Exercises 9.1

1. (a) Since f is already given by its Fourier series, we have from (4)

$$u(r,\,\theta) = 100 + 100r\cos\theta = 100 + 100\,x.$$

(b) The equation of the isotherms is given by

$$100 + 100\,x = T \Rightarrow x = \frac{T-100}{100},$$

where T is a constant between 0 and 200 (these are the maximum and minimum values of $f(\theta)$). Thus the isotherms are vertical lines inside the unit disk. The lines of heat flow are orthogonal to the isotherms. Thus the heat flows along the horizontal direction. You can also get the equation of the heat flow by appealing to (7). A harmonic conjugate of $100 + 100r\cos\theta$ is $v(r,\,\theta) = 100r\sin\theta$ or $v(r,\,\theta) = 100\,y$. Heat flows along the level curves of a harmonic conjugate; thus, the lines of heat flow are given by $100\,y = C$ or $y = c$. These describe horizontal lines, as we said previously.

5. (a) Let us compute the Fourier coefficients of f. We have

$$a_0 = \frac{50}{\pi}\int_0^{\pi/4} d\theta = \frac{25}{2};$$

$$a_n = \frac{100}{\pi}\int_0^{\pi/4}\cos n\theta\,d\theta = \frac{100}{n\pi}\sin n\theta\Big|_0^{\pi} = \frac{100}{n\pi}\sin\frac{n\pi}{4};$$

$$b_n = \frac{100}{\pi}\int_0^{\pi/4}\sin n\theta\,d\theta = -\frac{100}{n\pi}\cos n\theta\Big|_0^{\pi} = \frac{100}{n\pi}(1-\cos\frac{n\pi}{4}).$$

Hence

$$f(\theta) = \frac{25}{2} + \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sin\frac{n\pi}{4}\cos n\theta + (1-\cos\frac{n\pi}{4})\sin n\theta\right);$$

and

$$u(r,\,\theta) = \frac{25}{2} + \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sin\frac{n\pi}{4}\cos n\theta + (1-\cos\frac{n\pi}{4})\sin n\theta\right)r^n.$$

(b) The isotherms are the level curves of th solution. The curves of heat flow are the level curves of a harmonic conjugate of the solution, which is obtained by appealing to Proposition 1. We have

$$v(r,\,\theta) = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sin\frac{n\pi}{4}\sin n\theta - (1-\cos\frac{n\pi}{4})\cos n\theta\right)r^n.$$

9. The problem does not have a solution because the normal derivative on the boundary does not satisfy the compatibility condition

$$\int_0^{2\pi} f(\theta)\,d\theta = 0.$$

In fact,

$$\int_0^{2\pi} f(\theta)\,d\theta = \int_0^{2\pi}(50-50\cos\theta)\,d\theta = 100\pi.$$

13. Using the fact that the solutions must be bounded as $r\to\infty$, we see that $c_1 = 0$ in the first of the two equations in (3), and $c_2 = 0$ in the second of the two equations in (3). Thus

$$R(r) = R_n(r) = c_n r^{-n} = \left(\frac{r}{a}\right)^n \quad \text{for } n = 0,\,1,\,2,\,\ldots.$$

The general solution becomes

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n (a_n \cos n\theta + b_n \sin n\theta), \quad r > a.$$

Setting $r = a$ and using the boundary condition, we obtain

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta),$$

which implies that the a_n and b_n are the Fourier coefficients of f and hence are given by (5).

Solutions to Exercises 9.2

1. We appeal to the solution (5) with the coefficients (6). Since $f(r)=0$, then $A_n=0$ for all n. We have

$$\begin{aligned} B_n &= \frac{1}{\alpha_n J_1(\alpha_n)^2}\int_0^2 J_0\Big(\frac{\alpha_n r}{2}\Big) r\,dr \\ &= \frac{4}{\alpha_n^3 J_1(\alpha_n)^2}\int_0^{\alpha_n} J_0(s)s\,ds \quad (\text{let } s=\frac{\alpha_n}{2}r) \\ &= \frac{4}{\alpha_n^3 J_1(\alpha_n)^2}[sJ_1(s)]\Big|_0^{\alpha_n} \\ &= \frac{4}{\alpha_n^2 J_1(\alpha_n)} \quad \text{for all } n\geq 1. \end{aligned}$$

Thus

$$u(r,\,t)=4\sum_{n=1}^\infty \frac{J_0(\frac{\alpha_n r}{2})}{\alpha_n^2 J_1(\alpha_n)}\sin\Big(\frac{\alpha_n t}{2}\Big).$$

5. Since $g(r)=0$, we have $B_n=0$ for all n. We have

$$A_n=\frac{2}{J_1(\alpha_n)^2}\int_0^1 J_0(\alpha_1 r)J_0(\alpha_n r)r\,dr=0 \quad \text{for } n\neq 1 \text{ by orthogonality.}$$

For $n=1$,

$$A_1=\frac{2}{J_1(\alpha_1)^2}\int_0^1 J_0(\alpha_1 r)^2 r\,dr=1,$$

where we have used the orthogonality relation (12), Section 4.8, with $p=0$. Thus

$$u(r,\,t)=J_0(\alpha_1 r)\cos(\alpha_1 t).$$

9. (a) Modifying the solution of Exercise 3, we obtain

$$u(r,\,t)=\sum_{n=1}^\infty \frac{J_1(\alpha_n/2)}{\alpha_n^2 c J_1(\alpha_n)^2}J_0(\alpha_n r)\sin(\alpha_n ct).$$

(b) Under suitable conditions that allow us to interchange the limit and the summation sign (for example, if the series is absolutely convergent), we have, for a given $(r,\,t)$,

$$\begin{aligned} \lim_{c\to\infty} u(r,\,t) &= \lim_{c\to\infty}\sum_{n=1}^\infty \frac{J_1(\alpha_n/2)}{\alpha_n^2 c J_1(\alpha_n)^2}J_0(\alpha_n r)\sin(\alpha_n ct) \\ &= \sum_{n=1}^\infty \lim_{c\to\infty}\frac{J_1(\alpha_n/2)}{\alpha_n^2 c J_1(\alpha_n)^2}J_0(\alpha_n r)\sin(\alpha_n ct) \\ &= 0, \end{aligned}$$

because $\lim_{c\to\infty}\frac{J_1(\alpha_n/2)}{\alpha_n^2 c J_1(\alpha_n)^2}=0$ and $\sin(\alpha_n ct)$ is bounded. If we let $u_1(r,\,t)$ denote the solution corresponding to $c=1$ and $u_c(r,\,t)$ denote the solution for arbitrary $c>0$. Then, it is easy t check that

$$u_c(r,\,t)=\frac{1}{c}u_1(r,\,ct).$$

This shows that if c increases, the time scale speeds proportionally to c, while the displacement decreases by a factor of $\frac{1}{c}$.

