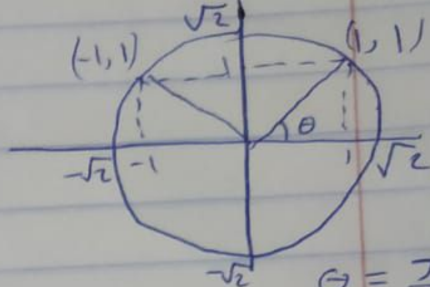
8. (C) $\int \vec{F}\cdot d\vec{r}$ along $x^2+y^2=2$ from $(-1,1)$ to $(1,1)$

$\vec{F} = (2x-3y)\hat{i} + (3x-2y)\hat{j}$



$\theta = \frac{\pi}{4}$

$\vec{F}\cdot d\vec{r} = (2x-3y)\,dx + (3x-2y)\,dy$

let $x = r\cos\theta$, $dx = -r\sin\theta\, d\theta$

$y = r\sin\theta$, $dy = r\cos\theta\, d\theta$

$r = \sqrt{2}$

$\int \vec{F}\cdot d\vec{r} = 2\int (2\cos\theta - 3\sin\theta)(-\sin\theta)\,d\theta + (3\cos\theta - 2\sin\theta)(\cos\theta)\,d\theta$

$= 2\int \left[-\sin(2\theta) + \underbrace{3\sin^2\theta + 3\cos^2\theta}_{3} - \sin(2\theta)\right] d\theta$

$= \int_{\theta=\pi/4}^{3\pi/4} 3 - 2\sin 2\theta \, d\theta = 3\theta + \cos(2\theta)\Big|_{\pi/4}^{3\pi/4}$

$= 2\times\left[\frac{9\pi}{4} + \overset{0}{\cos\left(\frac{3\pi}{2}\right)} - \frac{3\pi}{4} - \overset{0}{\cos\left(\frac{\pi}{2}\right)}\right]$

$= 2\times\frac{6\pi}{4} = \cancel{2}\times\frac{3\pi}{\cancel{2}} = 3\pi$

1. $x = a\cosh(u)\cos(v)$ $\quad q_1 = u$
$y = a\sinh(u)\sin(v)$ $\quad q_2 = v$
$z = z$ $\quad q_3 = z$

let $\vec{V} = V_1\vec{u} + V_2\vec{v} + V_3\vec{z}$; where $\vec{u}$, $\vec{v}$ and $\vec{z}$ are unit vectors

$$\nabla\cdot\vec{V} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}(V_1h_2h_3) + \frac{\partial}{\partial v}(V_2h_1h_3) + \frac{\partial}{\partial z}(V_3h_1h_2)\right]$$

$h_1 = h_u = \sqrt{\frac{dr}{du}\cdot\frac{dr}{du}}$

$dx = -a\cosh(u)\sin(v)\,dv + a\sinh(u)\cos(v)\,du$
$dy = a\sinh(u)\cos(v)\,dv + a\cosh(u)\cos(v)\,du$
$dz = dz$
$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$; where dx, dy and dz from above

back to $h_1 = h_u = \sqrt{\frac{dr}{du}\cdot\frac{dr}{du}} = \left(a^2\left[\sinh^2u\cos^2v + \cosh^2u\sin^2v\right]\right)^{1/2}$

$h_2 = h_v = \left|\frac{dr}{dv}\right| = \left(a^2\left[\cosh^2u\sin^2v + \sinh^2u\cos^2v\right]\right)^{1/2}$

$h_3 = h_z = 1$

Back to $\nabla\cdot\vec{V} =$

Continnone 5 :

$$\sin(M) = \mathrm{Im}(e^{iM})$$

$$= \mathrm{Im}\left[\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix} + a_1 i \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - a_2 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\right]$$

$$= \dots = \begin{pmatrix} 0 & \sin 1 & 0 \\ \sin 1 & 0 & \sin 1 \\ 0 & \sin 1 & 0 \end{pmatrix}$$

