1 Boas, problem p.578, 12.7-5

Show that $\int_{-1}^{1} dx\, P_l(x) = 0,\ l > 0:$

This is immediate once we remember that $P_0(x) = 1$: then

$$\int_{-1}^{1} dx\, P_l(x) = \int_{-1}^{1} dx\, P_l(x) \cdot 1 = \int_{-1}^{1} dx\, P_l(x) P_0(x) = 0\,, \quad l \neq 0 \tag{1}$$

because of the orthogonality of the Legendre polynomials in the interval $(-1, 1)$.

2 Boas, problem p.580, 12.8-5

Find the norm of $xe^{-x^2/2}$ on the interval $(0, +\infty)$ and state the normalized function:

To find the norm N, we have to calculate

$$N^2 = \int_0^\infty xe^{-x^2/2} \cdot xe^{-x^2/2} dx = \int_0^\infty x^2 e^{-x^2} dx\,; \tag{2}$$

this can be written as

$$\int_0^\infty x^2 e^{-kx^2} dx\,, \quad \text{with k=1;} \tag{3}$$

Now, if we call $I(k) = \int_0^\infty e^{-kx^2}$, we have $N^2 = -\frac{d}{dk} I(k)\big|_{k=1}$. $I(k)$ can be retrieved from the Gaussian integral

$$\int_{-\infty}^\infty e^{-kx^2} dx = \sqrt{\frac{\pi}{k}}\,, \implies I(k) = \frac{1}{2}\sqrt{\frac{\pi}{k}} \tag{4}$$

Finally, we find

$$N^2 = -\frac{d}{dk} I(k)\bigg|_{k=1} = \frac{1}{2}\sqrt{\pi}\frac{1}{2}k^{-3/2}\bigg|_{k=1} = \frac{\sqrt{\pi}}{4} \qquad \implies \qquad N = \frac{1}{2}\pi^{1/4} \tag{5}$$

The normalized function is then

$$\frac{1}{N} xe^{-x^2/2} = 2\pi^{-1/4} xe^{-x^2/2} \tag{6}$$

3 Boas, problem p.581, 12.9-5

Expand the following function in Legendre series:

$$f(x) = \begin{cases} x + 1, & -1 < x < 0 \\ 1 - x, & 0 < x < 1 \end{cases}\,. \tag{7}$$

Now, we write $f(x) = \sum_{l=0}^\infty c_l P_l(x)$, with unknown coefficients c_l; because of the orthogonality of the Legendre Polynomials, if we multiply $f(x)$ by $P_l(x)$ and integrate between -1 and 1, we have:

$$\int_{-1}^{1} dx\, f(x) P_l(x) = \sum_{m=0}^\infty c_m \int_{-1}^{1} dx\, P_m(x) P_l(x) = c_l \frac{2}{2l + 1} \tag{8}$$

Because $f(x)$ is an even function of x, all the odd coefficients will be zero. We want to find an explicit expression for the coefficient c_l:

$$\begin{aligned} c_l &= \frac{2l+1}{2}\int_{-1}^{1} dx\, f(x)P_l(x) \\ &= \frac{2l+1}{2}\left\{\int_{-1}^{0}(1+x)P_l(x)dx + \int_{0}^{1}(1-x)P_l(x)dx\right\} \\ &= \frac{2l+1}{2}\left\{\int_{-1}^{1}P_l(x) + \int_{-1}^{0}xP_l(x)dx - \int_{0}^{1}xP_l(x)dx\right\} \end{aligned} \tag{9}$$

We know from problem 1 on this homework set that

$$\int_{-1}^{1}P_l(x)dx = \frac{2}{2l+1}\,\delta_{l0} = \begin{cases} 2\,, & \text{for } l=0\,, \\ 0\,, & \text{for } l\neq 0\,. \end{cases} \tag{10}$$