Solutions to Exercises 9.3

1. The condition $g(r,\theta) = 0$ implies that $a^*_{mn} = 0 = b^*_{mn}$. Since $f(r,\theta)$ is proportional to $\sin 2\theta$, only $b_{2,n}$ will be nonzero, among all the a_{mn} and b_{mn}. This is similar to the situation in Example 2. For $n = 1, 2, \ldots$, we have

$$\begin{aligned}
b_{2,n} &= \frac{2}{\pi J_3(\alpha_{2,n})^2} \int_0^1 \int_0^{2\pi} (1-r^2)r^2 \sin 2\theta J_2(\alpha_{2,n} r) \sin 2\theta r \, d\theta \, dr \\
&= \frac{2}{\pi J_3(\alpha_{2,n})^2} \int_0^1 \overbrace{\int_0^{2\pi} \sin^2 2\theta \, d\theta}^{=\pi} (1-r^2) r^3 J_2(\alpha_{2,n} r) \, dr \\
&= \frac{2}{J_3(\alpha_{2,n})^2} \int_0^1 (1-r^2) r^3 J_2(\alpha_{2,n} r) \, dr \\
&= \frac{2}{J_3(\alpha_{2,n})^2} \frac{2}{\alpha_{2,n}^2} J_4(\alpha_{2,n}) = \frac{4 J_4(\alpha_{2,n})}{\alpha_{2,n}^2 J_3(\alpha_{2,n})^2},
\end{aligned}$$

where the last integral is evaluated with the help of formula (15), Section 4.3. We can get rid of the expression involving J_4 by using the identity

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x).$$

With $p = 3$ and $x = \alpha_{2,n}$, we get

$$\overbrace{J_2(\alpha_{2,n})}^{=0} + J_4(\alpha_{2,n}) = \frac{6}{\alpha_{2,n}} J_3(\alpha_{2,n}) \quad \Rightarrow \quad J_4(\alpha_{2,n}) = \frac{6}{\alpha_{2,n}} J_3(\alpha_{2,n}).$$

So

$$b_{2,n} = \frac{24}{\alpha_{2,n}^3 J_3(\alpha_{2,n})}.$$

Thus

$$u(r,\,\theta,\,t) = 24 \sin 2\theta \sum_{n=1}^{\infty} \frac{J_2(\alpha_{2,n} r)}{\alpha_{2,n}^3 J_3(\alpha_{2,n})} \cos(\alpha_{2,n} t).$$

5. We have $a_{mn} = b_{mn} = 0$. Also, all a^*_{mn} and b^*_{mn} are zero except $b^*_{2,n}$. We have

$$b^*_{2,n} = \frac{2}{\pi \alpha_{2,n} J_3(\alpha_{2,n})^2} \int_0^1 \int_0^{2\pi} (1-r^2) r^2 \sin 2\theta J_2(\alpha_{2,n} r) \sin 2\theta r \, d\theta \, dr.$$

The integral was computed in Exercise 1. Using the computations of Exercise 1, we find

$$b^*_{2,n} = \frac{24}{\alpha_{2,n}^4 J_3(\alpha_{2,n})}.$$

hus

$$u(r,\,\theta,\,t) = 24 \sin 2\theta \sum_{n=1}^{\infty} \frac{J_2(\alpha_{2,n} r)}{\alpha_{2,n}^4 J_3(\alpha_{2,n})} \sin(\alpha_{2,n} t).$$

9. (a) For $l = 0$ and all $k \geq 0$, the formula follows from (7), Section 4.8, with $p = k$:

$$\int r^{k+1} J_k(r) \, dr = r^{k+1} J_{k+1}(r) + C.$$

(b) Assume that the formula is true for l (and all $k \geq 0$). Integrate by parts, using $u = r^{2l}$, $dv = r^{k+1}J_{k+1}(r)\,dr$, and hence $du = 2lr^{2l-1}dr$ and $v = r^{k+1}J_{k+1}(r)$:

$$\begin{aligned}
\int r^{k+1+2l}J_k(r)\,dr &= \int r^{2l}[r^{k+1}J_k(r)]\,dr \\
&= r^{2l}\,r^{k+1}J_{k+1}(r) - 2l\int r^{2l-1}r^{k+1}J_{k+1}(r)\,dr \\
&= r^{k+1+2l}J_{k+1}(r) - 2l\int r^{k+2l}J_{k+1}(r)\,dr \\
&= r^{k+1+2l}J_{k+1}(r) - 2l\int r^{(k+1)+1+2(l-1)}J_{k+1}(r)\,dr
\end{aligned}$$

and so, by the induction hypothesis, we get

$$\begin{aligned}
\int r^{k+1+2l}J_k(r)\,dr &= r^{k+1+2l}J_{k+1}(r) - 2l\sum_{n=0}^{l-1}\frac{(-1)^n 2^n (l-1)!}{(l-1-n)!}r^{k+2l-n}J_{k+n+2}(r) + C \\
&= r^{k+1+2l}J_{k+1}(r) \\
&\quad + \sum_{n=0}^{l-1}\frac{(-1)^{n+1}2^{n+1}l!}{(l-(n+1))!}r^{k+1+2l-(n+1)}J_{k+(n+1)+1}(r) + C \\
&= r^{k+1+2l}J_{k+1}(r) + \sum_{m=1}^{l}\frac{(-1)^m 2^m l!}{(l-m)!}r^{k+1+2l-m}J_{k+m+1}(r) + C \\
&= \sum_{m=0}^{l}\frac{(-1)^m 2^m l!}{(l-m)!}r^{k+1+2l-m}J_{k+m+1}(r) + C,
\end{aligned}$$

which completes the proof by induction for all integers $k \geq 0$ and all $l \geq 0$.

13. The proper place for this problem is in the next section, since its solution invovles solving a Dirichlet problem on the unit disk. The initial steps are similar to the solution of the heat problem on a rectangle with nonzero boundary data (Exercise 11, Section 3.8). In order to solve the problem, we consider the following two subproblems: Subproblem #1 (Dirichlet problem)

$$\begin{aligned}
(u_1)_{rr} + \frac{1}{r}(u_1)_r + \frac{1}{r^2}(u_1)_{\theta\theta} &= 0, \qquad 0 < r < 1,\ 0 \leq \theta < 2\pi, \\
u_1(1,\,\theta) &= \sin 3\theta, \qquad 0 \leq \theta < 2\pi.
\end{aligned}$$

Subproblem #2 (to be solved after finding $u_1(r,\,\theta)$ from Subproblem #1)

$$\begin{aligned}
(u_2)_t &= (u_2)_{rr} + \tfrac{1}{r}(u_2)_r + \tfrac{1}{r^2}(u_2)_{\theta\theta}, \quad 0 < r < 1,\ 0 \leq \theta < 2\pi,\ t > 0, \\
u_2(1,\,\theta,\,t) &= 0, \qquad 0 \leq \theta < 2\pi,\ t > 0, \\
u_2(r,\,\theta,\,0) &= -u_1(r,\,\theta), \qquad 0 < r < 1,\ 0 \leq \theta < 2\pi.
\end{aligned}$$

You can check, using linearity (or superposition), that

$$u(r,\,\theta,\,t) = u_1(r,\,\theta) + u_2(r,\,\theta,\,t)$$

is a solution of the given problem.