Continue 5:

for M, $b = -\sqrt{2}, 0, \sqrt{2}$

for iM, $b = -i\sqrt{2}, 0, i\sqrt{2}$

for $-iM$, $b = i\sqrt{2}, 0, -i\sqrt{2}$

$$e^{iM} = a_0 iM^0 + a_1 iM^1 + a_2 M^2(-1)$$

$b=0$: $e^0 = \underline{a_0 = 1}$

$b = i\sqrt{2}$: ~~$e^{i\sqrt{2}} = a_0(iI) + a_1(iM) + a_2(iM)^2$~~

$$e^{i\sqrt{2}} = a_0(i\sqrt{2}) + a_1(i\sqrt{2}) + a_2(i\sqrt{2})^2$$

~~$e^{i\sqrt{2}} = i\sqrt{2}(a_0 + a_1 + i\sqrt{2}a_2)$~~

$$\cos\sqrt{2} + i\sin\sqrt{2} = -2a_2 + i(\sqrt{2}a_0 + \sqrt{2}a_1)$$

$$\underline{a_2} = \frac{\cos\sqrt{2}}{-2}$$

$$\sqrt{2}(a_0 + a_1) = \sin\sqrt{2}$$

$$\underline{a_1} = \frac{\sin\sqrt{2}}{\sqrt{2}} - 1$$

so, we find a_0, a_1, a_2

- Same procedure to calculate $e^{-iM} = a_0(-iM^0) - a_1 iM - a_2 M^2$

$b=0$: $a_0 = 1$

$b = i\sqrt{2}$: $e^{+i\sqrt{2}} = \ldots$

$b = -i\sqrt{2}$: $e^{-i\sqrt{2}} = \ldots$

Finally, $\cos(M) = \dfrac{e^{iM} + e^{-iM}}{2}$

$$\sin(M) = \frac{e^{iM} - e^{-iM}}{2i}$$

$\underline{\text{or}}$ $\cos(M) = \mathrm{Re}(e^{iM})$

$$= \mathrm{Re}(iI + a_1 iM - a_2 M^2)$$

$$= \mathrm{Re}\left\{\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix} + a_1 i \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - a_2 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\right\}$$

$$\ldots = \begin{pmatrix} 1 & \cos 1 & 1 \\ \cos 1 & 1 & \cos 1 \\ 1 & \cos 1 & 1 \end{pmatrix} = \cos(M)$$

8. (a) $\oint \vec{F}\cdot d\vec{r}$ around $x^2+y^2+2x=0$, $\vec{F} = y\hat{i} - x\hat{j}$

$dr = dx\, i + dy\, \hat{j}$

$F\cdot dr = y\, dx - x\, dy$

$\oint F\cdot dr = \oint y\,dx - x\,dy$

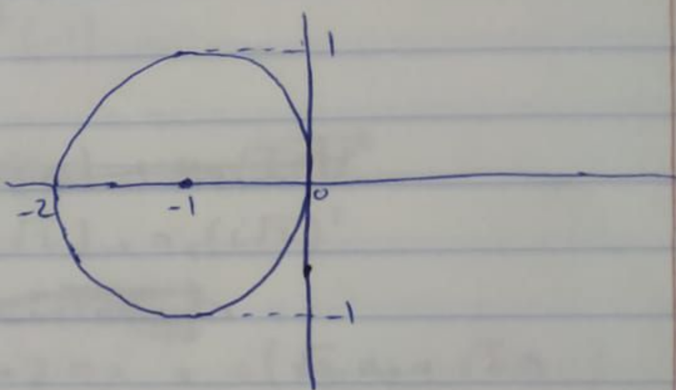
$x^2+y^2+2x=0$

$x^2+2x+1+y^2 = 1$

$(x+1)^2+y^2=1$

let $x = \cos\theta$, $y = \sin\theta$

$dx = -\sin\theta\, d\theta$; $dy = \cos\theta\, d\theta$



$\oint F\cdot dr = \oint \sin\theta(-\sin\theta\, d\theta) - \cos\theta(\cos\theta\, d\theta)$

$= \oint -\sin^2\theta\, d\theta - \cos^2\theta\, d\theta$

$= -\oint \sin^2\theta + \cos^2\theta\, d\theta$

$= -\oint_0^{2\pi} d\theta = -2\pi$

(b) $\iint \vec{V}\cdot\hat{n}\, d\sigma$ over surface $(x-3)^2+(y-2)^2+(z-1)^2 = 9$