The case of $l \neq 0$ was treated in problem 1. For $l = 0$, we have $P_0(x) = 1$ and the integral is trivial. In the second integral on the right hand side of (9), change variables $x \to -x$ and then use $P_l(-x) = (-1)^l P_l(x)$. It follows that:

$$\int_{-1}^{0} xP_l(x)dx = -\int_{0}^{1} xP_l(-x)dx = -(-1)^l\int_{0}^{1} xP_l(x)dx\,. \tag{11}$$

Hence, we conclude that:

$$c_l = \delta_{l0} - \frac{2l+1}{2}\left[1+(-1)^l\right]\int_{0}^{1} xP_l(x)dx\,. \tag{12}$$

Note that (12) implies that $c_l = 0$ for odd l. Thus, $c_{2l+1} = 0$ as expected since $f(x)$ is an even function of x. Thus, its expansion in terms of Legendre polynomials must involve only even functions which correspond to even l. Since $1 + (-1)^l = 2$ for even l, we can rewrite (12) as

$$c_0 = \tfrac{1}{2}\,, \tag{13}$$

$$c_{2l} = -(4l+1)\int_{0}^{1} xP_{2l}(x)dx\,, \qquad \text{for } l = 1,2,3,\ldots\,, \tag{14}$$

$$c_{2l+1} = 0\,, \qquad \text{for } l = 0,1,2,3,\ldots\,, \tag{15}$$

where c_0 has been obtained by inserting $P_0(x) = 1$ into (12) and computing the integral explicitly.

Our remaining task is to compute:

$$\int_{0}^{1} xP_{2l}(x)dx\,. \tag{16}$$

There are many ways to do this. Perhaps the simplest is to make use of the recursion relation given in eq. (5.8a) on p. 570 of Boas:

$$lP_l(x) = (2l-1)xP_{l-1}(x) - (l-1)P_{l-2}(x)\,. \tag{17}$$

Replacing $\ell \longrightarrow 2l+1$, the above recursion relation can be rewritten as:

$$(2l+1)P_{2l+1}(x) = (4l+1)xP_{2l}(x) - 2lP_{2l-1}(x)\,. \tag{18}$$

We can now solve this equation for $xP_{2l}(x)$,

$$xP_{2l}(x) = \frac{2l+1}{4l+1}P_{2l+1}(x) + \frac{2l}{4l+1}P_{2l-1}(x)\,. \tag{19}$$

Inserting this result into (16) yields:

$$\int_0^1 xP_{2l}(x)dx = \frac{2l+1}{4l+1}\int_0^1 P_{2l+1}(x) + \frac{2l}{4l+1}\int_0^1 P_{2l-1}(x)\,. \tag{20}$$

This can now be evaluated by making use a the following result derived in class (this result is also given in problem 23-3 on p. 615 of Boas):

$$\int_0^1 P_{2l+1}(x)dx = \frac{(-1)^l P_{2l}(0)}{2l+2} = \frac{(-1)^l(2l-1)!!}{(2l+2)!!} = \frac{(-1)^l(2l-1)!!}{2^{l+1}(l+1)!}\,. \tag{21}$$

Therefore,

$$\int_0^1 xP_{2l}(x)dx = \frac{2l+1}{4l+1}\frac{(-1)^l(2l-1)!!}{2^{l+1}(l+1)!} + \frac{2l}{4l+1}\frac{(-1)^{l-1}(2l-3)!!}{2^l l!} \tag{22}$$

Noting that $(l+1)! = (l+1)l!$ and $(2l-1)!! = (2l-1)(2l-3)!!$, one can rewrite (22) as:

$$\begin{aligned}\int_0^1 xP_{2l}(x)dx &= \frac{(-1)^l}{4l+1}\frac{(2l-3)!!}{2^l l!}\left[\frac{(2l+1)(2l-1)}{2(l+1)} - 2l\right]\\ &= -(-1)^l\frac{(2l-3)!!}{2^l l!}\frac{1}{2(l+1)}\\ &= \frac{(-1)^{l+1}(2l-3)!!}{2^{l+1}(l+1)!}\,.\end{aligned}$$

