1}{\tau} \quad \text{or} \quad c\tau = \frac{\lambda^2}{\Delta\lambda}$$

which is the coherence length l of Problem 10.11.

10.13

Use the relation $\Delta\lambda = \frac{\lambda_0^2}{c}\Delta\nu$, we have:

$$\Delta\lambda = \frac{(6.936\times10^{-7})^2}{3\times10^8}\times10^4 \approx 1.6\times10^{-17}[m]$$

Then, using the result in Problem 10.12, the coherence length is given by:

$$l = \frac{\lambda_0^2}{\Delta\lambda} = \frac{(6.936\times10^{-7})^2}{1.6\times10^{-17}} = 3\times10^4[m]$$

10.14

Referring to pages 46 and 47 of the text, and in particular to the example of the radiating atom, we see that the energy of the damped simple harmonic motion: $E = E_0 e^{-\omega_0 t/Q} = E_0 e^{-1}$ where $Q/\omega_0 = t$, the period for which the atom radiates before cut off at e^{-1}.

The length of the wave train radiated by the atom is $l = ct$ where c is the velocity of light and l is the coherence length which contains Q radians.

Since the coherence length is finite the radiation cannot be represented by a single angular frequency ω_0 but by a bandwidth $\Delta\omega$ centred about ω_0.

Now $Q = \omega_0/\Delta\omega$ so $Q/\omega_0 = t = 1/\Delta\omega$. Writing $t = \Delta t$ we have $\Delta\omega\Delta t = 1$ or $\Delta\nu\Delta t = 1/2\pi$.

The bandwidth effect on the spectral line is increased in a gas of radiating atoms at temperature T. Collisions between the atoms shorten the coherence length and the Doppler effect from atomic thermal velocities adds to $\Delta\nu$.

10.15

The Fourier transform of $f(t)$ gives:

$$F(\nu) = \int_{-\infty}^{+\infty} f(t)e^{-i2\pi\nu t}dt = \int_{-\infty}^{+\infty} f_0 e^{i2\pi\nu_0 t}e^{-t/\tau}e^{-i2\pi\nu t}dt = \int_{-\infty}^{+\infty} f_0 e^{[i2\pi(\nu_0-\nu)-1/\tau]t}dt$$

Noting that $t \geq 0$ for $f(t)$, we have:

$$F(\nu) = \int_0^{+\infty} f_0 e^{[i2\pi(\nu_0-\nu)-1/\tau]t} dt$$
$$= \frac{f_0}{i2\pi(\nu_0-\nu)-1/\tau}\left\{ e^{[i2\pi(\nu_0-\nu)-1/\tau]t}\Big|_0^{+\infty} \right\}$$
$$= -\frac{f_0}{i2\pi(\nu_0-\nu)-1/\tau} = \frac{f_0}{1/\tau + i2\pi(\nu-\nu_0)}$$

Hence, the energy distribution of frequencies in the region $\nu - \nu_0$ is given by:

$$|F(\nu)|^2 = \left|\frac{f_0}{1/\tau + i2\pi(\nu-\nu_0)}\right|^2 = \frac{f_0^2}{r^2} = \frac{f_0^2}{(1/\tau)^2 + [2\pi(\nu-\nu_0)]^2} = \frac{f_0^2}{(1/\tau)^2 + (\omega-\omega_0)^2}$$

10.16

In the text of Chapter 3, the resonance power curve is given by the expression:

$$P_{av} = \frac{F_0^2}{2Z_m}\cos\phi = \frac{F_0^2 r}{2Z_m^2} = \frac{F_0^2 r}{2[r^2 + (\omega m - s/\omega)^2]}$$

In the vicinity of $\omega_0 = \sqrt{s/m}$, we have $\omega \approx \omega_0$, so the above equation becomes:

$$P_{av} \approx \frac{F_0^2 r}{2[r^2 + (\omega m - s/\omega)^2]} = \frac{F_0^2 r}{2[r^2 + m^2(\omega-\omega_0)^2]} = \frac{f_0^2}{(1/\tau)^2 + (\omega-\omega_0)^2} = |F(\nu)|^2$$

where $f_0^2 = \dfrac{F_0^2 r}{2m^2}$ and $\tau = \dfrac{m}{r}$

The frequency at half the maximum value of $|F(\nu)|^2$ is given by:

$$|F(\nu)|^2 = \frac{f_0^2}{(1/\tau)^2 + (\omega-\omega_0)^2} = \frac{(f_o\tau)^2}{2}$$

i.e. $$\omega - \omega_0 = \pm 1/\tau$$

so the frequency width at half maximum is given by:

$$\Delta\omega = 2/\tau \quad \text{or} \quad \Delta\nu = 1/\pi\tau$$

In Problem 10.12 the spectrum width is given by:

$$\Delta\nu' = 1/\tau$$

so we have the relation between the two respectively defined frequency spectrum widths given by:

$$\Delta\nu = \frac{\Delta\nu'}{\pi}$$

Noting that $\Delta\lambda = \frac{\lambda_0^2}{c}\Delta\nu$ and $\Delta\lambda' = \frac{\lambda_0^2}{c}\Delta\nu'$, we have:

$$\Delta\lambda = \frac{\Delta\lambda'}{\pi}$$

where $\Delta\lambda$ and $\Delta\lambda'$ are the wavelength spectrum widths defined here and in Problem 10.12, respectively.

If the spectrum line has a value $\Delta\lambda = 3\times10^{-9}m$ in Problem 10.12, the coherence length is given by:

$$l = \frac{\lambda_0^2}{\Delta\lambda'} = \frac{\lambda_0^2}{\Delta\lambda\pi} = \frac{(5.46\times10^{-7})^2}{3\times10^{-9}\times\pi} \approx 32\times10^{-6}[m]$$

10.17

The double slit function (upper figure) and its self convolution (lower figure) are shown below:

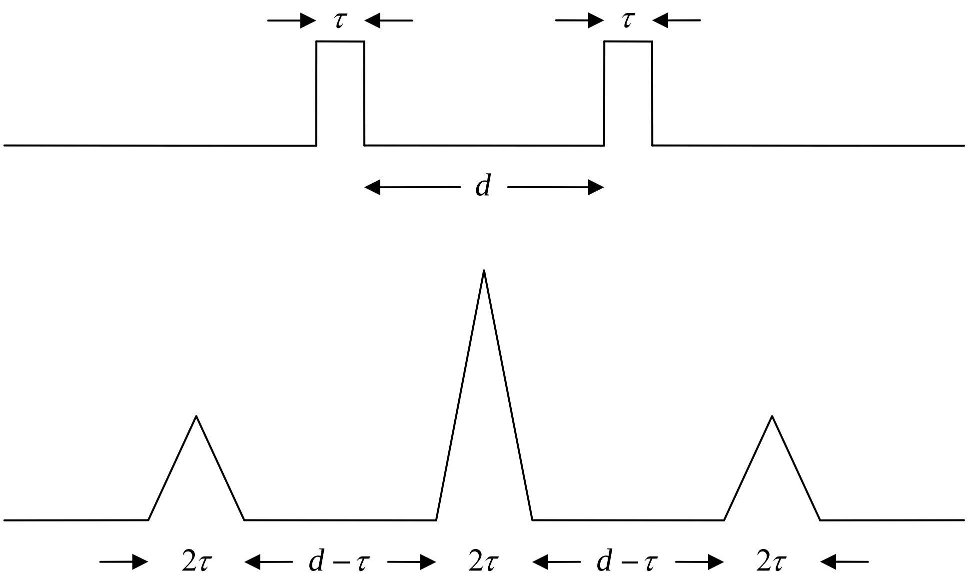


Fig. A.10.17

10.18

The convolution of the two functions is shown in Fig. A.10.18.1.

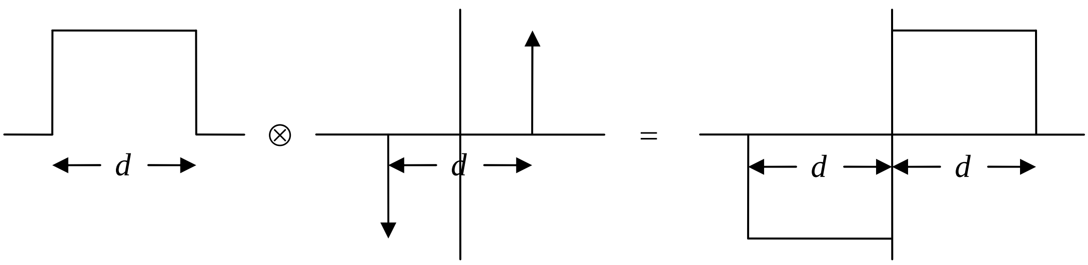


Fig. A.10.18.1

The respective Fourier transforms of the two functions are shown in Fig. A.10.18.2.

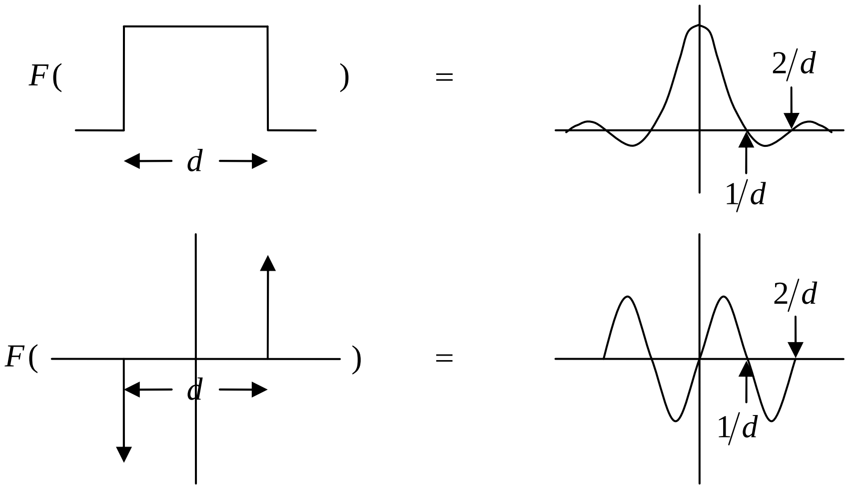


Fig. A.10.18.2

Hence, the Fourier transform of the convolution of the two functions is the product of the Fourier transform of the individual function, which is shown in Fig. A.10.18.3

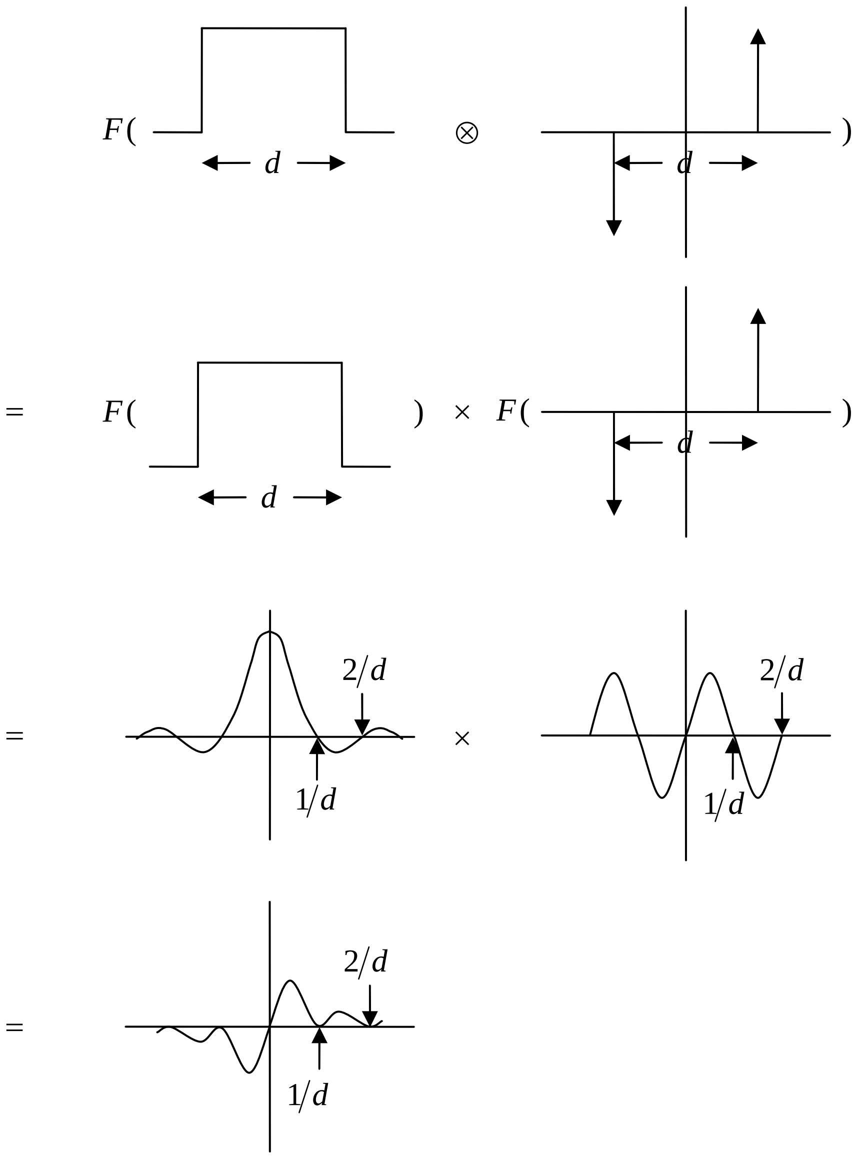


Fig. A.10.18.3

10.19

The area of the overlap is given by:

$$A = 2\left(\frac{1}{2}r \cdot 2\theta - \frac{1}{2} \cdot 2r\sin\theta \cdot r\cos\theta\right)$$
$$= r^2(2\theta - 2\sin\theta\cos\theta)$$

where $\cos\theta = \dfrac{R}{2r}$ and $\sin\theta = \sqrt{1 - \dfrac{R^2}{4r^2}}$