$\vec{V} = (3x - yz)\hat{i} + (z^2-y^2)\hat{j} + (2yz + x^2)\hat{k}$

$\text{div}\,\vec{V} = \nabla\cdot\vec{V} = 3 - 2y + 2y = 3$

by Divergence theorem in 3D :

$$\iint_{\partial\tau} \vec{V}\cdot\vec{n}\, d\sigma = \iiint_{\tau} \text{div}\,\vec{V}\, d\tau$$

$$= \iiint 3\, d\tau = \int_{z=} \int_{y=} \int_{x=} 3\, dx\, dy\, dz$$

$= 3 *$ Volume of the sphere

$$= 3 * \frac{4}{3} r^3 \pi = 4\pi r^3 = 4\pi 3^3 = 108\pi$$

~~(c) $\int \vec{F}\cdot d\vec{r}$ along $x^2+y^2=2$ from $(-1,1)$ to $(1,1)$~~

~~$\vec{F} = (2x-3y)\hat{i} + (3x-2y)\hat{j}$~~

~~$\vec{F}\cdot d\vec{r} = 2\,dx - 2\,dy$~~

6. The eigenvalues of Hermitian matrix are real

if $\lambda = \bar{\lambda}$ then λ is real

Starting with: $\lambda * \langle v, v \rangle = \langle \lambda * v, v \rangle$

$\lambda v = \lambda A$ (A is Hermitian matrix)

$$\Rightarrow \lambda * \langle v, v \rangle = \langle A * v, v \rangle$$
$$= \langle v, A^{+} * v \rangle$$
$$= \langle v, A * v \rangle$$
$$= \langle v, \lambda * v \rangle$$
$$= \bar{\lambda} * \langle v, v \rangle$$

$\Rightarrow LHS = \lambda * \langle v, v \rangle = RHS = \bar{\lambda} * \langle v, v \rangle$

divide both sides by $\langle v, v \rangle \Rightarrow \lambda = \bar{\lambda}$

so λ is real in Hermitian matrix.

7. $2\nabla(\vec{U}\cdot\vec{V}) = \vec{U} \times (\nabla \times \vec{V}) + (\vec{U}\cdot\nabla)\vec{V} + \vec{V} \times (\nabla \times \vec{U}) + (\vec{V}\cdot\nabla)\vec{U}$

~~RHS = Using this identity:~~ $\cancel{U \times (\vec{V} \times \nabla) = (U\cdot\nabla)V - (U\cdot V)\nabla}$

$V \times (\nabla \times U)$ $\cancel{= (V\cdot\nabla)V - (U}$

$\cancel{= (V\cdot U)\nabla - (V\cdot\nabla)U}$

$\cancel{RHS = (U\cdot\nabla)V - (U\cdot V)\nabla + (U\cdot\nabla)V}$

$$\vec{U} \times (\nabla \times \vec{V}) = (U\cdot V)\nabla - (U\cdot\nabla)V$$
$$\vec{V}(\nabla \times U) = (V\cdot U)\nabla - (V\cdot\nabla)U$$

$$RHS = (U\cdot V)\nabla - \cancel{(U\cdot\nabla)V} + \cancel{(U\cdot\nabla)V} + (V\cdot U)\nabla - \cancel{(V\cdot\nabla)U} + \cancel{(V\cdot\nabla)U}$$
$$= 2(U\cdot V)\nabla = LHS \checkmark$$

4. $\vec{F} = q\vec{E}$

F is conservative $\Leftrightarrow$ curl $F = \vec{\nabla}\times\vec{F} = 0$

Check 1: $\vec{E}_1 = k[xy\hat{i} + 2yz\hat{j} + 3xz\hat{k}]$

$\vec{F}_1 = q\vec{E}_1 = qk[xy\hat{i} + 2yz\hat{j} + 3xz\hat{k}]$

$\vec{\nabla}\times\vec{F} = qk\left(\frac{\partial(3xz)}{\partial y} - \frac{\partial(2yz)}{\partial z}\right)\hat{i} + \left(\frac{\partial(xy)}{\partial z} - \frac{\partial(3xz)}{\partial x}\right)\hat{j} + \left(\frac{\partial(2yz)}{\partial x} - \frac{\partial(xy)}{\partial y}\right)\hat{k}$

$= qk(0 - 2y\hat{i}) + (0 - 3z)\hat{j} + (0 - x)\hat{k} \neq \vec{0}$

$\Rightarrow$ $\vec{F}_1$ is not conservative.

