Birzeit University
Department of Physics
Mathematical Physics, Phys330
Fall 2020
Homework 3: Due date Oct. 12th 2020

1. In curvilinear coordinates systems, three coordinates(orthogonal is defined) q_1, q_2 and q_3. Any vector in 3D space is written as follows:

$$\vec{V} = V_1\hat{q}_1 + V_2\hat{q}_2 + V_3\hat{q}_3$$

The regular cartesian coordinates can be written as follows:

$$x = x(q_1, q_2, q_3)$$
$$y = y(q_1, q_2, q_3)$$
$$z = z(q_1, q_2, q_3)$$

   (a) apply the above equations for the case of spherical and cylindrical coordinates
   (b) Write the full derivative dx, dy and dz in terms of the three curvilinear coordinates.
   (c) Use the results of the previous part to write $\vec{dr} = dx\hat{i} + dy\hat{j} + dz\hat{k}$ as a summation in curvilinear coordinates
   (d) Calculate $ds^2 = \vec{dr} \cdot \vec{dr}$
   (e) Prove that

$$ds^2 = \sum_i (h_i dq_i)^2$$

   $h_i^2 = \frac{\partial \vec{r}}{\partial q_i} \cdot \frac{\partial \vec{r}}{\partial q_i}$.
   (f) Calculate h_r, h_θ, and h_ϕ (called scale factor)

2. The Cartesian surface element $dxdy$, which becomes an infinitesimal rectangle in the curvilinear coordinates q_1, q_2. The transformation between area element in cartesian coordinates (x,y,z) and curvilinear coordinates is given by:

$$dxdy = \begin{vmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} \end{vmatrix} dq_1 q_2$$

and the volume element is given by:

$$dxdydz = \begin{vmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \frac{\partial x}{\partial q_3} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \frac{\partial y}{\partial q_3} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3} \end{vmatrix} dq_1 dq_2 dq_3$$

Apply the area element for polar coordinate and the volume element for spherical coordinate. Comment on the results.

3. In curvilinear coordinates the gradient, divergence, curl, and the lamarckian are give by:

$$\begin{aligned} \vec{\nabla}\psi &= \sum_i \hat{q}_i \frac{1}{h_i}\frac{\partial \psi}{\partial q_i} \\ \vec{\nabla} \cdot \vec{V} &= \frac{1}{h_1 h_2 h_3}[\frac{\partial}{\partial q_1}(V_1 h_2 h_3) + \frac{\partial}{\partial q_2}(V_2 h_1 h_3) + \frac{\partial}{\partial q_3}(V_3 h_1 h_2)] \\ \vec{\nabla} \times \vec{V} &= \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \hat{q}_1 & \hat{q}_2 & \hat{q}_3 \\ \partial q_1 & \partial q_2 & \partial q_3 \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix} \end{aligned}$$

Apply these three for cylindrical and spherical coordinates, check your answers with the ones in the book. In addition find the laplacian