Hence the convolution is given by:

$$O(R) = r^2\left[2\cos^{-1}\frac{R}{2r} - 2\left(1 - \frac{R^2}{4r^2}\right)^{\frac{1}{2}}\frac{R}{2r}\right]$$

The convolution $O(R)$ in the region $[0,2r]$ is sketched below:

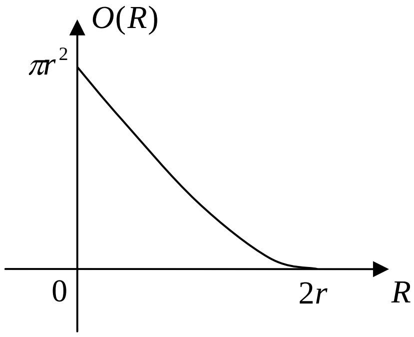


Fig. A.10.19

SOLUTIONS TO CHAPTER 11

11.1

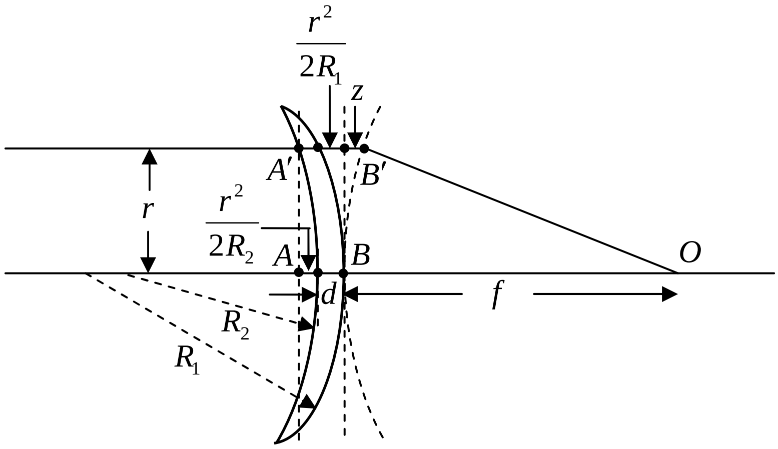


Fig A.11.1

In a bi-convex lens, as shown in Fig A.11.1, the time taken by the wavefront to travel through path AB is the same as through path $A'B'$, so we have:

$$\frac{nd}{c}+\frac{r^2}{2R_2}\cdot\frac{1}{c}=\left(\frac{n}{c}\right)\left(d+\frac{r^2}{2R_2}-\frac{r^2}{2R_1}\right)+\left(\frac{1}{c}\right)\left(z+\frac{r^2}{2R_1}\right)$$

which yields:

$$z=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{r^2}{2}$$

i.e.

$$P=\frac{1}{f}=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

11.2

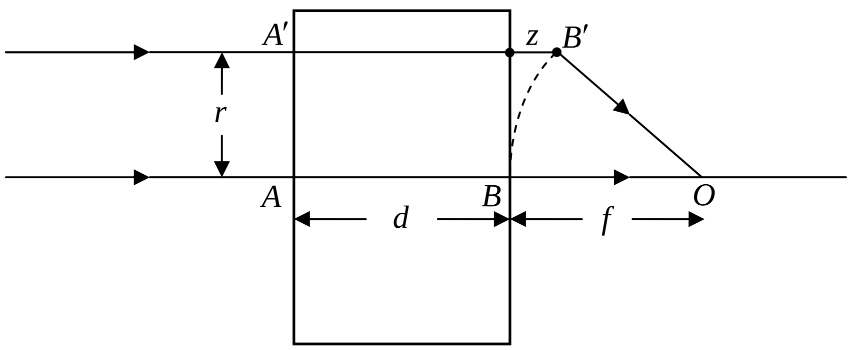


Fig A.11.2

As shown in Fig A.11.2, the time taken by the wavefront to travel through path AB

is the same as through path $A'B'$, so we have:

$$\frac{n_0 d}{c} = \frac{(n_0 - \alpha r^2)d}{c} + \frac{z}{c}$$

which yields:

$$z = \alpha r^2 d$$

i.e. $$f = \frac{1}{2\alpha d}$$

11.3

Choosing the distance $PF = \lambda/2$ then, for the path difference $BF - BF'$, the phase difference is π radians.
Similarly for the path difference $AF' - AF$ the phase difference is π radians.
Thus for the path difference $AF' - BF'$ the phase difference is 2π radians and the resulting amplitude of the secondary waves is zero.
Writing $F'F = x/2$, we then have in the triangle $F'FP$: $\frac{x}{2}\sin\theta = \frac{\lambda}{2}$, so the width of the focal spot is $x = \frac{\lambda}{\sin\theta}$.

11.4
If a man's near point is 40cm from his eye, his eye has a range of accommodation of:

$$\frac{1}{0.4} = 2.5[\text{dioptres}]$$

Noting that a healthy eye has a range of accommodation of 4 dioptres, he needs spectacles of power:

$$P = 4 - 2.5 = 1.5[\text{dioptres}]$$

If anther man is unable to focus at distance greater than 2m, his eye's minimum accommodation is:

$$\frac{1}{2} = 0.5[\text{dioptres}]$$

Therefore, he needs diverging spectacles with a power of -0.5 dioptres for clear image of infinite distance.

11.5

Noting that $\frac{y'}{y} = \frac{l'}{l}$, we have the transverse magnification given by: $M_T = \frac{l'}{l}$. The angular magnification is given by: $M_\alpha = \frac{\beta}{\gamma} = \frac{y/l}{y/d_0} = \frac{d_0}{l}$. Using the thin lens power equation: $P = \frac{1}{-l'} - \frac{1}{-l}$, we have $\frac{1}{l} = P + \frac{1}{l'}$, i.e. $M_\alpha = d_0(P + 1/l') = Pd_0 + 1$

11.6

The power of the whole two-lens telescope system is zero, so we have:

$$P = P_1 + P_2 - LP_1P_2 = 0$$

where L is the separation of the two lenses. Noting that $P_1 = \frac{1}{f_0}$ and $P_2 = \frac{1}{f_e}$, we have:

$$\frac{1}{f_0} + \frac{1}{f_e} - L\frac{1}{f_o}\frac{1}{f_e} = 0$$

which gives: $L = f_0 + f_e$

Suppose the image height at point I is h, we have:

$$|\alpha| = d/2L = h/|f_o| \text{ and } |\alpha'| = D/2L = h/|f_e|$$

which yields:

$$M_\alpha = \left|\frac{\alpha'}{\alpha}\right| = \left|\frac{f_o}{f_e}\right| = \frac{D}{d}$$

11.7

As shown from Figure 11.20, the magnification of objective lens is given by: $M_o = -\frac{x'}{f'_o}$. Suppose the objective lens is a thin lens, we have: $P_o = \frac{1}{f'_o}$ i.e. $M_o = -P_o x'$.

Similarly, the magnification of eye lens is given by: $M_e = \frac{d_o}{f'_e}$. Suppose the eye lens is a thin lens, we have: $P_e = \frac{1}{f'_e}$ i.e. $M_o = P_e d_o$.

So we have the total magnification given by:

$$M = M_o M_e = -P_o P_e d_o x'$$

11.8

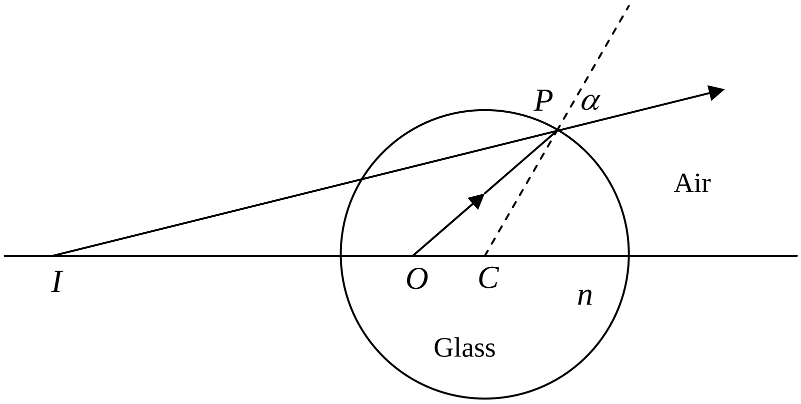


Fig.A.11.8(a)

As shown in Fig.A.11.8(a), Snell's law gives:

$$n \sin \angle OPC = \sin \alpha = \sin \angle IPC$$

In triangle OCP, we have:

$$\frac{|OC|}{\sin \angle OPC} = \frac{|PC|}{\sin \angle POC}$$

i.e. $$\frac{R}{n \sin \angle OPC} = \frac{R}{\sin \angle POC}$$

i.e. $$\frac{R}{\sin \angle IPC} = \frac{R}{\sin \angle POC}$$

i.e. $$\angle IPC = \angle POC$$

i.e. Δ's OPC and PIC are similar

i.e. $$\frac{|OC|}{|PC|} = \frac{|PC|}{|IC|}$$

i.e. $$|IC| = \frac{|PC|^2}{|OC|} = \frac{R^2}{R/n} = nR$$

Alternative proof using Fermat's Principle:

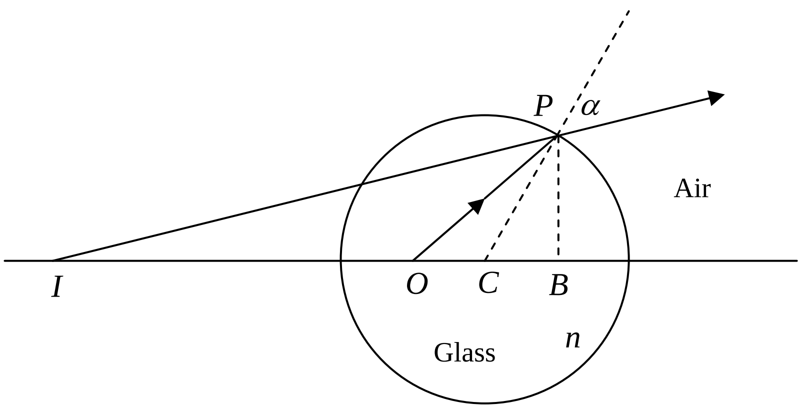


Fig.A.11.8(b)

Equate optical paths $|IP|^2 = n^2|OP|^2$. Let $|IC| = k$, $|CB| = l$ and $|PB| = d$, then:

$$|IP|^2 = (k+c)^2 + d^2 = n^2(OP)^2 = n^2\left[\left(\frac{R}{n}+l\right)^2 + d^2\right]$$

i.e. $$k^2 + 2kl + l^2 + d^2 = R^2 + 2nRl + n^2(l^2 + d^2)$$

i.e. $$k^2 + 2kl + R^2 = R^2 + 2nRl + n^2R^2$$

that is $$k = |IC| = nR$$

11.9

(a) The powers of the two spherical surfaces are given by:

$$P_1 = \frac{n'-n}{R} = \frac{1.5-1}{-1} = -0.5 \text{ and } P_2 = \frac{n'-n}{R} = \frac{1-1.5}{\infty} = 0$$

Suppose a parallel incident ray $(\overline{\alpha}_1 = 0)$ strikes the front surface of the system at a height of y_1. By using matrix method, we can find the ray angle $\overline{\alpha}_2'$ and height y_2' at the back surface of the system given by:

$$\begin{bmatrix}\overline{\alpha}_2' \\ y_2'\end{bmatrix} = R_2T_{12}R_1\begin{bmatrix}\overline{\alpha}_1 \\ y_1\end{bmatrix} = \begin{bmatrix}1 & P_2 \\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 \\ -\overline{d}_1' & 1\end{bmatrix}\begin{bmatrix}1 & P_1 \\ 0 & 1\end{bmatrix}\begin{bmatrix}\overline{\alpha}_1 \\ y_1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 \\ -0.3 & 1\end{bmatrix}\begin{bmatrix}1 & -0.5 \\ 0 & 1\end{bmatrix}\begin{bmatrix}0 \\ y_1\end{bmatrix} = \begin{bmatrix}-0.5y_1 \\ 1.15y_1\end{bmatrix}$$

So the focal length is given by:

$$f = \frac{y_1}{|\overline{\alpha}_2'|} = \frac{y_1}{0.5y_1} = 2[m]$$

The principal plane is located at a distance d to the left side of the right-end surface of the system, which is given by:

$$d = \frac{|y_2' - y_1|}{|\overline{\alpha}_2'|} = \frac{|1.15y_1 - y_1|}{0.5y_1} = 0.3[m]$$