Inserting this result back into (14) yields:

$$\boxed{c_{2l} = \frac{(-1)^l(4l+1)(2l-3)!!}{2^{l+1}(l+1)!}\,, \qquad \text{for } l = 1,2,3,\ldots} \tag{23}$$

Combining this result with (13) and (15), it follows that the first few coefficients in the Legendre series for $f(x)$ are given by:[1]

$$c_0 = \tfrac{1}{2}\,,\ c_2 = -\tfrac{5}{8}\,,\ c_4 = \tfrac{3}{16}\,,\ c_6 = -\tfrac{13}{128}\,,\ \ldots, \tag{24}$$

whereas $c_1 = c_3 = c_5 = \cdots = 0$. Hence, we conclude that

$$f(x) = \tfrac{1}{2}P_0(x) - \tfrac{5}{8}P_2(x) + \tfrac{3}{16}P_4(x) - \tfrac{13}{128}P_6(x) + \cdots\,. \tag{25}$$

4 Boas, problem p.582, 12.9-16

Prove the least square approximation property of the Legendre polynomials:
given $f(x)$ the function to be approximated and $p_l(x)$ the orthonormal Legendre polynomials, we can expand $f(x)$ in this basis:

$$f(x) = c_0p_0(x) + c_1p_1(x) + c_2p_2(x) + \ldots = \sum_{l=0} c_l p_l(x) \tag{26}$$

[1]Of course, we can always compute the first few terms of the Legendre series explicitly. For example,

$$2c_0 = \int_{-1}^1 dx\, f(x)P_0(x) = \int_{-1}^0 (x+1)dx + \int_0^1 (1-x)dx = \frac{x^2}{2}+x\Big|_{-1}^0 + x - \frac{x^2}{2}\Big|_0^1 = 1 \qquad \Longrightarrow c_0 = \tfrac{1}{2}$$

$$\frac{2}{5}c_2 = \int_{-1}^1 dx\, f(x)P_2(x) = \int_{-1}^0 (x+1)\frac{1}{2}(3x^2-1)dx + \int_0^1 (1-x)\frac{1}{2}(3x^2-1)dx = -\frac{1}{4} \qquad \Longrightarrow c_2 = -\frac{5}{8}$$

$$\frac{2}{9}c_4 = \int_{-1}^1 dx\, f(x)P_4(x) = \int_{-1}^0 (x+1)\frac{1}{8}(35x^4-30x^2+3)dx + \int_0^1 (1-x)\frac{1}{8}(35x^4-30x^2+3)dx = \frac{1}{24} \Longrightarrow c_4 = \frac{3}{16}$$

etc.

The coefficient c_l are found by multiplying $f(x)$ by p_l and integrating between -1 and 1:

$$\int_{-1}^{1} dx\, f(x)p_l(x) = \int_{-1}^{1} dx \sum_{m=0} c_m p_m(x)p_l(x) = \sum_m c_m \delta_{ml} = c_l \tag{27}$$

Now let $F(x) = b_0p_0(x) + b_1p_1(x) + b_2p_2(x)$ be the (unknown) quadratic polynomial satisfying the least square condition, that is, such that

$$I = \int_{-1}^{1} dx\, [f(x) - F(x)]^2 \tag{28}$$

is a minimum. Squaring the bracket and using the orthonormality of the p_l's we can rewrite I as

$$\begin{aligned} I = \int_{-1}^{1} dx\, [f^2(x) + F^2(x) - 2f(x)F(x)] &= \int_{-1}^{1} dx\, \left[f^2(x)\right] + b_0^2 + b_1^2 + b_2^2 - 2b_0c_0 - 2b_1c_1 - 2b_2c_2 = \\ = \int_{-1}^{1} f^2(x)dx + (b_0 - c_0)^2 + (b_1 - c_1)^2 + (b_2 - c_2)^2 - c_0^2 - c_1^2 - c_2^2. \end{aligned} \tag{29}$$

We are looking for the unknown coefficients b_l that minimize I; now, there are only three terms in I that depend on the b's, and they form a sum of squared numbers: then, I is minimum when these terms are zero, that is, when $b_l = c_l$. Finally, we have found that the coefficients of the quadratic polynomial that best approximates a function $f(x)$ are the coefficients of the Legendre expansion of the function itself.