Check 2: $\vec{F}_2 = q\vec{E}_2 = qk[y^2\hat{i} + (2xy + z^2)\hat{j} + 2yz\hat{k}]$

$\frac{\vec{\nabla}\times\vec{F}}{qk} = \left(\frac{\partial(2yz)}{\partial y} - \frac{\partial(2xy+z^2)}{\partial z}\right)\hat{i}$

$+ \left(\frac{\partial(y^2)}{\partial z} - \frac{\partial(2yz)}{\partial x}\right)\hat{j}$

$+ \left(\frac{\partial(2xy+z^2)}{\partial x} - \frac{\partial(y^2)}{\partial y}\right)\hat{k}$

$= (2z - 2z)\hat{i} + (0 - 0)\hat{j} + (2y - 2y)\hat{k} = \vec{0}$

$\Rightarrow$ $\vec{F}_2$ is conservative

5. $M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, Find $\cos(M)$, $\sin(M)$

▫ Finding the eigenvalues:

$Mx = bx$

$|M - bI| = 0$

$\begin{vmatrix} -b & 1 & 0 \\ 1 & -b & 1 \\ 0 & 1 & -b \end{vmatrix} = -b\begin{vmatrix} -b & 1 \\ 1 & -b \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & -b \end{vmatrix} = -b(b^2-1) + b$

$= -b^3 + b + b$

$= 2b - b^3 = 0$

$b = 0$, or $2 - b^2 = 0 \Rightarrow b = \pm\sqrt{2}$

$b = -\sqrt{2}, 0, \sqrt{2}$

$\Rightarrow e^M = a_0 M^0 + a_1 M^1 + a_2 M^2$

To find a_0, a_1, a_2, substitute $M = b$:

$b = 0$: $e^0 = a_0 = 1$

$b = \sqrt{2}$: $e^{\sqrt{2}} = a_0 + \sqrt{2}a_1 + 2a_2$

$b = -\sqrt{2}$: $e^{-\sqrt{2}} = a_0 - \sqrt{2}a_1 + 2a_2$

3. $z^3+(3+i)z^2+2z+(5+i)=0$

let $z=(x+iy)$

$(x+iy)^3+(3+i)(x+iy)^2+2(x+iy)+(5+i)=0$

$x^3+3ix^2y-3xy^2-iy^3+(3+i)x^2-(2-6i)xy-(3+i)y^2$

$+2x+2iy+(5+i)=0$

$i(3x^2y-y^3+x^2+6xy-y^2+2y+1)$

$+(x^3-3xy^2+3x^2-2xy-3y^2+2x+5)=0$ stuck :(

or let $z=re^{i\theta}$

$(3+i)$: $r\approx\sqrt{10}\approx 3.16$

$\theta=\tan^{-1}(\frac{1}{3})\approx 0.32$ radian

$(5+i)$: $r=\sqrt{26}\approx 5.1$

$\theta=\tan^{-1}(\frac{1}{5})\approx 0.2$ radian

$\Rightarrow r^3e^{3i\theta}+(3.16e^{0.32i})(r^2e^{2i\theta})+2re^{i\theta}+5.1e^{0.2i}=0$

Also stuck XD

1. (d) $\sum_{n=0}^{\infty} p^n \overset{\overset{\sin(nx)}{\uparrow}}{\cancel{\cos(nx)}}$; $|p|<1$

From the previous question (1c), I obtain:

$$\sum_{n=0}^{\infty} p^n e^{inx} = \frac{1-pe^{-ix}}{1-2p\cos x+p^2}$$

Note that $\sum_{n=0}^{\infty} p^n \sin(nx) = \operatorname{Im}\left[\sum_{n=0}^{\infty} p^n e^{inx}\right]$

$$= \operatorname{Im}\left[\frac{1-pe^{-ix}}{1-2p\cos x+p^2}\right] = \operatorname{Im}\left[\frac{1-\cos x * p + ip\sin x}{1-2p\cos x+p^2}\right]$$

$$= \frac{p\sin x}{1-2p\cos x+p^2}$$

2. Prove $\left(\frac{ic-1}{ic+1}\right)^{id} = e^{-2d\cot^{-1}(c)}$; $c, d \in \mathbb{R}$

$$LHS = \left(\frac{ic-1}{ic+1} * \frac{ic-1}{ic-1}\right)^{id} = \left(\frac{-c^2-2ci+1}{c^2+1}\right)^{id} = \Bigg(\underbrace{\frac{1-c^2}{c^2+1} - \frac{2c}{c^2+1}i}_{z}\Bigg)^{id}$$

$$z = re^{i\theta}, \quad r = \left(\frac{1-c^2}{c^2+1}\right)^2 + \left(\frac{-2c}{c^2+1}\right)^2 = \frac{(1-c^2)^2+4c^2}{(c^2+1)^2}$$

$$= \frac{1+2c^2+c^4}{1+2c^2+c^4} = 1$$

$$, \theta = \tan^{-1}\left(\frac{-2c}{1-c^2}\right)$$

$$\Rightarrow LHS = z^{id} = (1.e^{i\theta})^{id} = e^{-d\theta} = e^{-d\tan^{-1}\left(\frac{-2c}{1-c^2}\right)}$$