(b) The powers of the four spherical surfaces are given by:

$$P_1 = \frac{n'-n}{R} = \frac{1.5-1}{\infty} = 0$$

$$P_2 = \frac{n-n'}{R} = \frac{1-1.5}{-0.5} = 1$$

$$P_3 = \frac{n'-n}{R} = \frac{1.5-1}{-1} = -0.5$$

$$P_4 = \frac{n-n'}{R} = \frac{1-1.5}{\infty} = 0$$

Suppose a parallel incident ray $(\overline{\alpha}_1 = 0)$ strikes the front surface of the system at a height of y_1. By using matrix method, we can find the ray angle $\overline{\alpha}_4'$ and height y_4' at the back surface of the system given by:

$$\begin{bmatrix} \overline{\alpha}_4' \\ y_4' \end{bmatrix} = R_4 T_{34} R_3 T_{23} R_2 T_{12} R_1 \begin{bmatrix} \overline{\alpha}_1 \\ y_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & P_4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\overline{d}_3' & 1 \end{bmatrix} \begin{bmatrix} 1 & P_3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\overline{d}_2' & 1 \end{bmatrix} \begin{bmatrix} 1 & P_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\overline{d}_1' & 1 \end{bmatrix} \begin{bmatrix} 1 & P_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{\alpha}_1 \\ y_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -0.15 & 1 \end{bmatrix} \begin{bmatrix} 1 & -0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -0.2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -0.15 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ y_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6y_1 \\ 0.71y_1 \end{bmatrix}$$

So the focal length is given by:

$$f = \frac{y_1}{|\overline{\alpha}_4'|} = \frac{y_1}{0.6y_1} = 1.67[m]$$

The principal plane is located at a distance d to the left side of the right-end surface of the system, which is given by:

$$d = \frac{|y_4' - y_1|}{|\overline{\alpha}_4'|} = \frac{|0.71y_1 - y_1|}{0.6y_1} = 0.48[m]$$

(c) The powers of the four spherical surfaces are given by:

$$P_1 = \frac{n'-n}{R} = \frac{1.5-1}{\infty} = 0$$

$$P_2 = \frac{n-n'}{R} = \frac{1-1.5}{-0.5} = 1$$

$$P_3 = \frac{n'-n}{R} = \frac{1.5-1}{0.5} = 1$$

$$P_4 = \frac{n-n'}{R} = \frac{1-1.5}{\infty} = 0$$

Suppose a parallel incident ray $(\overline{\alpha}_1 = 0)$ strikes the front surface of the system at a height of y_1. By using matrix method, we can find the ray angle $\overline{\alpha}_4'$ and height y_4' at the back surface of the system given by:

$$\begin{bmatrix} \overline{\alpha}_4' \\ y_4' \end{bmatrix} = R_4 T_{34} R_3 T_{23} R_2 T_{12} R_1 \begin{bmatrix} \overline{\alpha}_1 \\ y_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & P_4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\overline{d}_3' & 1 \end{bmatrix} \begin{bmatrix} 1 & P_3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\overline{d}_2' & 1 \end{bmatrix} \begin{bmatrix} 1 & P_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\overline{d}_1' & 1 \end{bmatrix} \begin{bmatrix} 1 & P_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{\alpha}_1 \\ y_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -0.15 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -0.6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -0.15 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ y_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.4 y_1 \\ 0.19 y_1 \end{bmatrix}$$

So the focal length is given by:

$$f = \frac{y_1}{|\overline{\alpha}_4'|} = \frac{y_1}{1.4 y_1} = 0.71[m]$$

The principal plane is located at a distance d to the left side of the right-end surface of the system, which is given by:

$$d = \frac{|y_4' - y_1|}{|\overline{\alpha}_4'|} = \frac{|0.19 y_1 - y_1|}{1.4 y_1} = 0.58[m]$$

SOLUTIONS TO CHAPTER 12

12.1

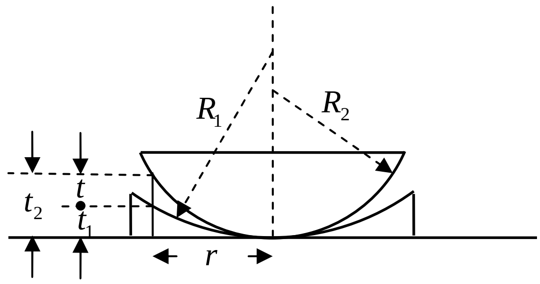


Fig.A.12.1

As shown in Fig.A.12.1, the air gap thickness t is given by:

$$t = t_2 - t_1 = \frac{r^2}{2R_2} - \frac{r^2}{2R_1}$$

Noting that there is a π rad of phase shift upon the reflection at the lower surface of the air gap, the thickness of air gap at dark rings should satisfy:

$$2t = n\lambda$$

i.e.

$$\frac{r^2}{R_2} - \frac{r^2}{R_1} = n\lambda$$

which yields the radius r_n of the nth dark ring given by:

$$r_n^2 = \frac{R_1 R_2 n\lambda}{R_1 - R_2}$$

12.2

The matrix relating reflection coefficient r and transmission coefficient t for the $\lambda/4$ film is given by:

$$M = \begin{bmatrix} \cos\delta & i\sin\delta/n_2 \\ in_2\sin\delta & \cos\delta \end{bmatrix} = \begin{bmatrix} 0 & i/n_2 \\ in_2 & 0 \end{bmatrix}$$

where the phase change $\delta = \pi/2$ for the $\lambda/4$ film.

Following the analysis in text page 352, we can find the coefficient A and B are given by:

$$A = n_1(M_{11} + M_{12}n_3) = in_1n_3/n_2$$

$$B = (M_{21} + M_{22} n_3) = i n_2$$

A perfect anti-reflector requires:

$$R = \frac{A - B}{A + B} = \frac{i n_1 n_3 / n_2 - i n_2}{i n_1 n_3 / n_2 + i n_2} = 0$$

which gives:

$$n_2^2 = n_1 n_3$$

12.3

As shown in page 357 of the text, the intensity distribution of the interference pattern is given by:

$$I = 4a^2 \cos^2 \frac{\delta}{2}$$

where δ is the phase difference between the two waves transmitted from the two radio masts to a point P and is given by:

$$\delta = kf \sin\theta = \frac{2\pi}{\lambda} f \sin\theta = \frac{2\pi}{3\times 10^8 / 1500\times 10^3} \times 400 \times \sin\theta = 4\pi \sin\theta$$

so we have:

$$I = 4a^2 \cos^2 \frac{4\pi \sin\theta}{2} = 2I_0 [1 + \cos(4\pi \sin\theta)]$$

where $I_0 = a^2$ represents the radiated intensity of each mast.

The intensity distribution is shown in the polar diagram below:

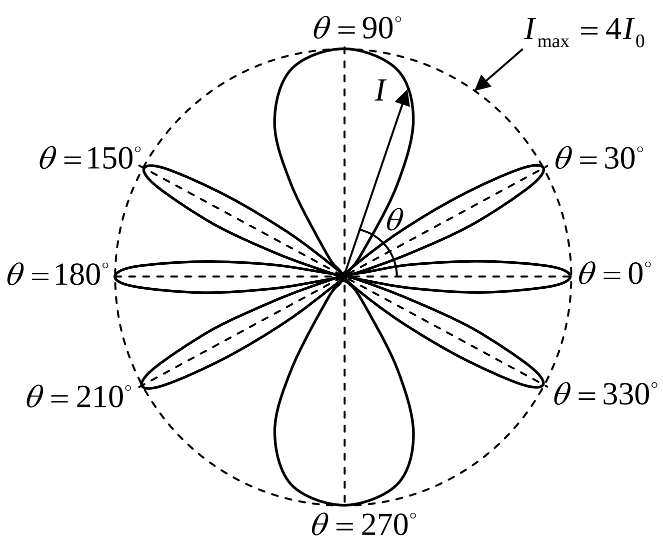


Fig.A.12.3

12.4

(a)

Analysis is the same as Problem 12.3 except:

$$\delta = \delta_0 + kf\sin\theta = \pi + \frac{2\pi}{\lambda}\cdot\frac{\lambda}{2}\sin\theta = \pi + \pi\sin\theta$$

Hence, the intensity distribution is given by:

$$I = 4a^2\cos^2\left(\frac{\delta}{2}\right) = 4a^2\cos^2\left(\frac{\pi+\pi\sin\theta}{2}\right) = 4I_s\sin^2\left(\frac{\pi\sin\theta}{2}\right)$$

where $I_s = a^2$ is the intensity of each source.

The polar diagram for I versus θ is shown below:

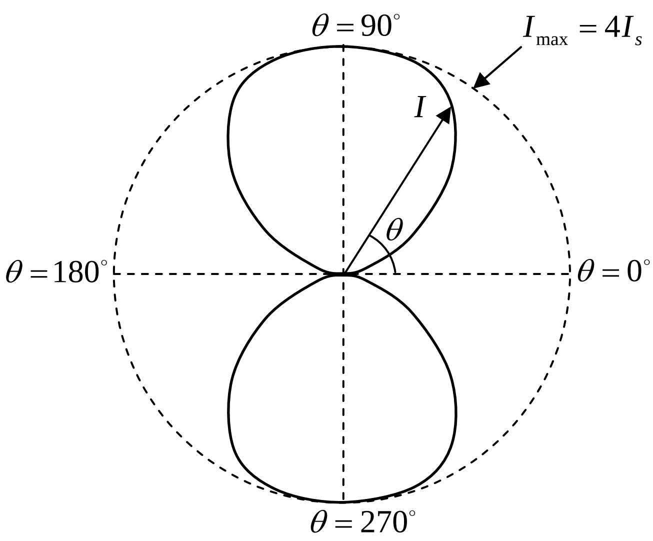


Fig.A.12.4(a)

(b)

In this case, the phase difference is given by:

$$\delta = \delta_0 + kf\sin\theta = \frac{\pi}{2} + \frac{2\pi}{\lambda}\cdot\frac{\lambda}{4}\sin\theta = \frac{\pi+\pi\sin\theta}{2}$$

Hence, the intensity distribution is given by:

$$I = 4a^2\cos^2\frac{\delta}{2} = 4a^2\cos^2\frac{\pi+\pi\sin\theta}{4} = 4I_s\left[\cos^2\frac{\pi}{4}(1+\sin\theta)\right]$$

where $I_s = a^2$ is the intensity of each source.

The polar diagram for I versus θ is shown below:

(b)

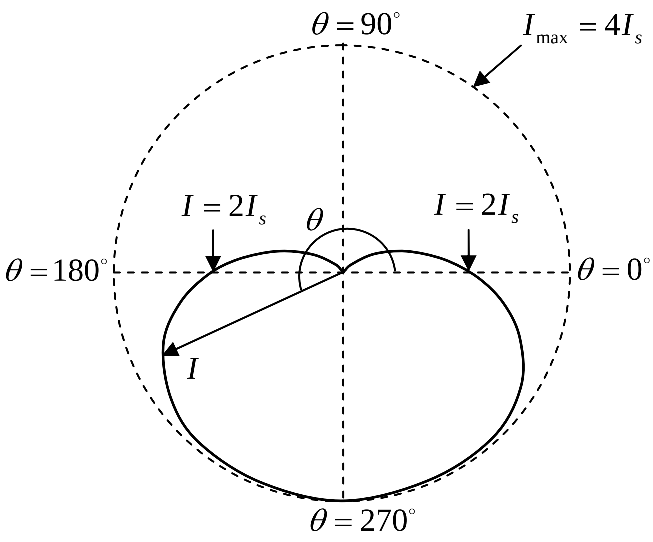


Fig.A.12.4(b)

12.5
(a)

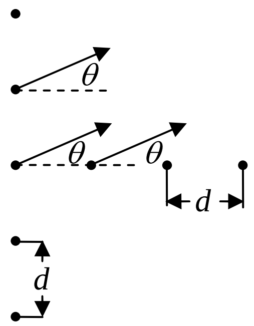


Fig.A.12.5(a)

Fig.A.12.5(a) shows elements of a vertical column and a horizontal row of radiators in a rectangular lattice with unit square cells of side d. Rays leave each lattice point at an angle θ to reach a distant point P. If P is simultaneously the location of the m*th* spectral order of interference from the column radiation and the n*th* spectral order of interference from the row radiation, we have from pages 364/5 the relations:

$$d \sin\theta = m\lambda \quad \text{and} \quad d \cos\theta = n\lambda$$

Thus

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{m}{n}$$

where m and n are integers.

(b)

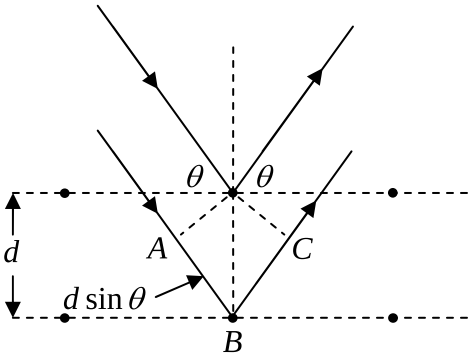


Fig.A.12.5(b)

Waves scattered elastically (without change of λ) by successive planes separated by a distance d in a crystal reinforce to give maxima on reflection when the path difference $2d\sin\theta = n\lambda$. In Fig.A.12.5(b), the path difference ABC between the incident and the reflected rays $= 2d\sin\theta$.

12.6

Using the Principal Maximum condition:

$$f\sin\theta = n\lambda$$

at $\theta = \pm\frac{\pi}{2}$, we have: $f = n\lambda$, which shows the minimum separation of equal sources is given by: $f = \lambda$.