The solution of subproblem #1 follows immediately from the method of Section 4.5. We have

$$u_2(r,\,\theta) = r^3 \sin 3\theta.$$

We now solve subproblem #2, which is a heat problem with 0 boundary data and initial temperature distribution given by $-u_2(r,\,\theta) = -r^3 \sin 3\theta$. reasoning as in Exercise 10, we find that the solution is

$$u_2(r,\,\theta,\,t) = \sum_{n=1}^{\infty} b_{3n} J_3(\alpha_{3n} r) \sin(3\theta) e^{-\alpha_{3n}^2 t},$$

where

$$\begin{aligned}
b_{3n} &= \frac{-2}{\pi J_4(\alpha_{3n})^2} \int_0^1 \int_0^{2\pi} r^3 \sin^2 3\theta J_3(\alpha_{3n} r) r \, d\theta \, dr \\
&= \frac{-2}{J_4(\alpha_{3n})^2} \int_0^1 r^4 J_3(\alpha_{3n} r) \, dr \\
&= \frac{-2}{J_4(\alpha_{3n})^2} \frac{1}{\alpha_{3n}^5} \int_0^{\alpha_{3n}} s^4 J_3(s) \, ds \quad \text{(where } \alpha_{3n} r = s\text{)} \\
&= \frac{-2}{J_4(\alpha_{3n})^2} \frac{1}{\alpha_{3n}^5} s^4 J_4(s) \Big|_0^{\alpha_{3n}} \\
&= \frac{-2}{\alpha_{3n} J_4(\alpha_{3n})}.
\end{aligned}$$

Hence

$$u(r,\,\theta,\,t) = r^3 \sin 3\theta - 2\sin(3\theta) \sum_{n=1}^{\infty} \frac{J_3(\alpha_{3n} r)}{\alpha_{3n} J_4(\alpha_{3n})} e^{-\alpha_{3n}^2 t}.$$

Solutions to Exercises 9.4

1. Using (2) and (3), we have that

$$u(\rho, z) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n \rho) \sinh(\lambda_n z), \quad \lambda_n = \frac{\alpha_n}{a},$$

where $\alpha_n = \alpha_{0,n}$ is the nth positive zero of J_0, and

$$\begin{aligned}
A_n &= \frac{2}{\sinh(\lambda_n h) a^2 J_1(\alpha_n)^2} \int_0^a f(\rho) J_0(\lambda_n \rho) \rho \, d\rho \\
&= \frac{200}{\sinh(2\alpha_n) J_1(\alpha_n)^2} \int_0^1 J_0(\alpha_n \rho) \rho \, d\rho \\
&= \frac{200}{\sinh(2\alpha_n) \alpha_n^2 J_1(\alpha_n)^2} \int_0^{\alpha_n} J_0(s) s \, ds \quad (\text{let } s = \alpha_n \rho) \\
&= \frac{200}{\sinh(2\alpha_n) \alpha_n^2 J_1(\alpha_n)^2} [J_1(s) s] \Big|_0^{\alpha_n} \\
&= \frac{200}{\sinh(2\alpha_n) \alpha_n J_1(\alpha_n)}.
\end{aligned}$$

So

$$u(\rho, z) = 200 \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \rho) \sinh(\alpha_n z)}{\sinh(2\alpha_n) \alpha_n J_1(\alpha_n)}.$$

5. (a) We proceed exactly as in the text and arrive at the condition $Z(h) = 0$ which leads us to the solutions

$$Z(z) = Z_n(z) = \sinh(\lambda_n (h - z)), \quad \text{where } \lambda_n = \frac{\alpha_n}{a}.$$

So the solution of the problem is

$$u(\rho, z) = \sum_{n=1}^{\infty} C_n J_0(\lambda_n \rho) \sinh(\lambda_n (h - z)),$$

where

$$C_n = \frac{2}{a^2 J_1(\alpha_n)^2 \sinh(\lambda_n h)} \int_0^a f(\rho) J_0(\lambda_n \rho) \rho \, d\rho.$$

(b) The problem can be decomposed into the sum of two subproblems, one treated in the text and one treated in part (a). The solution of the problem is the sum of the solutions of the subproblems:

$$u(\rho, z) = \sum_{n=1}^{\infty} \Big(A_n J_0(\lambda_n \rho) \sinh(\lambda_n z) + C_n J_0(\lambda_n \rho) \sinh(\lambda_n (h - z)) \Big),$$

where

$$A_n = \frac{2}{a^2 J_1(\alpha_n)^2 \sinh(\lambda_n h)} \int_0^a f_2(\rho) J_0(\lambda_n \rho) \rho \, d\rho,$$

and

$$C_n = \frac{2}{a^2 J_1(\alpha_n)^2 \sinh(\lambda_n h)} \int_0^a f_1(\rho) J_0(\lambda_n \rho)\rho \, d\rho.$$

9. We use the solution in Exercise 8 with $a = 1$, $h = 2$, $f(z) = 10z$. Then

$$\begin{aligned} B_n &= \frac{1}{I_0\left(\frac{n\pi}{2}\right)} \int_0^2 0z \sin\frac{n\pi z}{2}\, dz \\ &= \frac{40}{n\pi\, I_0\left(\frac{n\pi}{2}\right)} (-1)^{n+1}. \end{aligned}$$

Thus

$$u(\rho, z) = \frac{40}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\, I_0\left(\frac{n\pi}{2}\right)} I_0\left(\frac{n\pi}{2}\rho\right) \sin\frac{n\pi z}{2}.$$

Solutions to Exercises 9.5

1. Write (1) in polar coordinates:

$$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} = -k\phi \qquad \phi(a,\,\theta) = 0.$$

Consider a product solution $\phi(r,\,\theta) = R(r)\Theta(\theta)$. Since θ is a polar angle, it follows that

$$\Theta(\theta + 2\pi) = \Theta(\theta);$$

in other words, Θ is 2π-periodic. Plugging the product solution into the equation and simplifying, we find

$$\begin{aligned} R''\Theta + \tfrac{1}{r}R'\Theta + \tfrac{1}{r^2}R\Theta'' &= -kR\Theta; \\ \left(R'' + \tfrac{1}{r}R' + kR\right)\Theta &= -\tfrac{1}{r^2}R\Theta''; \\ r^2\tfrac{R''}{R} + r\tfrac{R'}{R} + kr^2 &= -\tfrac{\Theta''}{\Theta}; \end{aligned}$$

hence

$$r^2\frac{R''}{R} + r\frac{R'}{R} + kr^2 = \lambda,$$

and

$$-\frac{\Theta''}{\Theta} = \lambda \quad \Rightarrow \quad \Theta'' + \lambda\Theta = 0,$$

where λ is a separation constant. Our knowledge of solutions of second order linear ode's tells us that the last equation has 2π-periodic solutions if and only if

$$\lambda = m^2, \quad m = 0,\, \pm1,\, \pm2,\, \ldots.$$

This leads to the equations

$$\Theta'' + m^2\Theta = 0,$$

and

$$r^2\frac{R''}{R} + r\frac{R'}{R} + kr^2 = m^2 \quad \Rightarrow \quad r^2R'' + rR' + (kr^2 - m^2)R = 0.$$

These are equations (3) and (4). Note that the condition $R(a) = 0$ follows from $\phi(a,\,\theta) = 0 \Rightarrow R(a)\Theta(\theta) = 0 \Rightarrow R(a) = 0$ in order to avoid the constant 0 solution.

5. We proceed as in Example 1 and try

$$u(r,\,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn}r)(A_{mn}\cos m\theta + B_{mn}\sin m\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r,\,\theta),$$

where $\phi_{mn}(r,\,\theta) = J_m(\lambda_{mn}r)(A_{mn}\cos m\theta + B_{mn}\sin m\theta)$. We plug this solution into the equation, use the fact that $\nabla^2(\phi_{mn}) = -\lambda_{mn}^2\phi_{mn}$, and get

$$\begin{aligned} \nabla^2\left(\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r,\,\theta)\right) &= 1 - \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r,\,\theta) \\ &\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\nabla^2\left(\phi_{mn}(r,\,\theta)\right) = 1 - \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r,\,\theta) \\ &\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} -\lambda_{mn}^2\phi_{mn}(r,\,\theta) = 1 - \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r,\,\theta) \\ &\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(1 - \alpha_{mn}^2)\phi_{mn}(r,\,\theta) = 1. \end{aligned}$$