Now we can generalize this result to approximate any function to a polynomial of degree n; writing the polynomial as $F(x) = \sum_{l=0}^{n} b_l p_l(x)$ and trying to minimize the integral $I = \int_{-1}^{1} dx\, [f(x) - F(x)]^2$, working as we did above we find terms depending on the b's of this form:

$$(b_0 - c_0)^2 + (b_1 - c_1)^2 + \ldots + (b_n - c_n)^2.$$

Again, this sum is minimal for $b_l = c_l$, that is, when the approximated polynomial is given by the Legendre expansion of the function itself.

5 Boas, problem p.584, 12.10-3

Show that the functions $P_l^m(x)$ for each m are a set of orthogonal functions on $(-1, 1)$, that is, show that

$$\int_{-1}^{1} dx\, P_l^m(x)P_n^m(x) = 0\,, \qquad l \neq n\,: \tag{30}$$

We recall the associated Legendre equation

$$(1 - x^2)P_l^{m\,\prime\prime} - 2xP_l^{m\,\prime} + \left[l(l+1) - \frac{m^2}{1 - x^2}\right] P_l^m = 0 \tag{31}$$

and rewrite it as

$$\frac{d}{dx}\left[(1 - x^2)P_l^{m\,\prime}\right] + \left[l(l+1) - \frac{m^2}{1 - x^2}\right] P_l^m = 0 \tag{32}$$

Writing this equation for $P_n^m(x)$, multiplying it by $P_l^m(x)$, multiplying (32) by $P_n^m(x)$ and subtracting the two resulting equations we find

$$P_n^m \frac{d}{dx}\left[(1 - x^2)P_l^{m\,\prime}\right] - P_l^m \frac{d}{dx}\left[(1 - x^2)P_n^{m\,\prime}\right] + [l(l+1) - n(n+1)]\, P_l^m P_n^m = 0 \tag{33}$$

$$\frac{d}{dx}\left[(1 - x^2)(P_n^m P_l^{m\,\prime} - P_l^m P_n^{m\,\prime})\right] + [l(l+1) - n(n+1)]\, P_l^m P_n^m = 0 \tag{34}$$

Integrating between -1 and 1, we have

$$(1-x^2)(P_n^m P_l^{m\,\prime} - P_l^m P_n^{m\,\prime})\Big|_{-1}^{1} + [l(l+1) - n(n+1)] \int_{-1}^{1} dx\, P_l^m P_n^m = 0 \tag{35}$$

The integrated term is zero, so we have proven the orthogonality relation (30) for $l \neq n$.

6 Boas, problem p.584, 12.10-8

Write the definition of the associated Legendre function by Rodrigues' formula

$$P_l^m(x) = \frac{1}{2^l l!}(1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l \tag{36}$$

with m replaced by $-m$.

We have

$$P_l^{-m} = \frac{1}{2^l l!}(1-x^2)^{-m/2} \frac{d^{l-m}}{dx^{l-m}}(x^2-1)^l\,; \tag{37}$$

we quote the following relation from problem 12.10.7

$$\frac{d^{l-m}}{dx^{l-m}}(x^2-1)^l = \frac{(l-m)!}{(l+m)!}(x^2-1)^m \frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l \tag{38}$$

and substituting (38) in (37) we find

$$P_l^{-m} = \frac{1}{2^l l!}(1-x^2)^{-m/2} \frac{(l-m)!}{(l+m)!}(x^2-1)^m \frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l = \tag{39}$$

$$= \frac{(l-m)!}{(l+m)!}\frac{1}{2^l l!}(1-x^2)^{-m/2}(-1)^m(1-x^2)^m \frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x) \tag{40}$$

Because P_l^{-m} is proportional to P_l^m, it also solves the equation (31).

7 Boas, problem p.587, 12.11-13

Solve $y'' + y'/x^2 = 0$ by power series.