When $N = 4$, the intensity distribution as a function of θ is given by:

$$I = I_s \frac{\sin^2(4\pi\sin\theta)}{\sin^2(\pi\sin\theta)}$$

The $N-1=3$ points of zero intensity occur when:

$$f\sin\theta = \frac{\lambda}{4}, \frac{\lambda}{2}, \frac{3\lambda}{4}$$

i.e. $$\sin\theta = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$$

The position of the $N-2=2$ points of secondary intensity maxima should occur between the zero intensity points and should satisfy:

$$\frac{dI}{d\theta} = 0$$

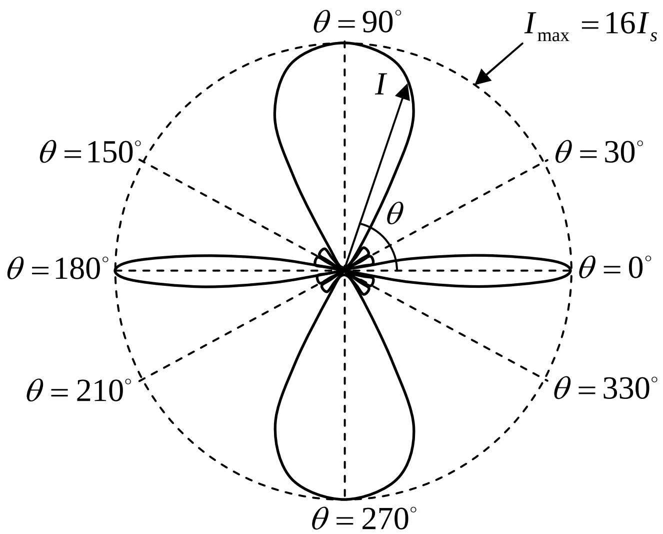
i.e. $$\frac{dI}{d\theta} = \frac{d}{d\theta}\left[I_s \frac{\sin^2(4\pi\sin\theta)}{\sin^2(\pi\sin\theta)}\right] = 0$$

i.e. $$6\cos^2(\pi\sin\theta) - 1 = 0$$

which yields: $\sin\theta = \frac{1}{\pi}\arccos\left(\pm\frac{1}{\sqrt{6}}\right)$

i.e. secondary intensity maxima occur when $\theta = 21.5^\circ$ and $\theta = 39.3^\circ$.

The angular distribution of the intensity is shown below:



12.7

The angular width of the central maximum $\delta\theta$ is the angular difference between +1 and -1 order zero intensity position and should satisfy:

$$\sin\delta\theta = \frac{2\lambda}{Nf} = \frac{2\times 0.21}{32\times 7} = 1.875\times 10^{-3} \quad \text{or} \quad \delta\theta = 6'$$

The angular separation between successive principal maxima $\Delta\theta$ is given by:

$$\sin\Delta\theta = \frac{\lambda}{f} = \frac{0.21}{7} = 0.03 \quad \text{or} \quad \Delta\theta = 1^\circ 42'$$

12.8

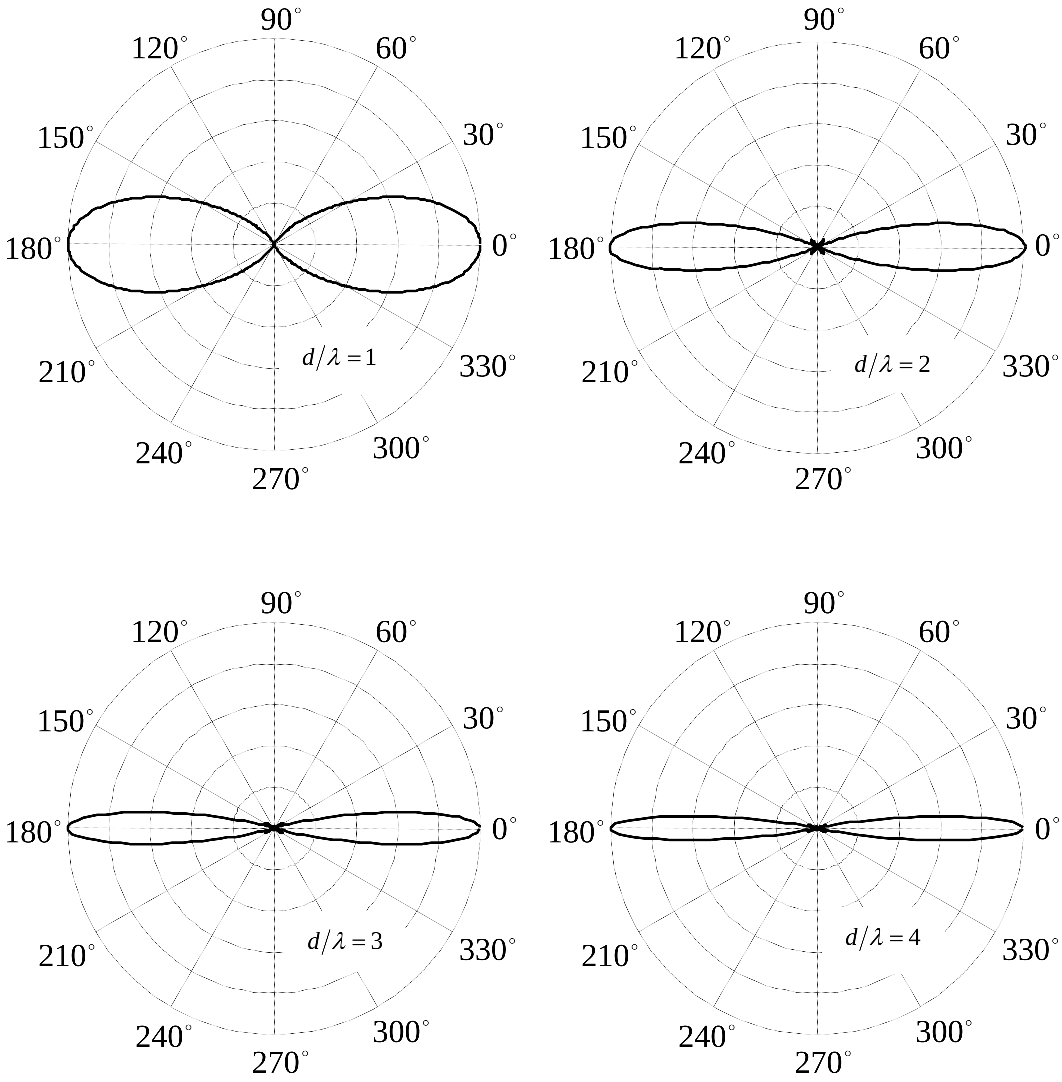


Fig.A.12.8

The above polar diagrams show the traces of the tip of the intensity of diffracted light I for monochromatic light normally incident on a single slit when the ratio of slit width to the wavelength d/λ changes from 1 to 4. It is evidently shown that the polar diagram becomes concentrated along the direction $\theta = 0$ as d/λ becomes larger.

12.9

It is evident that $\alpha = 0$ satisfies the condition: $\alpha = \tan\alpha$.

By substitution of $\alpha = 3\pi/2 - \delta$ into the condition: $\alpha = \tan\alpha$ we have:

$$3\pi/2 - \delta = \tan(3\pi/2 - \delta)$$

i.e. $$3\pi/2 - \delta = \cot\delta$$

i.e. $$(3\pi/2 - \delta)\sin\delta = \cos\delta$$

when δ is small, we have:

$$(3\pi/2 - \delta)\delta = 1 - \frac{\delta^2}{2}$$

The solution to the above equation is given by: $\delta = 0.7\pi$.

Using the similar analysis for $\alpha = 5\pi/2 - \delta$ and $\alpha = 7\pi/2 - \delta$, we can find $\delta = 0.041\pi$ and $\delta = 0.029\pi$ respectively. Therefore the real solutions for α are $\alpha = 0, \pm 1.43\pi, \pm 2.459\pi, 3.471\pi, \text{etc}$.

12.10

If only interference effects are considered the intensity of this grating is given by:

$$I = I_0 \frac{\sin^2 3\beta}{\sin^2 \beta}$$

The intensity of the principal maximum is given by: $I_{\max} = 9I_0$ when $\beta = 0$.

The β for the secondary maximum should satisfy:

$$\frac{d}{d\beta}\left(\frac{\sin^2 3\beta}{\sin^2 \beta}\right) = 0$$

i.e. $$\sin^2 \beta = 1$$

i.e. $$\beta_{\text{sec\_max}} = (2n+1)\frac{\pi}{2}, \text{ where } n \text{ is integer}$$

Hence, at the secondary maximum:

$$I_{\text{sec\_max}} = I_0 \frac{\sin^2 3\beta_{\text{sec\_max}}}{\sin^2 \beta_{\text{sec\_max}}} = I_0 = \frac{1}{9} I_{\max}$$

12.11

Suppose a monochromatic light incident on a grating, the phase change $d\beta$ required to move the diffracted light from the principal maximum to the first minimum is given by:

$$d\beta = d\left(\frac{\pi f \sin\theta}{\lambda}\right) = \frac{\pi f}{\lambda} d(\sin\theta) = \frac{\pi f}{\lambda} \cdot \frac{\lambda}{Nf} = \frac{\pi}{N}$$

Since N is a very large number, we have:

$$d\beta = \frac{\pi}{N} \approx 0$$

Then, suppose a non-monochromatic light, i.e. λ is not constant, incident on the same grating, the phase change $d\beta$ required to move the diffracted light from the principal maximum to the first minimum should be the same value as given above, so we have:

$$d\beta = d\left(\frac{\pi f \sin\theta}{\lambda}\right) = \frac{\pi f}{\lambda} d(\sin\theta) + \pi f \sin\theta d\left(\frac{1}{\lambda}\right)$$
$$= \frac{\pi f}{\lambda}\cos\theta d\theta - \frac{\pi f \sin\theta}{\lambda^2} d\lambda = \frac{\pi}{N} \approx 0$$

which gives:

$$d\theta = (nN \cot\theta)^{-1}$$

12.12

(a)

The derivative of the equation:

$$f \sin\theta = n\lambda$$

gives:

$$f \cos\theta d\theta = nd\lambda$$

when θ is a small angle we have:

$$\frac{d\theta}{d\lambda} = \frac{n}{f}$$

When the diffracted light from the grating is projected by a lens of focal length F on the screen, the relation between linear spacing on the screen l and the diffraction angle θ is given by:

$$l = F\theta$$

Its derivative over λ gives:

$$\frac{dl}{d\lambda} = F\frac{d\theta}{d\lambda} = \frac{nF}{f}$$

(b)

Using the result given above, the change in linear separation per unit increase in spectral order is given by:

$$\frac{dl}{n} = \frac{Fd\lambda}{f} = \frac{2\times(5.2\times10^{-7} - 5\times10^{-7})}{2\times10^{-6}} = 2\times10^{-2}[m]$$

12.13

(a)

Using the resolving power equation:

$$\frac{\lambda}{d\lambda} = nN$$

we have:

$$N = \frac{\lambda}{nd\lambda} = \frac{(5.89\times10^{-7} + 5.896\times10^{-7})/2}{3\times(5.896\times10^{-7} - 5.89\times10^{-7})} \approx 328$$

(b)

Using the resolving power equation:

$$\frac{\lambda}{d\lambda} = nN$$

we have:

$$d\lambda = \frac{\lambda}{nN} = \frac{6.5\times10^{-7}}{3\times9\times10^{4}} = 2.4\times10^{-12}[m]$$

12.14

When the objects O and O' are just resolved at I and I' the principal maximum of O and the first minimum of O' are located at I. Rayleigh's criterion thus defines the path difference:

$$O'BI - O'AI = O'B - O'A = 1.22\lambda \quad (BI = AI)$$

Also $OB = OA$ giving

$$(O'B - OB) + (OA - O'A) = 1.22\lambda$$

Fig.Q.12.14 shows OA parallel to $O'A$ and OB parallel to $O'B$, so:

$$OA - O'A = O'C = OO'\sin i = s\sin i$$

and

$$O'B - OB = O'C' = OO'\sin i = s\sin i$$

We therefore write:

$$s \approx \frac{1.22\lambda}{2\sin i} \quad \text{or} \quad s = \frac{1.22\lambda}{2\sin i} \quad \text{if} \quad i = 45^\circ$$

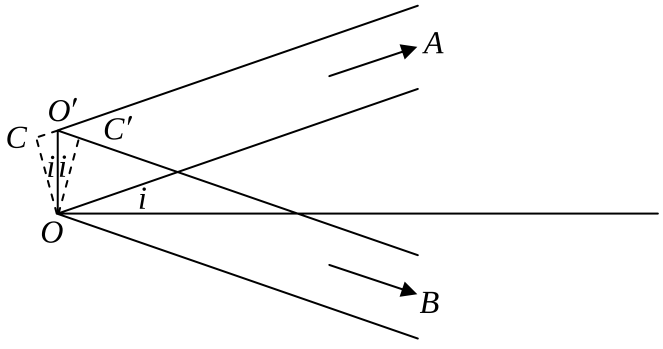


Fig.A.12.14

SOLUTIONS TO CHAPTER 13

13.1

For such an electron, the uncertainty of momentum Δp roughly equals the magnitude of momentum p, and the uncertainty of radius Δr roughly equals the magnitude of radius r. So we have:

$$p \approx \Delta p = \frac{\hbar}{\Delta r} \approx \frac{\hbar}{r}$$

By substitution of the above equation into the expression of electron energy, we have:

$$E = \frac{p^2}{2m} - \frac{e^2}{4\pi\varepsilon_0 r} = \frac{\hbar^2/r^2}{2m} - \frac{e^2}{4\pi\varepsilon_0 r}$$

(13.1.1)

The minimum energy occurs when $dE/dr = 0$, i.e.:

$$\frac{d}{dr}\left(\frac{\hbar^2/r^2}{2m} - \frac{e^2}{4\pi\varepsilon_0 r}\right) = 0$$

i.e. $$-\frac{\hbar^2}{mr^3} + \frac{e^2}{4\pi\varepsilon_0 r^2} = 0$$

which yields the minimum Bohr radius given by:

$$r = \frac{4\pi\varepsilon_0\hbar^2}{me^2} = \frac{\varepsilon_0 h^2}{\pi me^2}$$