We recognize this expansion as the expansion of the function 1 in terms of the functions ϕ_{mn}. Because the right side is independent of θ, it follows that all A_{mn} and B_{mn} are zero, except $A_{0,n}$. So

$$\sum_{n=1}^{\infty}(1-\alpha_{mn}^2)A_{0,n}J_0(\alpha_{0,n})r)=1,$$

which shows that $(1-\alpha_{mn}^2)A_{0,n}=a_{0,n}$ is the nth Bessel coefficient of the Bessel series expansion of order 0 of the function 1. This series is computed in Example 1, Section 4.8. We have

$$1=\sum_{n=1}^{\infty}\frac{2}{\alpha_{0,n}J_1(\alpha_{0,n})}J_0(\alpha_{0,n}r)\quad 0<r<1.$$

Hence

$$(1-\alpha_{mn}^2)A_{0,n}=\frac{2}{\alpha_{0,n}J_1(\alpha_{0,n})}\quad\Rightarrow\quad A_{0,n}=\frac{2}{(1-\alpha_{mn}^2)\alpha_{0,n}J_1(\alpha_{0,n})};$$

and so

$$u(r,\,\theta)=\sum_{n=1}^{\infty}\frac{2}{(1-\alpha_{mn}^2)\alpha_{0,n}J_1(\alpha_{0,n})}J_0(\alpha_{0,n}r).$$

9. Let

$$h(r)=\begin{cases} r & \text{if } 0<r<1/2,\\ 0 & \text{if } 1/2<r<1.\end{cases}$$

Then the equation becomes $\nabla^2 u=f(r,\,\theta)$, where $f(r,\,\theta)=h(r)\sin\theta$. We proceed as in the previous exercise and try

$$u(r,\,\theta)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m(\lambda_{mn}r)(A_{mn}\cos m\theta+B_{mn}\sin m\theta)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r,\,\theta),$$

where $\phi_{mn}(r,\,\theta)=J_m(\lambda_{mn}r)(A_{mn}\cos m\theta+B_{mn}\sin m\theta)$. We plug this solution into the equation, use the fact that $\nabla^2(\phi_{mn})=-\lambda_{mn}^2\phi_{mn}=-\alpha_{mn}^2\phi_{mn}$, and get

$$\begin{aligned}\nabla^2\left(\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r,\,\theta)\right) &= h(r)\sin\theta\\ &\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\nabla^2\left(\phi_{mn}(r,\,\theta)\right)=h(r)\sin\theta\\ &\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}-\alpha_{mn}^2\phi_{mn}(r,\,\theta)=h(r)\sin\theta.\end{aligned}$$

We recognize this expansion as the expansion of the function $h(r)\sin\theta$ in terms of the functions ϕ_{mn}. Because the right side is proportional to $\sin\theta$, it follows that all A_{mn} and B_{mn} are zero, except $B_{1,n}$. So

$$\sin\theta\sum_{n=1}^{\infty}-\alpha_{1n}^2B_{1,n}J_1(\alpha_{1n}r)=h(r)\sin\theta,$$

which shows that $-\alpha_{1n}^2B_{1,n}$ is the nth Bessel coefficient of the Bessel series expansion of order 1 of the function $h(r)$:

$$\begin{aligned}
-\alpha_{1n}^2 B_{1,n} &= \frac{2}{J_2(\alpha_{1,n})^2}\int_0^{1/2} r^2 J_1(\alpha_{1,n} r)\, dr \\
&= \frac{2}{\alpha_{1,n}^3 J_2(\alpha_{1,n})^2}\int_0^{\alpha_{1,n}/2} s^2 J_1(s)\, ds \\
&= \frac{2}{\alpha_{1,n}^3 J_2(\alpha_{1,n})^2} s^2 J_2(s)\Big|_0^{\alpha_{1,n}/2} \\
&= \frac{J_2(\alpha_{1,n}/2)}{2\alpha_{1,n} J_2(\alpha_{1,n})^2}.
\end{aligned}$$

Thus

$$u(r,\,\theta) = \sin\theta \sum_{n=1}^{\infty} -\frac{J_2(\alpha_{1,n}/2)}{2\alpha_{1,n}^3 J_2(\alpha_{1,n})^2} J_1(\alpha_{1,n} r).$$

Solutions to Exercises 9.6

1. Bessel equation of order 3. Using (7), the first series solution is

$$J_3(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+3)!} \left(\frac{x}{2}\right)^{2k+3} = \frac{1}{1\cdot 6}\frac{x^3}{8} - \frac{1}{1\cdot 24}\frac{x^5}{32} + \frac{1}{2\cdot 120}\frac{x^7}{128} + \cdots .$$

5. Bessel equation of order $\frac{3}{2}$. The general solution is

$$\begin{aligned} y(x) &= c_1 J_{\frac{3}{2}} + c_2 J_{-\frac{3}{2}} \\ &= c_1 \left(\frac{1}{1\cdot\Gamma(\frac{5}{2})}\left(\frac{x}{2}\right)^{\frac{3}{2}} - \frac{1}{1\cdot\Gamma(\frac{7}{2})}\left(\frac{x}{2}\right)^{\frac{7}{2}} + \cdots \right) \\ &\quad + c_2 \left(\frac{1}{1\cdot\Gamma(\frac{-1}{2})}\left(\frac{x}{2}\right)^{-\frac{3}{2}} - \frac{1}{1\cdot\Gamma(\frac{1}{2})}\left(\frac{x}{2}\right)^{\frac{1}{2}} + \cdots \right). \end{aligned}$$

Using the basic property of the gamma function and (15), we have

$$\begin{gathered} \Gamma(\tfrac{5}{2}) = \tfrac{3}{2}\Gamma(\tfrac{3}{2}) = \tfrac{3}{2}\tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \tfrac{3}{4}\sqrt{\pi} \\ \Gamma(\tfrac{7}{2}) = \tfrac{5}{2}\Gamma(\tfrac{5}{2}) = \tfrac{15}{8}\sqrt{\pi} \\ -\tfrac{1}{2}\Gamma(-\tfrac{1}{2}) = \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \ \Rightarrow\ \Gamma(-\tfrac{1}{2}) = -2\sqrt{\pi}. \end{gathered}$$

So

$$\begin{aligned} y(x) &= c_1\sqrt{\frac{2}{\pi x}}\left(\frac{4}{3}\frac{x^2}{4} - \frac{8}{15}\frac{x^4}{16} + \cdots\right) \\ &\quad c_2\sqrt{\frac{2}{\pi x}}(-1)\left(-\frac{1}{2}\frac{2}{x} - \frac{x}{2} - \cdots\right) \\ &= c_1\sqrt{\frac{2}{\pi x}}\left(\frac{x^2}{3} - \frac{x^4}{30} + \cdots\right) + c_2\sqrt{\frac{2}{\pi x}}\left(\frac{1}{x} + \frac{x}{2} - \cdots\right) \end{aligned}$$