Finding $\tan^{-1}\left(\frac{-2c}{1-c^2}\right)$: $\tan^{-1}\left(\frac{-2c}{1-c^2}\right) = -\tan^{-1}\left(\frac{2c}{1-c^2}\right)$

$$= -2\tan^{-1}(c)$$

$$\Rightarrow LHS = e^{2d\tan^{-1}(c)} = RHS$$

~~In the question, I think there is an error~~

Mathmetical Physics

Phys 330

1. (a) $\sum_{n=0}^{N-1} \cos(nx) = Re\left(\sum_{n=0}^{N-1} e^{inx} \right)$

$= Re\left[\frac{1-e^{iNx}}{1-e^{ix}} \right] = Re\left[\frac{1-e^{iNx}}{e^{ix/2}(e^{-ix/2}-e^{ix/2})} \right]$

$= Re\left[\frac{e^{-ix/2} - e^{iNx}e^{-ix/2}}{e^{-ix/2} - e^{ix/2}} \right] \rightarrow * \frac{-2i}{-2i}$

$= Re\left[\frac{e^{-ix/2} - e^{i(N-\frac{1}{2})x}}{-2i \sin\frac{x}{2}} \right] = Re\left[\frac{ie^{-ix/2} - ie^{i(N-\frac{1}{2})x}}{2\sin\frac{x}{2}} \right]$

$= Re\left[\frac{i\left(\cos\frac{x}{2} - i\sin\frac{x}{2}\right) - i\left(\cos(N-\frac{1}{2})x + i\sin(N-\frac{1}{2})x\right)}{2\sin x/2} \right]$

$= Re\left[\frac{i\cos\frac{x}{2} + \sin\frac{x}{2} - i\cos(N-\frac{1}{2})x + \sin(N-\frac{1}{2})x}{2\sin x/2} \right]$

$= \frac{\sin\frac{x}{2} + \sin(N-\frac{1}{2})x}{2\sin\frac{x}{2}}$

(b) $\sum_{n=0}^{N-1} \sin(nx)$

From the previous question (1a), I obtain:

$\sum_{n=0}^{N-1} e^{inx} = \frac{i\cos\frac{x}{2} + \sin\frac{x}{2} - i\cos(N-\frac{1}{2})x + \sin(N-\frac{1}{2})x}{2\sin x/2}$

But, ~~sin x~~ $\sum_{n=0}^{N-1} \sin(nx) = Im\left(\sum_{n=0}^{N-1} e^{inx} \right)$

$= \frac{\cos x/2 - \cos(N-\frac{1}{2})x}{2\sin x/2}$

$|pe^{ix}| < |p| \cdot 1 \le 1$
(geometric series)

↑

(c) $\sum_{n=0}^{\infty} p^n \cos(nx)$; $|p| < 1$

Note that $\sum_{n=0}^{\infty} p^n \cos(nx) = Re\left[\sum_{n=0}^{\infty} p^n e^{inx} \right] = Re\left[\sum_{n=0}^{\infty} (pe^{ix})^n \right]$

$= Re\left[\frac{1-\cancel{p}e^{-ix\cdot\infty}}{1-pe^{ix}} \right] = Re\left[\frac{1}{1-pe^{ix}} \right] \rightarrow * \frac{1-pe^{-ix}}{1-pe^{-ix}}$

$= Re\left[\frac{1-pe^{-ix}}{1-pe^{-ix}-pe^{ix}+p^2} \right] = Re\left[\frac{1-pe^{-ix}}{1-p(e^{-ix}+e^{ix})+p^2} \right]$

$= Re\left[\frac{1-pe^{-ix}}{1-2p\cos x+p^2} \right] = \frac{1-p\cos x}{1-2p\cos x+p^2}$