By substitution into equation 13.1.1, we find the electron's ground state energy given by:

$$E_0 = \frac{\hbar^2}{2m}\left(\frac{\pi me^2}{\varepsilon_0 h^2}\right)^2 - \frac{e^2}{4\pi\varepsilon_0}\frac{\pi me^2}{\varepsilon_0 h^2} = \frac{-me^4}{8\varepsilon_0^2 h^2}$$

13.2

Use the uncertainty relation $\Delta p \Delta x \approx h$ we have:

$$\Delta x \approx \frac{h}{\Delta p} \geq \frac{h}{p}$$

Photons' energy converted from mass m is given by:

$$E = pc = mc^2$$

So, the momentum of these photons is given by:

$$p = mc$$

Therefore, these photons' spatial uncertainty should satisfy:

$$\Delta x \geq \frac{h}{mc}$$

which shows the short wavelength limit on length measurement, i.e. the Compton wavelength, is given by:

$$\lambda = \frac{h}{mc}$$

By substitution of electron mass: $m_e = 9.1 \times 10^{-31}[kg]$ into the above equation, we have the Compton wavelength for an electron given by:

$$\lambda = \frac{h}{m_e c} = \frac{6.63 \times 10^{-34}}{9.1 \times 10^{-31} \times 3 \times 10^8} \approx 2.42 \times 10^{-12}[m]$$

13.3

The energy of a simple harmonic oscillation at frequency ω should satisfy:

$$E = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \geq \frac{\overline{\Delta p^2}}{2m} + \frac{1}{2} m\omega^2 \overline{\Delta x^2}$$

The relation: $(\overline{\Delta x^2})(\overline{\Delta p^2}) \approx \frac{\hbar^2}{4}$ gives: $\overline{\Delta p^2} \approx \frac{\hbar^2}{4\overline{\Delta x^2}}$, by substitution into the above equation, we have:

$$E \geq \frac{\overline{\Delta p^2}}{2m} + \frac{1}{2} m\omega^2 \overline{\Delta x^2} = \frac{\hbar^2}{8m\overline{\Delta x^2}} + \frac{1}{2} m\omega^2 \overline{\Delta x^2} \geq 2\sqrt{\frac{\hbar^2}{8m\overline{\Delta x^2}}\left(\frac{1}{2} m\omega^2 \overline{\Delta x^2}\right)}$$
$$= \frac{1}{2}\hbar\omega = \frac{1}{2} h\nu$$

i.e. the simple harmonic oscillation has a minimum energy of $\frac{1}{2} h\nu$.

13.4

When an electron passes through a slit of width Δx, the intensity distribution of diffraction pattern is given by:

$$I = I_0 \frac{\sin^2 \alpha}{\alpha^2}, \text{ where } \alpha = \frac{\pi}{\lambda} \Delta x \sin\theta$$

The first minimum of the intensity pattern occurs when $\alpha = \pi$,

i.e. $$\alpha = \frac{\pi}{\lambda} \Delta x \sin\theta = \pi$$

Noting that $\lambda = h/p$, we have:

$$\alpha = \frac{\pi}{h}\Delta xp\sin\theta = \pi$$

i.e. $$\Delta x\Delta p = h$$

where $\Delta p = p\sin\theta$ is the change of the electron's momentum in the direction parallel to the plane of the slit. This relation is in accordance with Heisenberg's uncertainty principle.

13.5

The angular spread due to diffraction can be seen as the half angular width of the principal maximum $\Delta\theta$ of the diffraction pattern. Use the same analysis as Problem 13.4, we have:

$$\alpha = \frac{\pi}{\lambda}d\sin\theta \approx \frac{\pi}{\lambda}d\Delta\theta = \pi$$

i.e. $$\Delta\theta = \frac{\lambda}{d} = \frac{10^{-5}}{10^{-4}} = 0.1 \approx 5^{\circ}44'$$

13.6

The energy of the electron after acceleration across a potential difference V is given by: $E = eV$, so its momentum is given by: $p = \sqrt{2m_eE} = \sqrt{2m_eeV}$, therefore its de Broglie wavelength is given by:

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m_eeV}} = \frac{6.63\times10^{-34}}{\sqrt{2\times9.1\times10^{-31}\times1.6\times10^{-19}V}} = 1.23\times10^{-9}V^{-1/2}[m]$$

13.7

From problem 13.6 we have:

$$V^{1/2} = \frac{1.23\times10^{-9}}{\lambda} = \frac{1.23\times10^{-9}}{3\times10^{-10}} = 4.1$$

$$\therefore V = 16.81[\mathrm{V}]$$

13.8

The energy per unit volume of electromagnetic wave is given by: $E = \frac{1}{2}\varepsilon_0E_0^2$, where E_0 is the electric field amplitude. For photons of zero rest mass, the energy is given by: $E = mc^2 = pc$, where p is the average momentum per unit volume associated with this electromagnetic wave. So we have:

$$\frac{1}{2}\varepsilon_0E_0^2 = pc$$

i.e. $$p = \frac{1}{2}\varepsilon_0 E_0^2 \Big/ c$$

The dimension the above equation is given by:

$$\frac{\mathrm{F \cdot V^2 \cdot m^{-2}}}{\mathrm{m \cdot s^{-1}}} = \frac{\mathrm{C \cdot V \cdot m^{-2}}}{\mathrm{m \cdot s^{-1}}} = \frac{\mathrm{C \cdot W \cdot m^{-2}}}{\mathrm{m \cdot s^{-1} \cdot A}} = \frac{\mathrm{A \cdot s \cdot kg \cdot m^2 \cdot m^{-2}}}{\mathrm{m \cdot s^{-1} \cdot s^3 \cdot A}} = \mathrm{kg \cdot m^{-1} \cdot s^{-1}}$$

which is the dimension of momentum.

13.9

When the wave is normally incident on a perfect absorber, all the photons' velocity changes from c to 0, the radiation pressure should equal the energy density of the incident wave, i.e.:

$$P = cp - 0 = cp = \frac{1}{2}\varepsilon_0 E_0^2$$

When the wave is normally incident on a perfect reflector, all the photons' velocity changes to the opposite directing but keeps the same value, hence, the radiation pressure is given by:

$$P = cp - (-cp) = 2cp = \varepsilon_0 E_0^2$$

13.10

Using the result of Problem 13.9, we have the radiation pressure from the sun incident upon the perfectly absorbing surface of the earth given by:

$$P = \frac{1}{3} \times \frac{1}{2}\varepsilon_0 E_0^2 = \frac{1}{3} \times \frac{I}{c} = \frac{1}{3} \times \frac{1.4 \times 10^3}{3 \times 10^8} = 1.5 \times 10^{-6}[\mathrm{Pa}] \approx 10^{-11}[\mathrm{atm}]$$

13.11

Using the result of Problem 13.3, we have the minimum energy, i.e. the zero point energy, of such an oscillation given by:

$$\frac{1}{2}h\nu = \frac{1}{2} \times 6.63 \times 10^{-34} \times 6.43 \times 10^{11} = 2.13 \times 10^{-22}[\mathrm{J}] = 1.33 \times 10^{-3}[\mathrm{eV}]$$

13.12

The probability of finding the mass in the box is given by the integral:

$$\int_{-a}^{a} |\psi(x)|^2 dx = \int_{-a}^{a} \frac{1}{a}\left(1 - \frac{\pi^2 x^2}{8a^2}\right)^2 dx$$

$$= \int_{-a}^{a} \frac{1}{a}\left(1 - \frac{\pi^2 x^2}{4a^2} + \frac{\pi^4 x^4}{64a^4}\right) dx$$

$$= 2 - \frac{2\pi^2}{12} + \frac{2\pi^4}{320} \approx 0.96$$

The general expression of the wave function is given by: $\psi = Ce^{ikx} + De^{-ikx}$, where A, B are constants. Using boundary condition at $x = a$ and $x = -a$, we have:

$$\psi(a) = Ce^{ika} + De^{-ika} = 0$$
$$\psi(-a) = Ce^{-ika} + De^{ika} = 0$$

which gives: $C = D$, so we have:

$$\psi = Ce^{ikx} + Ce^{-ikx} = A\cos kx$$

where $A = 2C$.

Boundary condition: $\psi = 0$ at $x = a$ gives: $\cos ka = 0$, i.e.:

$$k = \left(n + \frac{1}{2}\right)\frac{\pi}{a}\text{, where } n = 0,1,2,3,\cdots$$

Hence, the ground state equation is given by letting $n = 0$, i.e.:

$$\psi = A\cos\left(\frac{\pi x}{2a}\right)$$

By normalization of the wave function, we have:

$$\int_{-\infty}^{+\infty}|\psi(x)|^2\,dx = 1$$

i.e. $$\int_{-\infty}^{+\infty} A^2\cos^2\left(\frac{\pi x}{2a}\right)dx = 1$$

i.e. $$\int_{-a}^{+a} A^2\,\frac{1+\cos(\pi x/a)}{2}\,dx = 1$$

i.e. $$A = 1/\sqrt{a}$$

Therefore the normalized ground state wave function is:

$$\psi(x) = (1/\sqrt{a})\cos(\pi x/2a)$$

which can be expanded as:

$$\psi(x) = \frac{1}{\sqrt{a}}\left[1 - \frac{1}{2}\left(\frac{\pi x}{2a}\right)^2 + \cdots\right] \approx \frac{1}{\sqrt{a}}\left(1 - \frac{\pi^2 x^2}{8a^2}\right)$$

13.13

At ground state, i.e. at the bottom of the deep potential well, $n_1 = n_2 = n_3 = 1$.

By normalization of the wave function at ground state, we have:

$$\iiint |\psi(xyz)|^2 dV = \int_0^c \int_0^b \int_0^a \left| A\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\sin\frac{\pi z}{c} \right|^2 dxdydz$$

$$= A^2 \int_0^a \left(\sin\frac{\pi x}{a}\right)^2 dx \cdot \int_0^a \left(\sin\frac{\pi y}{b}\right)^2 dy \cdot \int_0^a \left(\sin\frac{\pi z}{c}\right)^2 dz$$

$$= A^2 \int_0^a \left(\frac{1}{2} - \frac{1}{2}\cos\frac{2\pi x}{a}\right) dx \cdot \int_0^b \left(\frac{1}{2} - \frac{1}{2}\cos\frac{2\pi y}{b}\right) dy \cdot \int_0^c \left(\frac{1}{2} - \frac{1}{2}\cos\frac{2\pi z}{c}\right) dz$$

$$= A^2 \cdot \frac{a}{2} \cdot \frac{b}{2} \cdot \frac{c}{2} = 1$$

i.e. $$A = \sqrt{8/abc}$$

13.14

Text in page 426 shows number of electrons per unit volume in energy interval dE is given by:

$$dn = \frac{2 \times 4\pi(2m^3)^{1/2} E^{1/2}}{h^3} dE$$

and the total number of electrons given by:

$$N = \frac{16\pi(2m_e^3)^{1/2} E_F^{3/2}}{3h^3}$$

so we have the total energy of these electrons given by:

$$U = \int E dn = \int_0^{E_f} E \frac{dn}{dE} dE$$

$$= \int_0^{E_f} \frac{2 \times 4\pi(2m^3)^{1/2} E^{3/2}}{h^3} dE$$

$$= \frac{16\pi(2m^3)^{1/2} E_F^{5/2}}{5h^3} = \frac{3}{5} NE_F$$

13.15

Noting that Copper has one conduction electron per atom and one atom has a mass of $m_0 = 1.66 \times 10^{-27}\,\text{kg}$, the number of free electrons per unit volume in Copper is given by:

$$n_0 = \frac{\rho}{64m_0} = \frac{9 \times 10^3}{64 \times 1.66 \times 10^{-27}} \approx 8 \times 10^{28} [\text{m}^{-3}]$$

Using the expression of number of electrons per unit volume in text of page 426, we have the Fermi energy level of Copper given by:

$$E_F = \left(\frac{n_0 \cdot 3h^3}{16\pi\sqrt{2m_e^3}}\right)^{2/3} = \left[\frac{8 \times 10^{28} \times 3 \times (6.63 \times 10^{-34})^3}{16\pi \times \sqrt{2 \times (9.1 \times 10^{-31})^3}}\right]^{2/3} \approx 1.08 \times 10^{-18} [\text{J}] = 7[\text{eV}]$$

13.16

By substitution of values of x, $V-E$, and m into the expression $e^{-2\alpha x}$, we have:

For an electron:

$$e^{-2\alpha x} = e^{-2\left[\sqrt{2m_e(V-E)}/\hbar\right]x} = e^{-2\left[\sqrt{2\times 9.1\times 10^{-31}\times 1.6\times 10^{-19}}/\left(6.63\times 10^{-34}/2\pi\right)\right]\times 2\times 10^{-10}} = e^{-2.05} \approx 0.1$$

For a proton:

$$e^{-2\alpha x} = e^{-2\left[\sqrt{2m_p(V-E)}/\hbar\right]x} = e^{-2\left[\sqrt{2\times 1.67\times 10^{-27}\times 1.6\times 10^{-19}}/\left(6.63\times 10^{-34}/2\pi\right)\right]\times 2\times 10^{-10}} = e^{-87.4} \approx 10^{-38}$$

13.17

Text in page 432-434 shows the amplitude reflection and transmission coefficients for such a particle are given by:

$$r = \frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2} \quad \text{and} \quad t = \frac{C}{A} = \frac{2k_1}{k_1 + k_2}$$

where,

$$k_1 = \frac{\sqrt{2mE}}{\hbar} \quad \text{and} \quad k_2 = \frac{\sqrt{2m(E-V)}}{\hbar}$$

If V is a very large negative value at $x > 0$, we have the amplitude reflection coefficient given by:

$$r = \lim_{V\to -\infty} \frac{\sqrt{2mE} - \sqrt{2m(E-V)}}{\sqrt{2mE} + \sqrt{2m(E-V)}} = -1$$

and the amplitude transmission coefficient given by:

$$t = \lim_{V\to -\infty} \frac{2\sqrt{2mE}}{\sqrt{2mE} + \sqrt{2m(E-V)}} = 0$$

i.e. the amplitude of reflected wave tends to unity and that of transmitted wave to zero.