9. Divide the equation through by x^2 and put it in the form

$$y'' + \frac{1}{x}y' + \frac{x^2-9}{x^2}y = 0 \quad \text{for } x > 0.$$

Now refer to Appendix A.6 for terminology and for the method of Frobenius that we are about to use in this exercise. Let

$$p(x) = \frac{1}{x} \quad \text{for} \quad q(x) = \frac{x^2-9}{x^2}.$$

The point $x = 0$ is a singular point of the equation. But since $x\,p(x) = 1$ and $x^2\,q(x) = x^2 - 9$ have power series expansions about 0 (in fact, they are already given by their power series expansions), it follows that $x = 0$ is a regular singular point. Hence we may apply the Frobenius method. We have already found one series solution in Exercise 1. To determine the second series solution, we consider the indicial equation

$$r(r-1) + p_0 r + q_0 = 0,$$

where $p_0 = 1$ and $q_0 = -9$ (respectively, these are the constant terms in the series expansions of $xp(x)$ and $x^2q(x)$). Thus the indicial equation is

$$r - 9 = 0 \quad \Rightarrow \quad r_1 = 3,\ r_2 = -3.$$

The indicial roots differ by an integer. So, according to Theorem 2, Appendix A.6, the second solution y_2 may or may not contain a logarithmic term. We have, for $x > 0$,

$$y_2 = ky_1 \ln x + x^{-3} \sum_{m=0}^{\infty} b_m x^m = ky_1 \ln x + \sum_{m=0}^{\infty} b_m x^{m-3},$$

where $a_0 \neq 0$ and $b_0 \neq 0$, and k may or may not be 0. Plugging this into the differential equation

$$x^2 y'' + xy' + (x^2 - 9)y = 0$$

and using the fact that y_1 is a solution, we have

$$\begin{aligned}
y_2 &= ky_1 \ln x + \sum_{m=0}^{\infty} b_m x^{m-3} \\
y_2' &= ky_1' \ln x + k\frac{y_1}{x} + \sum_{m=0}^{\infty} (m-3) b_m x^{m-4}; \\
y_2'' &= ky_1'' \ln x + k\frac{y_1'}{x} + k\frac{xy_1' - y_1}{x^2} + \sum_{m=0}^{\infty} (m-3)(m-4) b_m x^{m-5} \\
&= ky_1'' \ln x + 2k\frac{y_1'}{x} - k\frac{y_1}{x^2} + \sum_{m=0}^{\infty} (m-3)(m-4) b_m x^{m-5};
\end{aligned}$$

$$\begin{aligned}
&x^2 y_2'' + xy_2' + (x^2 - 9)y_2 \\
&= kx^2 y_1'' \ln x + 2kxy_1' - ky_1 + \sum_{m=0}^{\infty} (m-3)(m-4) b_m x^{m-3} \\
&\quad + kxy_1' \ln x + ky_1 + \sum_{m=0}^{\infty} (m-3) b_m x^{m-3} \\
&\quad + (x^2 - 9)\left[ky_1 \ln x + \sum_{m=0}^{\infty} b_m x^{m-3}\right] \\
&= k \ln x \big[\overbrace{x^2 y_1'' + xy_1' + (x^2 - 9)y_1}^{=0}\big] \\
&\quad + 2kxy_1' + \sum_{m=0}^{\infty} \big[(m-3)(m-4)b_m + (m-3)b_m - 9b_m\big] x^{m-3} \\
&\quad + x^2 \sum_{m=0}^{\infty} b_m x^{m-3} \\
&= 2kxy_1' + \sum_{m=0}^{\infty} (m-6) m b_m x^{m-3} + \sum_{m=0}^{\infty} b_m x^{m-1}.
\end{aligned}$$

To combine the last two series, we use reindexing as follows

$$\begin{aligned}
&\sum_{m=0}^{\infty}(m-6)mb_m x^{m-3} + \sum_{m=0}^{\infty} b_m x^{m-1}\\
&\quad = \quad -5b_1x^{-2} + \sum_{m=2}^{\infty}(m-6)mb_m x^{m-3} + \sum_{m=2}^{\infty} b_{m-2}x^{m-3}\\
&\quad = \quad -5b_1x^{-2} + \sum_{m=2}^{\infty}\big[(m-6)mb_m + b_{m-2}\big]x^{m-3}.
\end{aligned}$$

Thus the equation

$$x^2y_2'' + xy_2' + (x^2-9)y_2 = 0$$

implies that

$$2kxy_1' - 5b_1x^{-2} + \sum_{m=2}^{\infty}\big[(m-6)mb_m + b_{m-2}\big]x^{m-3} = 0.$$

This equation determines the coefficients b_m $(m \geq 1)$ in terms of the coefficients of y_1. Furthermore, it will become apparent that k cannot be 0. Also, b_0 is arbitrary but by assumption $b_0 \neq 0$. Let's take $b_0 = 1$ and determine the the first five b_m's.

Recall from Exercise 1

$$y_1 = \frac{1}{1\cdot 6}\frac{x^3}{8} - \frac{1}{1\cdot 24}\frac{x^5}{32} + \frac{1}{2\cdot 120}\frac{x^7}{128} + \cdots.$$

So

$$y_1' = \frac{3}{1\cdot 6}\frac{x^2}{8} - \frac{5}{1\cdot 24}\frac{x^4}{32} + \frac{7}{2\cdot 120}\frac{x^6}{128} + \cdots$$

and hence (taking $k = 1$)

$$2kxy_1' = \frac{6k}{1\cdot 6}\frac{x^3}{8} - \frac{10k}{1\cdot 24}\frac{x^5}{32} + \frac{14k}{2\cdot 120}\frac{x^7}{128} + \cdots.$$

The lowest exponent of x in

$$2kxy_1' - 5b_1x^{-2} + \sum_{m=2}^{\infty}\big[(m-6)mb_m + b_{m-2}\big]x^{m-3}$$

is x^{-2}. Since its coefficient is $-5b_1$, we get $b_1 = 0$ and the equation becomes

$$2xy_1' + \sum_{m=2}^{\infty}\big[(m-6)mb_m + b_{m-2}\big]x^{m-3}.$$

Next, we consider the coefficient of x^{-1}. It is $(-4)2b_2 + b_0$. Setting it equal to 0, we find

$$b_2 = \frac{b_0}{8} = \frac{1}{8}.$$

Next, we consider the constant term, which is the $m = 3$ term in the series. Setting its coefficient equal to 0, we obtain

$$(-3)3b_3 + b_1 = 0 \quad \Rightarrow \quad b_3 = 0$$

because $b_1 = 0$. Next, we consider the term in x, which is the $m = 4$ term in the series. Setting its coefficient equal to 0, we obtain

$$(-2)4b_4 + b_2 = 0 \quad \Rightarrow \quad b_4 = \frac{1}{8}b_2 = \frac{1}{64}.$$

Next, we consider the term in x^2, which is the $m = 5$ term in the series. Setting its coefficient equal to 0, we obtain $b_5 = 0$. Next, we consider the term in x^3, which is the $m = 6$ term in the series plus the first term in $2kxy_1'$. Setting its coefficient equal to 0, we obtain

$$0 + b_4 + \frac{k}{8} = 0 \quad \Rightarrow \quad k = -8b_4 = -\frac{1}{8}.$$

Next, we consider the term in x^4, which is the $m = 7$ term in the series. Setting its coefficient equal to 0, we find that $b_7 = 0$. It is clear that $b_{2m+1} = 0$ and that

$$y_2 \approx -\frac{1}{8}y_1 \ln x + \frac{1}{x^3} + \frac{1}{8\,x} + \frac{1}{64}x + \cdots$$

Any nonzero constant multiple of y_2 is also a second linearly independent solution of y_1. In particular, $384\,y_2$ is an alternative answer (which is the answer given in the text).

13. The equation is of the form given in Exercise 10 with $p = 3/2$. Thus its general solution is

$$y = c_1 x^{3/2} J_{3/2}(x) + c_2 x^{3/2} Y_{3/2}(x).$$

Using Exercise 22 and (1), you can also write this general solution in the form

$$\begin{aligned} y &= c_1 x \left[\frac{\sin x}{x} - \cos x\right] + c_2 x \left[-\frac{\cos x}{x} - \sin x\right] \\ &= c_1 \left[\sin x - x\cos x\right] + c_2 \left[-\cos x - x \sin x\right]. \end{aligned}$$

In particular, two linearly independent solution are

$$y_1 = \sin x - x\cos x \quad \text{and} \quad y_2 = \cos x + x\sin x.$$

This can be verified directly by using the differential equation (try it!).

