counter-example :

let
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 23 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 3 & 3 \\ 5 & 55 & 26 \\ 5 & 23 & 25 \end{pmatrix}$$

$$AB \neq BA$$

$\Rightarrow e^A e^B - e^{A+B} \neq 0$ [I used Wolfram Mathmetica 12 code to solve it quickly]

5 P8 - section 15

$$A = \begin{pmatrix} 1 & 0 & 2i \\ i & -3 & 0 \\ 1 & 0 & i \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & i & 1 \\ 0 & -3 & 0 \\ +2i & 0 & i \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -2i \\ -i & -3 & 0 \\ 1 & 0 & -i \end{pmatrix} = A^* = \bar{A}$$

$$A^+ = \begin{pmatrix} 1 & -i & 1 \\ 0 & -3 & 0 \\ -2i & 0 & -i \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -1 & 0 & 2 \\ -1/3 & -1/3 & 2/3 \\ -i & 0 & i \end{pmatrix}$$

1 back to question 1 :

$e^{A+B} = e^A e^B \iff$ A and B are commutative matrices

$e^{A+B} \neq e^A e^B$ for non-commutative matrices :

$$e^{A+B} = \sum_{i=0}^{\infty} \frac{(A+B)^i}{i!}$$

$$e^A e^B = \left(\sum_{i=0}^{\infty} \frac{A^i}{i!}\right)\left(\sum_{j=0}^{\infty} \frac{B^j}{j!}\right) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{A^i}{i!} \frac{B^j}{j!}$$

but in e^{A+B}, we cannot use bionomial expansion theorem to expand $(A+B)^i$ that is, $(A+B)^i \neq \sum_{k=0}^{i} \binom{i}{k} A^{i-k} B^k$ unless A and B are commutative.

To verify that :

$e^A e^B = \left(1 + A + \frac{A^2}{2!} + \dots\right)\left(1 + B + \frac{B^2}{2!} + \dots\right)$, if only multiply the first three terms :

$$= 1 + B + \frac{B^2}{2!} + A + AB + \frac{AB^2}{2!} + \frac{A^2}{2!} + \frac{A^2 B}{2!} + \frac{A^2 B^2}{2!\,2!}$$

but $e^{A+B} = 1 + (A+B) + \frac{(A+B)^2}{2!} + \frac{(A+B)^3}{3!} + \dots$

$$\Rightarrow \frac{(A+B)^2}{2!} = \frac{A^2 + B^2 + AB + BA}{2!} \neq \frac{A^2 + B^2 + 2AB}{2!} \text{ (for non-commutative)}$$

$$\Rightarrow \frac{(A+B)^3}{3!} = \left[A^3 + A^2 B + ABA + AB^2 + BAA + BAB + B^2 A + B^3\right] \frac{1}{3!}$$
$$\neq A^3 + 3A^2 B + 3AB^2 + B^3 \text{ (for non-commutative)}$$

□ Note that in $e^A e^B$ the term BA doesn't exist, so $e^{A+B} \neq e^A e^B$

P17 - section 9:

a) A and B are symmetric $\Leftrightarrow$ $A = A^T$ and $B = B^T$

$(AB)^T = B^T A^T = BA$

if A and B are commutative matrices $\Rightarrow (AB)^T = BA = AB$

if not: $(AB)^T = BA \neq AB \Rightarrow AB$ is not symmetric

b) A and B are orthogonal $\Leftrightarrow A^{-1} = A^T$ and $B^{-1} = B^T$

$(AB)^{-1} = B^{-1}A^{-1} = B^T A^T = (AB)^T \Rightarrow AB$ is orthogonal

c) A and B are Hermitian $\Leftrightarrow A^+ = A$, $B^+ = B$

$(AB)^+ = ((AB)^T)^* = (B^T A^T)^* = (B^T)^*(A^T)^*$

$= BA$

$\Rightarrow AB$ is Hermitian $\Leftrightarrow$ A and B commute.