13.18

The potential energy of one dimensional simple harmonic oscillator of frequency ω is given by:

$$V = \frac{1}{2} m\omega^2 x^2$$

By substitution into Schrödinger’s equation, we have:

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left[E - \frac{1}{2} m\omega^2 x^2\right]\psi = 0$$

(13.18.1)

Try $\psi(x) = \sqrt{a/\sqrt{\pi}}\, e^{-a^2x^2/2}$ in $d\psi/dx$:

$$\frac{d\psi}{dx} = -a^2 x\sqrt{\frac{a}{\sqrt{\pi}}}e^{\frac{-a^2x^2}{2}}$$

so:

$$\frac{d^2\psi}{dx^2} = -a^2\sqrt{\frac{a}{\sqrt{\pi}}}e^{\frac{-a^2x^2}{2}} + a^4x^2\sqrt{\frac{a}{\sqrt{\pi}}}e^{\frac{-a^2x^2}{2}} = (a^4x^2 - a^2)\sqrt{\frac{a}{\sqrt{\pi}}}e^{\frac{-a^2x^2}{2}}$$

In order to satisfy the Schrödinger's equation (13.18.1), we should have:

$$a^2 = \frac{2mE}{\hbar^2} \quad \text{and} \quad \frac{m^2\omega^2}{\hbar^2} = a^4$$

which yields:

$$E_0 = \frac{\hbar^2 a^2}{2m} = \frac{1}{2}\hbar\omega$$

Try $\psi(x) = \sqrt{a/2\sqrt{\pi}}\,2axe^{-a^2x^2/2}$ in $d\psi/dx$:

$$\frac{d\psi}{dx} = 2a\sqrt{\frac{a}{2\sqrt{\pi}}}e^{\frac{-a^2x^2}{2}} - 2a^3x^2\sqrt{\frac{a}{2\sqrt{\pi}}}e^{\frac{-a^2x^2}{2}} = (2a - 2a^3x^2)\sqrt{\frac{a}{2\sqrt{\pi}}}e^{\frac{-a^2x^2}{2}}$$

so:

$$\frac{d^2\psi}{dx^2} = -4a^3x\sqrt{\frac{a}{2\sqrt{\pi}}}e^{\frac{-a^2x^2}{2}} + (2a^5x^3 - 2a^3x)\sqrt{\frac{a}{2\sqrt{\pi}}}e^{\frac{-a^2x^2}{2}} = (2a^5x^3 - 6a^3x)\sqrt{\frac{a}{2\sqrt{\pi}}}e^{\frac{-a^2x^2}{2}}$$

In order to satisfy the Schrödinger's equation (13.18.1), we should have:

$$3a^2 = \frac{2mE}{\hbar^2} \quad \text{and} \quad \frac{m^2\omega^2}{\hbar^2} = a^4$$

which yields:

$$E_1 = \frac{3\hbar^2 a^2}{2m} = \frac{3}{2}\hbar\omega$$

13.19

When $n = 0$:

$$N_0 = (a/\pi^{1/2}2^0 0!)^{1/2} = \sqrt{a/\sqrt{\pi}}$$

$$H_0(ax) = (-1)^0 e^{a^2x^2} e^{-a^2x^2} = 1$$

Hence:

$$\psi_0 = N_0 H_0(ax)e^{-a^2x^2/2} = \sqrt{a/\sqrt{\pi}}\,e^{-a^2x^2/2}$$

When $n = 1$:

$$N_1 = (a/\pi^{1/2}2^1 1!)^{1/2} = \sqrt{a/2\sqrt{\pi}}$$

$$H_1(ax) = (-1)^1 e^{a^2x^2} \frac{d}{d(ax)} e^{-a^2x^2} = -e^{a^2x^2} \cdot e^{-a^2x^2} \cdot (-2ax) = 2ax$$

Hence:

$$\psi_1 = N_1 H_1(ax) e^{-a^2x^2/2} = \sqrt{a/2\sqrt{\pi}}\, 2axe^{-a^2x^2/2}$$

When $n = 2$:

$$N_2 = (a/\pi^{1/2} 2^2 2!)^{1/2} = \sqrt{a/8\sqrt{\pi}} = \frac{\sqrt{a/2\sqrt{\pi}}}{2}$$

$$H_2(ax) = (-1)^2 e^{a^2x^2} \frac{d^2}{d(ax)^2} e^{-a^2x^2} = -2 + 4a^2x^2$$

Hence:

$$\psi_2 = N_2 H_2(ax) e^{-a^2x^2/2} = \sqrt{a/2\sqrt{\pi}}\,(2a^2x^2 - 1) e^{-a^2x^2/2}$$

When $n = 3$:

$$N_3 = (a/\pi^{1/2} 2^3 3!)^{1/2} = \sqrt{a/48\sqrt{\pi}} = \frac{\sqrt{a/3\sqrt{\pi}}}{4}$$

$$H_3(ax) = (-1)^3 e^{a^2x^2} \frac{d^3}{d(ax)^3} e^{-a^2x^2} = 8a^3x^3 - 12ax$$

Hence:

$$\psi_3 = N_3 H_3(ax) e^{-a^2x^2/2} = \sqrt{a/3\sqrt{\pi}}\,(2a^3x^3 - 3ax) e^{-a^2x^2/2}$$

13.20

The reflection angle θ_r and reflection wavelength λ_d should satisfy Bragg condition:

$$2a \sin\theta_r = \lambda_r$$

where a is separation of the atomic plane of the nickel crystal. Hence the reflected electron momentum p_r should satisfy:

$$p_r = \frac{h}{\lambda_r} = \frac{h}{2a\sin\theta_r}$$

Hence, the reflected electron energy is given by:

$$E_r = \frac{p_r^2}{2m_e} = \frac{h^2}{8m_e a^2 \sin^2\theta_r}$$
$$= \frac{(6.63\times10^{-34})^2}{8\times9.1\times10^{-31}\times(0.91\times10^{-10})^2\times\sin^2 65^\circ}$$
$$= 8.88\times10^{-19}[\mathrm{J}] = 55.5[\mathrm{eV}]$$

The difference between the incident and scattered kinetic energies is given by:

$$\frac{|E_r - E_i|}{E_i} = \frac{55.5-54}{54}\times100\% = 2.8\% < 3.9\%$$

13.21

For $\psi = \sin ka$:

Since ψ, ψ^*, V are all periodic functions with a period of a, we have:

$$\Delta E = \frac{\int \psi^* V \psi dx}{\int \psi^* \psi dx} = -\sum_{m=1}^{\infty} \frac{\int_0^a \sin^2 kx V_m \cos\frac{2\pi mx}{a} dx}{\int_0^a \sin^2 kx dx}$$
$$= -\sum_{m=1}^{\infty} V_m \frac{\int_0^a \frac{1-\cos(2\pi nx/a)}{2}\cos\frac{2\pi mx}{a}dx}{\int_0^a \frac{1-\cos(2\pi nx/a)}{2}dx}$$
$$= -\sum_{m=1}^{\infty} V_m \frac{-\frac{1}{2}\int_0^a \cos\frac{2\pi nx}{a}\cdot\cos\frac{2\pi mx}{a}dx}{a/2}$$
$$= \sum_{m=1}^{\infty} V_m \frac{\frac{1}{2}\int_0^a \left[\cos\frac{2\pi(m+n)x}{a} + \cos\frac{2\pi(m-n)x}{a}\right]dx}{a}$$

The above equation has non-zero term only when $m = n$, so we have:

$$\Delta E = V_n \frac{\int_0^a \left(\cos\frac{4\pi nx}{a} + 1\right)dx}{2a} = V_n \frac{a}{2a} = \frac{1}{2}V_n$$

For $\psi = \cos ka$:

$$\Delta E=\frac{\int\psi^* V\psi dx}{\int\psi^*\psi dx}=-\sum_{m=1}^{\infty}\frac{\int_0^a \cos^2 kxV_m\cos\frac{2\pi mx}{a}dx}{\int_0^a\cos^2 kxdx}$$

$$=-\sum_{m=1}^{\infty}V_m\frac{\int_0^a\frac{1+\cos(2\pi nx/a)}{2}\cos\frac{2\pi mx}{a}dx}{\int_0^a\frac{1+\cos(2\pi nx/a)}{2}dx}$$

$$=-\sum_{m=1}^{\infty}V_m\frac{\frac{1}{2}\int_0^a\cos\frac{2\pi nx}{a}\cdot\cos\frac{2\pi mx}{a}dx}{a/2}$$

$$=-\sum_{m=1}^{\infty}V_m\frac{\frac{1}{2}\int_0^a\left[\cos\frac{2\pi(m+n)x}{a}+\cos\frac{2\pi(m-n)x}{a}\right]dx}{a}$$

The above equation has non-zero term only when $m=n$, so we have:

$$\Delta E=-V_n\frac{\int_0^a\left(\cos\frac{4\pi nx}{a}+1\right)dx}{2a}=-V_n\frac{a}{2a}=-\frac{1}{2}V_n$$

SOLUTIONS TO CHAPTER 14

14.1

For $\theta_0 < 30^\circ$, we have:

$$T < T_0\left(1+\frac{1}{4}\sin^2\frac{30^\circ}{2}\right) = 1.017T_0$$

i.e.
$$\frac{T-T_0}{T_0} = 1.7\% < 2\%$$

For $\theta_0 = 90^\circ$, we have:

$$T = T_0\left(1+\frac{1}{4}\sin^2\frac{90^\circ}{2}\right) = 1.125T_0$$

i.e.
$$\frac{T-T_0}{T_0} = 12.5\%$$

14.2

Multiplying the equation of motion by $2\,dx/dt$ and integrating with respect to t gives:

$$m\left(\frac{dx}{dt}\right)^2 = A - 2\int_0^x f(x)dx$$

where A is the constant of integration. The velocity $\frac{dx}{dt}$ is zero at the maximum displacement $x = x_0$, giving $A = 2\int_0^{x_0} f(x)dx$.

i.e.
$$m\left(\frac{dx}{dt}\right)^2 = 2\int_0^{x_0} f(x)dx - 2\int_0^{x} f(x)dx = 2F(x_0) - 2F(x)$$

i.e.
$$\frac{dx}{dt} = \sqrt{\frac{2}{m}[F(x_0) - F(x)]}$$

Upon integration of the above equation, we have:

$$t = \int \sqrt{\frac{m}{2}} \frac{dx}{\sqrt{F(x_0) - F(x)}}$$

If $x = 0$ at time $t = 0$ and τ_0 is the period of oscillation, then $x = x_0$ at $t = \tau_0/4$, so we have:

$$\tau_0 = 4\sqrt{\frac{m}{2}} \int_0^{x_0} \frac{dx}{\sqrt{F(x_0) - F(x)}}$$

14.3

By substitution of the solution into $\ddot{x}$:

$$\ddot{x} = \sum_{n=1}^{\infty} \left[-a_n \frac{n^2}{9} \cos\frac{n}{3}\phi - b_n \frac{n^2}{9} \sin\frac{n}{3}\phi \right]$$

Since $s_3 << s_1$, we have $s(x) \approx s_1 x$, so:

$$\ddot{x} + s(x) = \sum_{n=1}^{\infty} \left[a_n \left(s_1 - \frac{n^2}{9} \right) \cos\frac{n}{3}\phi + b_n \left(s_1 - \frac{n^2}{9} \right) \sin\frac{n}{3}\phi \right] = F_0 \cos\omega t$$

i.e. $$\sum_{n=1}^{\infty} \left[a_n \left(s_1 - \frac{n^2}{9} \right) \cos\frac{n}{3}\phi + b_n \left(s_1 - \frac{n^2}{9} \right) \sin\frac{n}{3}\phi \right] = F_0 \cos\phi$$

i.e.

The above equation is true only if $b_n = 0$ and the even numbered cosine terms are zero. By neglecting the zero terms, we have:

$$a_3(s_1 - 1)\cos\phi + a_9(s_1 - 9)\cos 3\phi + \cdots = F_0 \cos\phi$$

i.e. $$a_3(s_1 - 1)\cos\phi + a_9(s_1 - 9)(4\cos^3\phi - 3\cos\phi) + \cdots = F_0 \cos\phi$$

i.e. $$[a_3(s_1 - 1) - 3a_9(s_1 - 9)]\cos\phi + 4a_9(s_1 - 9)\cos^3\phi + \cdots = F_0 \cos\phi$$

As we can see, only a_3 and a_9 are the main coefficients in the solution, i.e. the fundamental frequency term and its third harmonic term are the significant terms in the solution.