17. We have

$$\begin{aligned} y &= x^{-p}u, \\ y' &= -px^{-p-1}u + x^{-p}u', \\ y'' &= p(p+1)x^{-p-2}u + 2(-p)x^{-p-1}u' + x^{-p}u'', \\ xy'' + (1+2p)y' + xy &= x\big[p(p+1)x^{-p-2}u - 2px^{-p-1}u' + x^{-p}u''\big] \\ &\quad +(1+2p)\big[-px^{-p-1}u + x^{-p}u'\big] + x\,x^{-p}u \\ &= x^{-p-1}\big[x^2u'' + [-2px + (1+2p)x]u' \\ &\quad +[p(p+1) - (1+2p)p + x^2]u\big] \\ &= x^{-p-1}\big[x^2u'' + xu' + (x^2 - p^2)u\big]. \end{aligned}$$

Thus, by letting $y = x^{-p}u$, we transform the equation

$$xy'' + (1 + 2p)y' + xy = 0$$

into the equation

$$x^{-p-1}\big[x^2u'' + xu' + (x^2 - p^2)u\big] = 0,$$

which, for $x > 0$, is equivalent to

$$x^2u'' + xu' + (x^2 - p^2)u = 0,$$

a Bessel equation of ordr $p > 0$ in u. The general solution of the last equation is

$$u = c_1 J_p(x) + c_2 Y_p(x).$$

Thus the general solution of the original equation is

$$Y = c_1 x^{-p} J_p(x) + c_2 x^{-p} Y_p(x).$$

21. Using (7),

$$\begin{aligned}
J_{-\frac{1}{2}}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k - \frac{1}{2} + 1)} \left(\frac{x}{2}\right)^{2k-\frac{1}{2}} \\
&= \sqrt{\frac{2}{x}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k + \frac{1}{2})} \frac{x^{2k}}{2^{2k}} \\
&= \sqrt{\frac{2}{x}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{2^{2k}k!}{(2k)!\sqrt{\pi}} \frac{x^{2k}}{2^{2k}} \quad \text{(by Exercise 44(a))} \\
&= \sqrt{\frac{2}{\pi\, x}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{x^{2k}}{(2k)!} = \sqrt{\frac{2}{\pi\, x}} \cos x.
\end{aligned}$$

22. (a) Using (7),

$$\begin{aligned}
J_{\frac{3}{2}}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\frac{3}{2}+1)} \left(\frac{x}{2}\right)^{2k+\frac{3}{2}} \\
&= \sqrt{\frac{2}{x}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+2+\frac{1}{2})} \frac{x^{2k+2}}{2^{2k+2}} \\
&= \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{2\, 2^{2k+1} k!}{(2k+3)(2k+1)!} \frac{x^{2k+2}}{2^{2k+2}} \\
& \quad (\Gamma(k+2+\frac{1}{2}) = \Gamma(k+1+\frac{1}{2})\Gamma(k+1+\frac{1}{2}) \text{ then use Exercise 44(b)}) \\
&= \sqrt{\frac{2}{\pi\, x}} \sum_{k=0}^{\infty} \frac{(-1)^k (2k+2)}{(2k+3)!} x^{2k+2} \quad \text{(multiply and divide by } (2k+2)) \\
&= \sqrt{\frac{2}{\pi\, x}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (2k)}{(2k+1)!} x^{2k} \quad \text{(change } k \text{ to } k-1) \\
&= \sqrt{\frac{2}{\pi\, x}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} [(2k+1)-1]}{(2k+1)!} x^{2k} \\
&= \sqrt{\frac{2}{\pi\, x}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k)!} x^{2k} - \sqrt{\frac{2}{\pi\, x}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k+1)!} x^{2k} \\
&= \sqrt{\frac{2}{\pi\, x}} \left(-\cos x + \frac{\sin x}{x}\right).
\end{aligned}$$

25. (a) Let $u = \frac{2}{a}\, e^{-\frac{1}{2}(at-b)}$, $Y(u) = y(t)$, $e^{-at+b} = \frac{a^2}{4}\, u^2$; then

$$\frac{dy}{dt} = \frac{dY}{du}\frac{du}{dt} = Y'\,(-e^{-\frac{1}{2}(at-b)}); \quad \frac{d^2y}{dt^2} = \frac{d}{du}\left(Y'\,(-e^{-\frac{1}{2}(at-b)})\right) = Y''e^{-at+b} + Y'\frac{a}{2}e^{-\frac{1}{2}(at-b)}.$$

So

$$Y''e^{-at+b} + Y'\frac{a}{2}e^{-\frac{1}{2}(at-b)} + Ye^{-at+b} = 0 \quad \Rightarrow \quad Y'' + \frac{a}{2}Y'e^{-\frac{1}{2}(at-b)} + Y = 0,$$

upon multiplying by e^{at-b}. Using $u = \frac{2}{a}e^{-\frac{1}{2}(at-b)}$, we get

$$Y'' + \frac{1}{u}Y' + Y = 0 \quad \Rightarrow \quad u^2Y'' + uY' + u^2Y = 0,$$

which is Bessel's equation of order 0.
(b) The general solution of $u^2Y'' + uY' + u^2Y = 0$ is $Y(u) = c_1J_0(u) + c_2Y_0(u)$. But $Y(u) = y(t)$ and $u = \frac{2}{a}e^{-\frac{1}{2}(at-b)}$, so

$$y(t) = c_1J_0(\frac{2}{a}e^{-\frac{1}{2}(at-b)}) + c_2Y_0(\frac{2}{a}e^{-\frac{1}{2}(at-b)}).$$

(c) (i) If $c_1 = 0$ and $c_2 \neq 0$, then

$$y(t) = c_2Y_0(\frac{2}{a}e^{-\frac{1}{2}(at-b)}).$$

As $t \to \infty$, $u \to 0$, and $Y_0(u) \to -\infty$. In this case, $y(t)$ could approach either $+\infty$ or $-\infty$ depending on the sign of c_2. $y(t)$ would approach infinity linearly as near 0, $Y_0(x) \approx \ln x$ so

$y(t) \approx \ln\left(\frac{2}{a}e^{-\frac{1}{2}(at-b)}\right) \approx At.$
(ii) If $c_1 \neq 0$ and $c_2 = 0$, then

$$y(t) = c_1 J_0(\frac{2}{a}e^{-\frac{1}{2}(at-b)}).$$

As $t \to \infty$, $u(t) \to 0$, $J_0(u) \to 1$, and $y(t) \to c_1$. In this case the solution is bounded.
(ii) If $c_1 \neq 0$ and $c_2 \neq 0$, as $t \to \infty$, $u(t) \to 0$, $J_0(u) \to 1$, $Y_0(u) \to -\infty$. Since Y_0 will dominate, the solution will behave like case (i).

It makes sense to have unbounded solutions because eventually the spring wears out and does not affect the motion. Newton's laws tell us the mass will continue with unperturbed momentum, i.e., as $t \to \infty$, $y'' = 0$ and so $y(t) = c_1 t + c_2$, a linear function, which is unbounded if $c_1 \neq 0$.