d) let: A and B unitary matrices $\Rightarrow A^{-1} = A^+$ and $B^{-1} = B^+$

$(AB)^{-1} = B^{-1}A^{-1} = B^+ A^+ = (B^T A^T)^* = ((AB)^T)^* = (AB)^+$

$\Rightarrow AB$ is unitary matrix

4 P18 - section 11

let $A = \begin{pmatrix} -1 & 1 & 3 \\ 1 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix}$, $|A - \lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 1 & 3 \\ 1 & 2-\lambda & 0 \\ 3 & 0 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda = 4, -3, 2$

* Eigenvectors:

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$$

P25 - section 11

$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$, eigenvalues: $-2, 2, 1$

eigenvectors: $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

P26 - section 11:

$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, eigenvalues: $4, 1, 1$

eigenvectors: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

P 15 - section 9 :

$\boxed{a}$ The Pauli spin matrices are:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(A^T)^* = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^T\right)^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A \checkmark$$

$$(B^T)^* = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = B \checkmark$$

$$(C^T)^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = C \checkmark$$

$\boxed{b}$ Jacobi identity :

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

$$\underbrace{A(AB - BA) - (AB - BA)A}_{} + \underbrace{B(CA - AC) - (CA - AC)B}_{} + \underbrace{C(AB - BA) - (AB - BA)C}_{} = 0$$

$$\Rightarrow BC = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad CB = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad CA = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad AC = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$AB = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad BA = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$[B, C] = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix}, \quad [C, A] = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, \quad [A, B] = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$$

$$\Rightarrow [A, [B, C]] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad [C, [A, B]] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad [B, [C, A]] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow [A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \checkmark$$

Mathamatical Method

HW2

Hamza AlHasan - 118l636

[2] (a) The square root is not a linear operator.

$\sqrt{A+B} \neq \sqrt{A}+\sqrt{B}$ and $\sqrt{KA} \neq K\sqrt{A}$

(b) $|A+B| \neq |A|+|B|$ and $|KA_{n\times n}| = K^n |A_{n\times n}| \neq K\,|A_{n\times n}|$

↳ let $A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $B=\begin{pmatrix}0&0\\0&1\end{pmatrix}$

$|A+B| = \begin{vmatrix}1&0\\0&1\end{vmatrix} = 1$

$|A|=0=|B|$

$\Rightarrow |A|+|B| \neq |A+B|$ #

[3] P 11 - section 9 :

□ Real Hermitian matrix is symmetric.

Real matrix $\Leftrightarrow A = A^*$

Hermitian matrix $\Leftrightarrow A^+ = (A^T)^* = A$

Real Hermitian matrix $\Leftrightarrow A^+ = (A^T)^* = \boxed{A^T = A}$

$\Leftrightarrow$ symmetric matrix #

□ Real unitary matrix is orthogonal.

Unitary matrix $\Leftrightarrow A^{-1} = A^+$

Reall matrix $\Leftrightarrow A = A^*$

Real unitary matrix $\Leftrightarrow \underline{A^{-1}} = A^+ = (A^T)^* = \underline{A^T}$

$\Leftrightarrow A^{-1} = A^T \Leftrightarrow$ orthogonal matrix #

P 12 - section 9 :

~~let A = [a_ij] is a matrix~~

let A is a matrix with elements a_{ij}

$A^T = a_{ji}$, $\bar{A} = \overline{a_{ij}}$ or a^*_{ij}

$\Rightarrow$ for Hermitian matrix : $A^+ = (A^T)^* = A$

$a^+_{ij} = (a_{ij}{}^T)^* = (a_{ji})^* = a^*_{ji}$ or $\overline{a_{ji}}$ #

Example of Hermitian matrix:

$A=\begin{pmatrix}2&-i\\i&1\end{pmatrix}$, $A^T=\begin{pmatrix}2&i\\-i&1\end{pmatrix}$, $(A^T)^*=\begin{pmatrix}2&-i\\i&1\end{pmatrix}=A$