14.4

Since $V = V_0$ at $r = r_0$, by expanding V at r_0, we have:

$$V = V_0 + \left(\frac{dV}{dr} \right)_{r_0} (r - r_0) + \left(\frac{d^2V}{dr^2} \right)_{r_0} (r - r_0)^2 + \cdots$$

Noting that:

$$\left(\frac{dV}{dr}\right)_{r_0} = 12V_0\left(\frac{r_0^6}{r_0^7} - \frac{r_0^{12}}{r_0^{13}}\right) = 0$$

$$\left(\frac{d^2V}{dr^2}\right)_{r_0} = 12V_0\left(\frac{13r_0^{12}}{r_0^{14}} - \frac{7r_0^6}{r_0^8}\right) = 72\frac{V_0}{r_0^2}$$

We have:

$$V = V_0 + \frac{72V_0}{r_0^2}(r - r_0)^2 + \cdots$$

The expression of potential energy for harmonic oscillation is given by: $V = V_0 + \frac{1}{2}sx^2$, hence $s = \frac{72V_0}{r_0^2}$, and the oscillation frequency is given by:

$$\omega^2 = \frac{s}{m} = \frac{72V_0}{mr_0^2}$$

14.5

The restoring force of this oscillator is given by:

$$F(x) = -\frac{dV(x)}{dx} = -kx + ax^2$$

Hence, the equation of motion is given by:

$$m\ddot{x} = F(x)$$

i.e.

$$\ddot{x} + \frac{k}{m}x - \frac{a}{m}x^2 = 0$$

At $\omega_0^2 = \omega^2 = k/m$, using $\alpha = a/m$, the equation of motion becomes:

$$\ddot{x} + \omega_0^2 x - \alpha x^2 = 0$$

try the solution $x = A\cos\omega_0 t + B\sin 2\omega_0 t + x_1$ in the above equation with $x_1 = \frac{A^2}{2\omega_0^2}$

and $B = -\frac{\alpha A^2}{6\omega_0^2} = -\frac{\alpha x_1}{3}$, we have:

$$\dot{x} = \dot{x}_1 - \omega_0 A\sin\omega_0 t + 2\omega_0 B\cos 2\omega_0 t$$

$$\ddot{x} = \ddot{x}_1 - \omega_0^2 A\cos\omega_0 t - 4\omega_0^2 B\sin 2\omega_0 t = \ddot{x}_1 - \omega_0^2 A\cos\omega_0 t - \frac{4}{3}\alpha\omega_0^2 x_1 \sin 2\omega_0 t \qquad \text{(a)}$$

$$x^2 = x_1^2 + A^2\cos^2\omega_0 t + B^2\sin^2 2\omega_0 t + 2AB\cos\omega_0 t\sin 2\omega_0 t + 2(A\cos\omega_0 t + B\sin 2\omega_0 t)x_1$$

$$-\alpha x^2 = -\alpha x_1^2 - \alpha A^2\cos^2\omega_0 t - \alpha B^2\sin^2 2\omega_0 t - 2\alpha AB\cos\omega_0 t\sin 2\omega_0 t$$
$$-2\alpha(A\cos\omega_0 t + B\sin 2\omega_0 t)x_1$$

where

$$-\alpha A^2\cos^2\omega_0 t = -2\alpha\omega_0^2 x_1\cos^2\omega_0 t \qquad \text{(b)}$$

$$-\alpha B^2\sin^2 2\omega_0 t = -\frac{\alpha^2}{9}x_1^2\sin^2 2\omega_0 t \qquad \text{(e)}$$

$$-2\alpha AB\cos\omega_0 t\sin 2\omega_0 t = \left(\frac{2}{3}\alpha^2 A\cos\omega_0 t\sin 2\omega_0 t\right)x_1 \qquad \text{(c)}$$

$$-2\alpha Ax_1\cos\omega_0 t - 2\alpha Bx_1\sin 2\omega_0 t = -2\alpha Ax_1\cos\omega_0 t \qquad \text{(d)}$$

$$+\frac{2}{3}\alpha^2 x_1^2\sin 2\omega_0 t \qquad \text{(f)}$$

Using (a)(b)(c)(d) the coefficients of x_1 are:

$$\omega_0^2 - \frac{4}{3}\alpha\omega_0^2\sin 2\omega_0 t - 2\alpha\omega_0^2\cos^2\omega_0 t - 2\alpha^2 A\cos\omega_0 t\sin 2\omega_0 t - 2\alpha A\cos\omega_0 t$$

which with $\alpha << \omega_0^2$ leaves $\omega_0^2 x_1$ as the only significant term.

Similarly using (e) and (f) the coefficients of x_1^2 are:

$$-\alpha - \frac{\alpha^2}{9}\sin 2\omega_0 t + \frac{2}{3}\alpha^2\sin 2\omega_0 t$$

with $-\alpha x_1^2$ the dominant term. (Note $x_1^2 \propto \frac{1}{\omega_0^4}$).

We therefore have $\ddot{x}_1 + \omega_0^2 x_1 - \alpha x_1^2 = 0$ as a good approximation to the original equation.

14.6

Extending the chain rule at the bottom of page 472 and noting that the fixed point x_0 is the origin of the cycle: $x_1^* \to x_2^* \to x_1^* \to x_2^*$ which are fixed points for f^2 when $\lambda > \frac{3}{4}$ and also noting that $x_1^* = f(x_2^*)$ and $x_2^* = f(x_1^*)$, we have:

$$f^{2'}(x_2^*) = f'(x_1^*)f'(x_2^*) \quad \text{and} \quad f^{2'}(x_1^*) = f'(x_2^*)f'(x_1^*)$$

So the slopes of f^2 at x_1^* and x_2^* are equal.

14.7

The fractal dimension of the Koch Snowflake is:

$$d = \frac{\log 4}{\log 3} = 1.262$$

The Hausdorff–Besicovitch definition uses a scaling process for both integral and fractal dimensions to produce the relation:

$$c = a^d$$

where c is the number of copies(including the original) produced when a shape of dimensions d has its side length increased by a factor a.

Thus, for $a = 2$

(i) a line $d = 1$ has $c = 2$

(ii) a square $d = 2$ has $c = 4$

(iii) a cube $d = 3$ has $c = 8$

(iv) an equilateral triangle with a horizontal base produces 3 copies to give $d = \frac{\log 3}{\log 2} = 1.5849$ (a fractal)

SOLUTIONS TO CHAPTER 15

15.1

In the energy conservation equation the internal energy:

$$e = c_V T = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

so the two terms:

$$e + \frac{p}{\rho} = \frac{\gamma}{\gamma - 1}\frac{p}{\rho}$$

In the reservoir there is no flow energy, so its total energy is internal $= \frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0} = \frac{1}{\gamma-1}c_0^2$,

where c_0 is the velocity of sound in the reservoir.

When the diaphragm where is flow along the tube of velocity u and energy $1/2u^2$ so the total energy of flow along the tube is:

$$\frac{1}{2}u^2 + \frac{\gamma}{\gamma-1}\frac{p^*}{\rho^*} = \frac{1}{2}c^{*2} + \frac{1}{\gamma-1}c^{*2} = \left(\frac{1}{2} + \frac{1}{\gamma-1}\right)c^{*2}$$

where $u = c^*$, $u^2 = c^{*2} = \frac{\gamma p^*}{\rho^*}$.

Hence,

$$\frac{1}{\gamma-1}c_0^2 = \frac{2+(\gamma-1)}{2(\gamma-1)}c^{*2} = \frac{\gamma+1}{2(\gamma-1)}c^{*2}$$

If the wavefront flows at a velocity u_1 with a local velocity of sound c_1, the energy conservation condition gives:

$$\frac{1}{2}u_1^2 + \frac{1}{\gamma-1}c_1^2 = \frac{\gamma+1}{2(\gamma-1)}c^{*2}$$

i.e. $$\frac{1}{2} + \frac{1}{\gamma-1}\left(\frac{c_1}{u_1}\right)^2 = \frac{\gamma+1}{2(\gamma-1)}\left(\frac{c^*}{u_1}\right)^2$$

i.e. $$\frac{1}{2} + \frac{1}{(\gamma-1)M_s^2} = \frac{\gamma+1}{2(\gamma-1)M^{*2}}$$

i.e. $$M^{*2} = \frac{(\gamma+1)M_s^2}{(\gamma-1)M_s^2+2}$$

15.2

Energy conservation gives

$$\frac{1}{\gamma-1}c_1^2 + \frac{1}{2}u_1^2 = \frac{1}{\gamma-1}c_2^2 + \frac{1}{2}u_2^2 = \frac{\gamma+1}{2(\gamma-1)}c^{*2}$$ from Problem 15.1

So

$$c_1^2 + \frac{\gamma-1}{2}u_1^2 = c_2^2 + \frac{\gamma-1}{2}u_2^2 = \frac{\gamma+1}{2}c^{*2}$$

(A)

Momentum conservation gives

$$c_1^2 + \gamma u_1^2 = \frac{\rho_2}{\rho_1}(c_2^2 + \gamma u_2^2) = \frac{u_1}{u_2}(c_2^2 + \gamma u_2^2)$$

(B)

Combine equations A and B to eliminate c_1^2 and c_2^2 and rearrange terms to give

$$\frac{\gamma+1}{2}c^{*2} - \frac{\gamma-1}{2}u_1^2 + \gamma u_1^2 = \frac{u_1}{u_2}\left[\frac{\gamma+1}{2}c^{*2} - \frac{\gamma-1}{2}u_2^2 + \gamma u_2^2\right]$$

i.e. $$\frac{\gamma+1}{2}(c^{*2} + u_1^2) = \frac{u_1}{u_2}\left(\frac{\gamma+1}{2}\right)(c^{*2} + u_2^2)$$

i.e. $$(u_1 - u_2)c^{*2} = (u_2u_1^2 - u_1u_2^2) = u_1u_2(u_1 - u_2)$$

i.e. $$c^{*2} = u_1u_2$$

15.3

The three conservation equations are given by:

$$\rho_1 u_1 = \rho_2 u_2$$

(15.3.1)

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$

(15.3.2)

$$\frac{1}{2}u_1^2 + e_1 + \frac{p_1}{\rho_1} = \frac{1}{2}u_2^2 + e_2 + \frac{p_2}{\rho_2}$$

(15.3.3)

Using equation 15.3.1 and 15.3.2 to eliminate u_1 gives:

$$\frac{\rho_2^2 - \rho_1\rho_2}{\rho_1} u_2^2 = p_2 - p_1$$

(15.3.4)

Using equation 15.3.1 and 15.3.3 and the relation

$$e = c_V T = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

to eliminate u_1 gives:

$$\frac{\rho_2^2 - \rho_1^2}{2\rho_1^2} u_2^2 = \frac{\gamma}{\gamma - 1}\left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right)$$

(15.3.5)

Then, using equation 15.3.4 and 15.3.5 to eliminate u_2, we have:

$$\frac{\rho_2 + \rho_1}{2} = \frac{\gamma}{\gamma - 1}(\rho_1 p_2 - \rho_2 p_1)\frac{1}{p_2 - p_1}$$

i.e. $$\frac{\rho_2/\rho_1 + 1}{2} = \frac{\gamma}{\gamma - 1}\frac{p_2/p_1 - \rho_2/\rho_1}{p_2/p_1 - 1}$$

i.e. $$[(1 + \rho_2/\rho_1)(\gamma - 1) - 2\gamma]\, p_2/p_1 = (1 + \rho_2/\rho_1)(\gamma - 1) - 2\gamma\beta$$

i.e. $$[\beta(\gamma - 1) - (\gamma + 1)]\, p_2/p_1 = (\gamma - 1) - \beta(\gamma + 1)$$

which yields:

$$\frac{p_2}{p_1} = \frac{\beta - \alpha}{1 - \beta\alpha}$$

where $\alpha = (\gamma - 1)/(\gamma + 1)$ and $\beta = \rho_2/\rho_1$.

15.4

Using the result of Problems 15.1 and 15.2, we have:

$$M^{*2} = \left(\frac{u_1}{c^*}\right)^2 = \frac{u_1^2}{u_1 u_2} = \frac{u_1}{u_2} = \frac{(\gamma + 1)M_s^2}{(\gamma - 1)M_s^2 + 2}$$

(15.4.1)

Using equation 15.3.1 and 15.3.2 to eliminate ρ_1 gives:

$$\frac{p_1 - p_2}{\rho_2 u_2^2} = 1 - \frac{u_1}{u_2}$$

(15.4.2)

Using equation 15.3.1 and 15.3.3 to eliminate ρ_1 gives:

$$\frac{1}{2}\left(1-\frac{u_1^2}{u_2^2}\right)=\frac{1}{\rho_2 u_2^2}\frac{\gamma}{\gamma-1}\left(\frac{u_1}{u_2}p_1-p_2\right)$$

(15.4.3)

Using equation 15.4.2 and 15.4.3 to eliminate ρ_2 gives:

$$\left(\frac{1}{\alpha}+y\right)\frac{u_1}{u_2}=\frac{y}{\alpha}+1$$

(15.4.4)

where $y = p_2/p_1$ and $\alpha=(\gamma-1)/(\gamma+1)$.