Solutions to Exercises 9.7

1. (a) Using the series definition of the Bessel function, (7), Section 9.6, we have

$$\begin{aligned}
\frac{d}{dx}[x^{-p}J_p(x)] &= \frac{d}{dx}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^p k!\Gamma(k+p+1)}\left(\frac{x}{2}\right)^{2k} \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{2^p k!\Gamma(k+p+1)}\frac{d}{dx}\left(\frac{x}{2}\right)^{2k} = \sum_{k=0}^{\infty}\frac{(-1)^k 2k}{2^p k!\Gamma(k+p+1)}\frac{1}{2}\left(\frac{x}{2}\right)^{2k-1} \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{2^p (k-1)!\Gamma(k+p+1)}\left(\frac{x}{2}\right)^{2k-1} \\
&= -\sum_{m=0}^{\infty}\frac{(-1)^m}{2^p m!\Gamma(m+p+2)}\left(\frac{x}{2}\right)^{2m+1} \qquad (\text{set } m=k-1) \\
&= -x^{-p}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+p+2)}\left(\frac{x}{2}\right)^{2m+p+1} = -x^{-p}J_{p+1}(x).
\end{aligned}$$

To prove (7), use (1):

$$\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x) \quad \Rightarrow \quad \int x^p J_{p-1}(x)\,dx = x^p J_p(x) + C.$$

Now replace p by $p+1$ and get

$$\int x^{p+1}J_p(x)\,dx = x^{p+1}J_{p+1}(x) + C,$$

which is (7). Similarly, starting with (2),

$$\begin{aligned}
\frac{d}{dx}[x^{-p}J_p(x)] = -x^{-p}J_{p+1}(x) \quad &\Rightarrow \quad -\int x^{-p}J_{p+1}(x)\,dx = x^{-p}J_p(x) + C \\
&\Rightarrow \quad \int x^{-p}J_{p+1}(x)\,dx = -x^{-p}J_p(x) + C.
\end{aligned}$$

Now replace p by $p-1$ and get

$$\int x^{-p+1}J_p(x)\,dx = -x^{-p+1}J_{p-1}(x) + C,$$

which is (8).
(b) To prove (4), carry out the differentiation in (2) to obtain

$$x^{-p}J_p'(x) - px^{-p-1}J_p(x) = -x^{-p}J_{p+1}(x) \quad \Rightarrow \quad xJ_p'(x) - pJ_p(x) = -xJ_{p+1}(x),$$

upon multiplying through by x^{p+1}. To prove (5), add (3) and (4) and then divide by x to obtain

$$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x).$$

To prove (6), subtract (4) from (3) then divide by x.

5. $\int J_1(x)\,dx = -J_0(x) + C$, by (8) with $p = 1$.

9.

$$\begin{aligned}
\int J_3(x)\,dx &= \int x^2[x^{-2}J_3(x)]\,dx \\
&\quad x^2 = u, x^{-2}J_3(x)\,dx = dv,\ 2x\,dx = du, v = -x^{-2}J_2(x) \\
&= -J_2(x) + 2\int x^{-1}J_2(x)\,dx = -J_2(x) - 2x^{-1}J_1(x) + C \\
&= J_0(x) - \frac{2}{x}J_1(x) - \frac{2}{x}J_1(x) + C(\text{use (6) with } p = 1) \\
&= J_0(x) - \frac{4}{x}J_1(x) + C.
\end{aligned}$$

13. Use (6) with $p = 4$. Then

$$\begin{aligned}
J_5(x) &= \frac{8}{x}J_4(x) - J_3(x) \\
&= \frac{8}{x}\left[\frac{6}{x}J_3(x) - J_2(x)\right] - J_3(x) \quad (\text{by (6) with } p = 3) \\
&= \left(\frac{48}{x^2} - 1\right)J_3(x) - \frac{8}{x}J_2(x) \\
&= \left(\frac{48}{x^2} - 1\right)\left(\frac{4}{x}J_2(x) - J_1(x)\right) - \frac{8}{x}J_2(x) \quad (\text{by (6) with } p = 2) \\
&= \left(\frac{192}{x^3} - \frac{12}{x}\right)J_2(x) - \left(\frac{48}{x^2} - 1\right)J_1(x) \\
&= \frac{12}{x}\left(\frac{16}{x^2} - 1\right)\left[\frac{2}{x}J_1(x) - J_0(x)\right] - \left(\frac{48}{x^2} - 1\right)J_1(x) \\
&\quad (\text{by (6) with } p = 1) \\
&= -\frac{12}{x}\left(\frac{16}{x^2} - 1\right)J_0(x) + \left(\frac{384}{x^4} - \frac{72}{x^2} + 1\right)J_1(x).
\end{aligned}$$

17. (a) From (17),

$$\begin{aligned}
A_j &= \frac{2}{J_1(\alpha_j)^2}\int_0^1 f(x)J_0(\alpha_j x)x\,dx = \frac{2}{J_1(\alpha_j)^2}\int_0^c J_0(\alpha_j x)x\,dx \\
&= \frac{2}{\alpha_j^2 J_1(\alpha_j)^2}\int_0^{c\alpha_j} J_0(s)s\,ds \quad (\text{let } \alpha_j x = s) \\
&= \frac{2}{\alpha_j^2 J_1(\alpha_j)^2}J_1(s)s\Big|_0^{c\alpha_j} = \frac{2cJ_1(\alpha_j)}{\alpha_j J_1(\alpha_j)^2}.
\end{aligned}$$

Thus, for $0 < x < 1$,

$$f(x) = \sum_{j=1}^{\infty} \frac{2cJ_1(\alpha_j)}{\alpha_j J_1(\alpha_j)^2}J_0(\alpha_j x).$$

(b) The function f is piecewise smooth, so by Theorem 2 the series in (a) converges to $f(x)$ for all $0 < x < 1$, except at $x = c$, where the series converges to the average value $\frac{f(c+)+f(c-)}{2} = \frac{1}{2}$.

21. (a) Take $m = 1/2$ in the series expansion of Exercise 20 and you'll get

$$\sqrt{x} = 2\sum_{j=1}^{\infty} \frac{J_{1/2}(\alpha_j x)}{\alpha_j\, J_{3/2}(\alpha_j)} \quad \text{for } 0 < x < 1,$$

where α_j is the jth positive zero of $J_{1/2}(x)$. By Example 1, Section 4.7, we have

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi\, x}}\,\sin x.$$

So

$$\alpha_j = j\pi \quad \text{for } j = 1,\, 2,\, \ldots.$$

(b) We recall from Exercise 11 that

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi\, x}}\left(\frac{\sin x}{x} - \cos x\right).$$

So the coefficients are

$$\begin{aligned} A_j &= \frac{2}{\alpha_j\, J_{3/2}(\alpha_j)} = \frac{2}{j\pi\, J_{3/2}(j\pi)} \\ &= \frac{2}{j\pi\,\sqrt{\frac{2}{\pi\, j\pi}}\left(\frac{\sin j\pi}{j\pi} - \cos j\pi\right)} \\ &= (-1)^{j-1}\sqrt{\frac{2}{j}} \end{aligned}$$

and the Bessel series expansion becomes, or $0 < x < 1$,

$$\sqrt{x} = \sum_{j=1}^{\infty} (-1)^{j-1}\sqrt{\frac{2}{j}}\, J_{1/2}(\alpha_j x).$$

(c) Writing $J_{1/2}(x)$ in terms of $\sin x$ and simplifying, this expansion becomes

$$\begin{aligned} \sqrt{x} &= \sum_{j=1}^{\infty} (-1)^{j-1}\sqrt{\frac{2}{j}}\, J_{1/2}(\alpha_j x) \\ &= \sum_{j=1}^{\infty} (-1)^{j-1}\sqrt{\frac{2}{j}}\sqrt{\frac{2}{\pi\,\alpha_j}}\,\sin\alpha_j \\ &= \frac{2}{\pi}\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j}\,\frac{\sin(j\pi x)}{\sqrt{x}}. \end{aligned}$$

Upon multiplying both sides by $\sqrt{x}$, we obtain

$$x = \frac{2}{\pi}\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j}\,\sin(j\pi x) \quad \text{for } 0 < x < 1,$$

which is the familiar Fourier sine series (half-range expansion) of the function $f(x) = x$.

25. By Theorem 2 with $p = 1$, we have

$$\begin{aligned} A_j &= \frac{2}{J_2(\alpha_{1,j})^2} \int_{\frac{1}{2}}^{1} J_1(\alpha_{1,j}x)\,dx \\ &= \frac{2}{\alpha_{1,j} J_2(\alpha_{1,j})^2} \int_{\frac{\alpha_{1,j}}{2}}^{\alpha_{1,j}} J_1(s)\,ds \quad (\text{let } \alpha_{1,j}x = s) \\ &= \frac{2}{\alpha_{1,j} J_2(\alpha_{1,j})^2} \big[-J_0(s)\Big|_{\frac{\alpha_{1,j}}{2}}^{\alpha_{1,j}} \quad (\text{by (8) with } p = 1) \\ &= \frac{-2\big[J_0(\alpha_{1,j}) - J_0(\frac{\alpha_{1,j}}{2})\big]}{\alpha_{1,j} J_2(\alpha_{1,j})^2} \\ &= \frac{-2\big[J_0(\alpha_{1,j}) - J_0(\frac{\alpha_{1,j}}{2})\big]}{\alpha_{1,j} J_0(\alpha_{1,j})^2}, \end{aligned}$$

where in the last equality we used (6) with $p = 1$ at $x = \alpha_{1,j}$ (so $J_0(\alpha_{1,j}) + J_2(\alpha_{1,j}) = 0$ or $J_0(\alpha_{1,j}) = -J_2(\alpha_{1,j})$). Thus, for $0 < x < 1$,

$$f(x) = -2 \sum_{j=1}^{\infty} \frac{-2\big[J_0(\alpha_{1,j}) - J_0(\frac{\alpha_{1,j}}{2})\big]}{\alpha_{1,j} J_0(\alpha_{1,j})^2} J_1(\alpha_{1,j}x).$$

29. By Theorem 2 with $p = 1$, we have

$$\begin{aligned} A_j &= \frac{1}{2\,J_2(\alpha_{1,j})^2} \int_0^2 J_1(\alpha_{2,j}x/2)x\,dx \\ &= \frac{2}{\alpha_{1,j}^2 J_2(\alpha_{1,j})^2} \int_0^{\alpha_{1,j}} J_1(s)s\,ds \quad (\text{let } \alpha_{1,j}x/2 = s). \end{aligned}$$

Since we cannot evaluate the definite integral in a simpler form, just leave it as it is and write the Bessel series expansion as

$$1 = \sum_{j=1}^{\infty} \frac{2}{\alpha_{1,j}^2 J_2(\alpha_{1,j})^2} \left[\int_0^{\alpha_{1,j}} J_1(s)s\,ds\right] J_1(\alpha_{1,j}x/2) \quad \text{for } 0 < x < 2.$$

33. $p = \frac{1}{2}$, $y = c_1 J_{\frac{1}{2}}(\lambda x) + c_2 Y_{\frac{1}{2}}(\lambda x)$. For y to be bounded near 0, we must take $c_2 = 0$. For $y(\pi) = 0$, we must take $\lambda = \lambda_j = \frac{\alpha_{\frac{1}{2},j}}{\pi} = j$, $j = 1, 2, \ldots$ (see Exercises 21); and so

$$y = y_i = c_{1,j} J_1\Big(\frac{\alpha_{\frac{1}{2},j}}{\pi} x\Big) = c_{1,j} \sqrt{\frac{2}{\pi x}} \sin(jx)$$

(see Example 1, Section 4.7).

One more formula. To complement the integral formulas from this section, consider the following interesting formula. Let a, b, c, and p be positive real numbers with $a \neq b$. Then

$$\int_0^c J_p(ax)\,J_p(bx)x\,dx = \frac{c}{b^2 - a^2}\big[aJ_p(bc)J_{p-1}(ac) - bJ_p(ac)J_{p-1}(bc)\big].$$

To prove this formula, we note that $y_1 = J_p(ax)$ satisfies

$$x^2 y_1'' + x y_1' + (a^2x^2 - p^2)y_1 = 0$$

and $y_2 = J_p(bx)$ satisfies

$$x^2 y_2'' + x y_2' + (b^2 x^2 - p^2) y_2 = 0.$$

Write these equations in the form

$$\big(x y_1'\big)' + y_1' + \frac{a^2 x^2 - p^2}{x} y_1 = 0$$

and

$$\big(x y_2'\big)' + y_2' + \frac{b^2 x^2 - p^2}{x} y_1 = 0.$$

Multiply the first by y_2 and the second by $y-1$, subtract, simplify, and get

$$y_2\big(xy_1'\big)' - y_1\big(xy_2'\big)' = y_1 y_2 (b^2 - a^2) x.$$

Note that

$$y_2\big(xy_1'\big)' - y_1\big(xy_2'\big)' = \frac{d}{dx}\big[y_2(xy_1') - y_1(xy_2')\big].$$

So

$$(b^2 - a^2) y_1 y_2 x = \frac{d}{dx}\big[y_2(xy_1') - y_1(xy_2')\big],$$

and, after integrating,

$$(b^2 - a^2)\int_0^c y_1(x) y_2(x) x\, dx = \big[y_2(xy_1') - y_1(xy_2')\big]\Big|_0^c = x\big[y_2 y_1' - y_1 y_2'\big]\Big|_0^c.$$

On the left, we have the desired integral times $(b^2 - a^2)$ and, on the right, we have

$$c\big[J_p(bc) a J_p'(ac) - b J_p(ac) J_p'(bc)\big] - c\big[a J_p(0) J_p'(0) - b J_p(0) J_p'(0)\big].$$

Since $J_p(0) = 0$ if $p > 0$ and $J_0'(x) = -J_1(x)$, it follows that $J_p(0)J_p'(0) - J_p(0)J_p'(0) = 0$ for all $p > 0$. Hence the integral is equal to

$$I = \int_0^c J_p(ax)\, J_p(bx) x\, dx = \frac{c}{b^2 - a^2}\big[a J_p(bc) J_p'(ac) - b J_p(ac) J_p'(bc)\big].$$

Now using the formula

$$J_p'(x) = \frac{1}{2}\big[J_{p-1}(x) - J_{p+1}(x)\big],$$

we obtain

$$I = \frac{c}{2(b^2 - a^2)}\big[a J_p(bc)\big(J_{p-1}(ac) - J_{p+1}(ac)\big) - b J_p(ac)\big(J_{p-1}(bc) - J_{p+1}(bc)\big)\big].$$

Simplify with the help of the formula

$$J_{p+1}(x) = \frac{2p}{x} J_p(x) - J_{p-1}(x)$$

and you get

$$\begin{aligned} I &= \frac{c}{2(b^2 - a^2)}\Big[a J_p(bc)\big(J_{p-1}(ac) - (\frac{2p}{ac} J_p(ac) - J_{p-1}(ac))\big) \\ &\quad - b J_p(ac)\big(J_{p-1}(bc) - (\frac{2p}{bc} J_p(bc) - J_{p-1}(bc))\big)\Big] \\ &= \frac{c}{b^2 - a^2}\big[a J_p(bc) J_{p-1}(ac) - b J_p(ac) J_{p-1}(bc)\big], \end{aligned}$$

as claimed.

Note that this formula implies the orthogonality of Bessel functions. In fact its proof mirrors the proof of orthogonality from Section 9.7.