Then, using equation 15.4.1 and 15.4.4 to eliminate u_1/u_2 we have:

$$M_s=\frac{u_1}{c_1}=\sqrt{\frac{y+\alpha}{1+\alpha}}$$

(15.4.5)

Frome equation 15.4.4 and 15.4.5 we have:

$$\frac{u_2}{c_1}=\frac{1+\alpha y}{\sqrt{1+\alpha}\sqrt{y+\alpha}}$$

(15.4.6)

Hence, from equations 15.4.5 and 15.4.6 we have the flow velocity behind the shock give by:

$$u=u_1-u_2=\frac{c_1(1-\alpha)(y-1)}{\sqrt{1(1+\alpha)(y+\alpha)}}$$

15.5

In the case of reflected shock wave, as shown in Fig.(b), the shock strength is p_3/p_2 and the velocity of sound ahead of the shock front is c_2. Hence, using the result of Problem 15.4, we have the flow velocity u_r behind the reflected wave given by:

$$\frac{u_r}{c_2}=\frac{(1-\alpha)(p_3/p_2-1)}{\sqrt{(1+\alpha)(p_3/p_2+\alpha)}}$$

(15.5.1)

In Fig.(a) the flow velocity u behind of the incident shock front is given by:

$$\frac{u}{c_1} = \frac{(1-\alpha)(y-1)}{\sqrt{(1+\alpha)(y+\alpha)}}$$

(15.5.2)

Using equation 15.5.1 and 15.5.2 together with the relation $u + u_r = 0$, $c_2/c_1 = (T_1/T_2)^{1/2}$ and $\frac{T_2}{T_1} = y\left(\frac{1+\alpha y}{\alpha + y}\right)$, where $y = p_2/p_1$, we have:

$$\frac{(y-1)^2(p_3/p_2+\alpha)}{(p_3/p_2-1)^2} = y(1+\alpha y)$$

which yields:

$$\frac{p_3}{p_2} = \frac{(2\alpha+1)y-\alpha}{\alpha y+1}$$

15.6

$$\frac{p_3-p_1}{p_2-p_1} = \frac{p_3/p_2 - p_1/p_2}{1-p_1/p_2} = \frac{p_3/p_2 - 1/y}{1-1/y} = \frac{y\,p_3/p_2 - 1}{y-1}$$

By substitution of $y = p_2/p_1$ and $\frac{p_3}{p_2} = \frac{(2\alpha+1)y-\alpha}{\alpha y+1}$ into the above equation, we have:

$$\frac{p_3-p_1}{p_2-p_1} = \frac{\frac{(2\alpha+1)y^2-\alpha y}{\alpha y+1} - 1}{y-1} = \frac{(2\alpha+1)y^2 - 2\alpha y - 1}{(y-1)(\alpha y+1)}$$

In the limit of very strong shock, i.e. $y >> 1$, we have:

$$\frac{p_3-p_1}{p_2-p_1} \approx \frac{(2\alpha+1)y^2}{\alpha y^2} = 2 + \frac{1}{\alpha}$$

15.7

$$u_t = -(u + t u_t)\, f'$$

i.e. $$u_t = -\frac{uf'}{1+tf'}$$

From equation 15.9 in the text, we have:

$$u_x = \frac{f'}{1+tf'}$$

so we have:

$$u_t + uu_x = -\frac{uf'}{1+tf'} + \frac{uf'}{1+tf'} = 0$$

15.8

Using $u = -2\nu\psi_x/\psi$, we have:

$$u_t = -2\nu \frac{\psi_{xt}\psi - \psi_t\psi_x}{\psi^2}$$

$$u_x = -2\nu \frac{\psi_{xx}\psi - \psi_x^2}{\psi^2}$$

$$u_{xx} = -2\nu\left[\frac{\psi_{xxx}}{\psi} - \frac{3\psi_x\psi_{xx}}{\psi^2} + \frac{2\psi_x^3}{\psi^3}\right]$$

By substitution of the above expression into Burger's equation, we have:

$$-2\nu \frac{\psi_{xt}\psi - \psi_t\psi_x}{\psi^2} + 4\nu^2 \frac{\psi\psi_x\psi_{xx} - \psi_x^3}{\psi^3} + 2\nu^2\left(\frac{\psi_{xxx}}{\psi} - \frac{3\psi_x\psi_{xx}}{\psi^2} + \frac{2\psi_x^3}{\psi^3}\right) = 0$$

i.e. $$\frac{\psi\psi_{xt} - \psi_t\psi_x}{\psi^2} = \nu\frac{\psi\psi_{xxx} - \psi_x\psi_{xx}}{\psi^2}$$

i.e. $$\left(\frac{\psi_t}{\psi}\right)_x = \left(\nu\frac{\psi_{xx}}{\psi}\right)_x$$

which yields:

$$\psi_t = \nu\psi_{xx}$$

15.9

Using the relation $\tanh'\phi \equiv \mathrm{sech}^2\phi$ and $\mathrm{sech}'\phi = \mathrm{sech}\phi\tanh\phi$, where $\phi = \alpha(x - ct)$, we have the derivatives:

$u_t = 4\alpha^3 c\, \mathrm{sech}^2\phi \tanh\phi = 2\alpha uc \tanh\phi$

$u_x = -4\alpha^3 \mathrm{sech}^2\phi \tanh\phi = -2\alpha u \tanh\phi$

$u_{xx} = 8\alpha^4 \mathrm{sech}^2\phi \tanh^2\phi - 4\alpha^4 \mathrm{sech}^4\phi = 4\alpha^2 u \tanh^2\phi - u^2$

$u_{xx} = -16\alpha^5 \mathrm{sech}^2\phi \tanh\phi + 48\alpha^5 \mathrm{sech}^4\phi \tanh\phi = -8\alpha^3 u \tanh\phi + 12\alpha u^2 \tanh\phi$

Then, using the relation $c = 4\alpha^2$ we have:

$u_t + 6uu_x + u_{xxx}$
$= 2\alpha uc \tanh\phi - 12\alpha u^2 \tanh\phi - 8\alpha^3 u \tanh\phi + 12\alpha u^2 \tanh\phi$
$= 0$

15.10

At the peak of Figure 15.5(a), $\phi = 2\alpha(x - ct) = 0$, and near the base of Figure 15.5(a), $\phi = 2\alpha(x - ct) >> 0$, i.e. $e^{-\phi} \approx 0$. Hence, the solution of the KdV equation can be written as:

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\log[1 + e^{-2\alpha(x-ct)}] \approx 2\frac{\partial^2}{\partial x^2}e^{-2\alpha(x-ct)} = 8\alpha^2 e^{-2\alpha(x-ct)}$$

Then, we have the derivatives,

$$u_t = 16\alpha^3 c e^{-2\alpha(x-ct)}$$

$$u_{xxx} = -64\alpha^5 e^{-2\alpha(x-ct)}$$

Therefore, if $c = 4\alpha^2$,

$$u_t + u_{xxx} = 0$$

15.11

Using the substitution $z = x - ct$ and the relation $\tanh'\phi \equiv \text{sech}^2\phi$ and $\text{sech}'\phi \equiv -\text{sech}\phi\tanh\phi$, where $\phi = \alpha(z - z_0)$, we have the derivatives:

$u_t = -4\alpha^3 c\, \text{sech}^2\phi \tanh\phi = 2\alpha uc \tanh\phi$

$u_x = -4\alpha^3\, \text{sech}^2\phi \tanh\phi = -2\alpha u \tanh\phi$

$u_{xx} = 4\alpha^4\, \text{sech}^4\phi + 4\alpha^2 u \tanh^2\phi = 4\alpha^2 u \tanh^2\phi + u^2$

$u_{xx} = -16\alpha^5\, \text{sech}^2\phi \tanh\phi + 48\alpha^5\, \text{sech}^4\phi \tanh\phi = -8\alpha^3 u \tanh\phi - 12\alpha u^2 \tanh\phi$

Then, using the relation $c = 4\alpha^2$, we have:

$u_t - 6uu_x + u_{xxx}$
$= 2\alpha uc \tanh\phi + 12\alpha u^2 \tanh\phi + 8\alpha^3 u \tanh\phi - 12\alpha u^2 \tanh\phi$
$= -8\alpha^3 u \tanh\phi + 8\alpha^3 u \tanh\phi$
$= 0$

15.12

If $v^2 + v_x = u$, then:

$$u_x = 2vv_x + v_{xx}$$

$$u_{xx} = 2(v_x^2 + vv_{xx}) + v_{xxx}$$

$$u_{xxx} = 2(2v_x v_{xx} + vv_{xxx} + v_x v_{xx}) + v_{xxxx} = 6v_x v_{xx} + 2vv_{xxx} + v_{xxxx}$$

$$u_t = 2vv_t + v_{xt}$$

The left side terms of equation mark in equation 15.13 give:

$$\begin{aligned}&\left(\frac{\partial}{\partial x} + 2v\right)(v_t - 6v^2 v_x + v_{xxx})\\ &= v_{xt} - 6(2vv_x^2 + v^2 v_{xx}) + v_{xxxx} + 2v(v_t - 6v^2 v_x + v_{xxx})\\ &= 2vv_t - 12v^3 v_x + v_{xt} - 12vv_x^2 - 6v^2 v_{xx} + 2vv_{xxx} + v_{xxxx}\end{aligned}$$

The right side terms of equation mark in equation 15.13 give:

$$\begin{aligned}&u_t - 6uu_x + u_{xxx}\\ &= 2vv_t + v_{xt} - 6(v^2 + v_x)(2vv_x + v_{xx}) + 6v_x v_{xx} + 2vv_{xxx} + v_{xxxx}\\ &= 2vv_t - 12v^3 v_x + v_{xt} - 12vv_x^2 - 6v^2 v_{xx} + 2vv_{xxx} + v_{xxxx}\end{aligned}$$

So we have:

$$\left(\frac{\partial}{\partial x} + 2v\right)(v_t - 6v^2 v_x + v_{xxx}) = u_t - 6uu_x + u_{xxx}$$

15.13

By substitution of $v = \psi_x / \psi$ into $u(x) = v_x + v^2$, we have:

$$u(x) = v_x + v^2 = \frac{\psi_{xx}\psi - \psi_x^2}{\psi^2} + \frac{\psi_x^2}{\psi^2} = \frac{\psi_{xx}}{\psi}$$

hence,

$$\psi_{xx} - u(x)\psi = \psi_{xx} - \psi\frac{\psi_{xx}}{\psi} = 0$$

15.14

$$\psi_x = -\alpha A \operatorname{sech}\alpha(x - x_0) \tanh\alpha(x - x_0)$$

$$\psi_{xx} = \alpha^2 A \operatorname{sech}\alpha(x - x_0) - 2\alpha^2 A \operatorname{sech}^3\alpha(x - x_0)$$

Using the soliton solution

$$u = -2\alpha^2 \operatorname{sech}^2\alpha(x - x_0)$$

we have:

$\psi_{xx} + (\lambda - u(x))\psi$

$= \alpha^2 A\,\mathrm{sech}\,\alpha(x - x_0) - 2\alpha^2 A\,\mathrm{sech}^3\alpha(x - x_0) + [-\alpha^2\,\mathrm{sech}^2\alpha(x - x_0)]A\,\mathrm{sech}\,\alpha(x - x_0)$

$= 0$

15.15

Using transformation $u \to u^* - \lambda$ and $x \to x^* + 6\lambda t$, where u^* and x^* are the variables before transformation, we have: $u^* = u + \lambda$ and $x^* = x - 6\lambda t$, hence:

$$u_t = (u^* - \lambda)_t = u_t^*$$

$$u_x = (u^* - \lambda)_{x^*} x_x^* = u_{x^*}^*$$

$$u_{xx} = u_{x^*x^*}^* x_x^* = u_{x^*x^*}^*$$

$$u_{xxx} = u_{x^*x^*x^*}^* x_x^* = u_{x^*x^*x^*}^*$$

Using the above relations, equation $u_t + 6uu_x + u_{xxx} = 0$ is transformed to its original form:

$$u_t^* + 6u^* u_{x^*}^* + u_{x^*x^*x^*}^* = 0$$

15.16

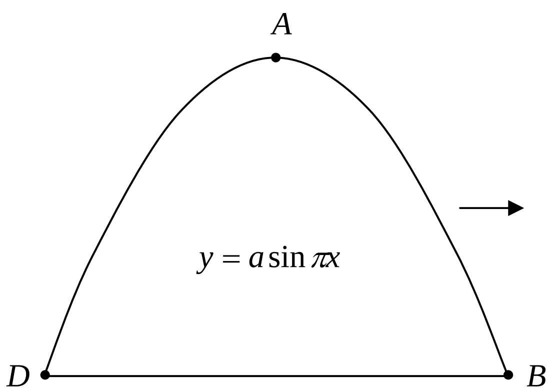


Fig.A.15.16

In Fig.A.15.16, A is the peak and B is the base point of the leading edge of the right going wave: $y = a\sin \pi x$. The phase velocity of any point on the leading edge is given by:

$$v = c_0\left(1 + \varepsilon a\frac{\partial y}{\partial x}\right)^{1/2} = c_0\left(1 + \frac{1}{2}\varepsilon a\frac{\partial y}{\partial x}\right) = c_0\left(1 + \frac{1}{2}\varepsilon a\pi\cos \pi x\right)$$

$\therefore$ phase velocity at $A: v_A = c_0 \quad \left(\cos \pi x = \cos\dfrac{\pi}{2} = 0\right)$

and phase velocity at $B: v_B = c_0\left(1 - \dfrac{1}{2}\varepsilon a \pi\right) \quad (\cos \pi x = \cos \pi = -1)$

$\therefore$ phase velocity of A related to B:

$$v_A - v_B = c_0 - c_0\left(1 - \frac{1}{2}\varepsilon a \pi\right) = \frac{1}{2}c_0 \varepsilon a \pi = |du|$$

The phase distance $|dx|$ between A and B $= \pi/2$

$\therefore$ Time for A to reach the position vertically above B is:

$$\left|\frac{dx}{du}\right| = \frac{\pi}{2}\frac{1}{1/2\, c_0 \varepsilon a \pi} = \frac{1}{c_0 \varepsilon a}\text{[secs]}$$