We take $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and substitute to find

$$\sum_{n=2} n(n-1)a_n x^{n-2} + \sum_{n=1} n a_n x^{n-3} = 0 \qquad \Longrightarrow \qquad \sum_{n=2} n(n-1)a_n x^{n-2} + \sum_{n=0} (n+1)a_{n+1} x^{n-2} = 0$$

$$a_1 x^{-2} + 2a_2 x^{-1} + \sum_{n=2}\left[n(n-1)a_n x^{n-2} + (n+1)a_{n+1}x^{n-2}\right] = 0 \qquad \Longrightarrow a_{n+1} = -\frac{n(n-1)}{n+1}a_n \tag{41}$$

Naively looking at the convergence of the series $\sum_{n=0}^{\infty} a_n x^n$ by the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} n = \infty \tag{42}$$

The series looks divergent and one is tempted to say that there is no power series solution of the equation.

But if we look back at (41) we see that the starting terms of the series have zero coefficients: we must have $a_1 = a_2 = 0$; in turn, this tells us that $a_n = 0$, for all values of $n > 0$: they are all zero. The only coefficient without constraints is a_0, implying $y(x) = a_0 = \text{const}$ and one sees that this is a solution of the equation, as $y' = y'' = 0$.

8 Boas, problem p.590, 12.12-8

Prove that

$$\lim_{x\to 0} x^{-3/2} J_{3/2}(x) = \frac{1}{3}\sqrt{2/\pi}: \tag{43}$$

We write down the series for $J_{3/2}$

$$J_{3/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+\frac{3}{2})} \left(\frac{x}{2}\right)^{2n+\frac{3}{2}}. \tag{44}$$

This series starts with a $x^{3/2}$ term; if we multiply by $x^{-3/2}$ and take the limit $x \to 0$, the other terms in the series are proportional to $x^{2n} \to 0$. Then

$$\lim_{x\to 0} x^{-3/2} J_{3/2}(x) = \frac{1}{\Gamma(1)\Gamma(\frac{3}{2}+1)} 2^{-3/2} = \frac{1}{\frac{3}{2}\Gamma(\frac{3}{2})} 2^{-3/2} = \frac{1}{\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})} 2^{-3/2} = \frac{1}{3}\sqrt{\frac{2}{\pi}}. \tag{45}$$

where we used $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

9 Boas, problem p.590, 12.12-9

Prove that

$$\sqrt{\frac{\pi x}{2}} J_{1/2}(x) = \sin x: \tag{46}$$

We simply substitute the series for $p = \frac{1}{2}$:

$$\sqrt{\frac{\pi x}{2}} J_{1/2}(x) = \sqrt{\frac{\pi x}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+\frac{1}{2})} \left(\frac{x}{2}\right)^{2n+\frac{1}{2}} = \tag{47}$$

$$= \sqrt{\frac{\pi}{2}} x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+\frac{1}{2})(n+\frac{1}{2}-1)\ldots\frac{1}{2}\Gamma(\frac{1}{2})2^{2n}\sqrt{2}} x^{2n} = \tag{48}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sin x \tag{49}$$

10 Boas, problem p.591, 12.13-6

Show from

$$N_p(x) = \frac{cos(\pi p) J_p(x) - J_{-p}(x)}{\sin(\pi p)} \tag{50}$$

that

$$N_{(2n+1)/2}(x) = (-1)^{n+1} J_{-(2n+1)/2}(x): \tag{51}$$

For $p = \frac{2n+1}{2}$ we have $\cos \pi p = 0$ and $\sin \pi p = (-1)^n$; plugging those values back in (50) we find

$$N_{(2n+1)/2}(x) = -(-1)^n J_{-(2n+1)/2}(x). \tag{52}$$

11 Boas, problem p.593, 12.15-7

(a) Using $\frac{d}{dx}[x^{-p}J_p(x)] = -x^{-p}J_{p+1}(x)$, show that

$$\int_0^\infty J_1(x)dx = -J_0(x)\Big|_0^\infty = 1: \tag{53}$$

That is immediate once we make the substitution and integrate by parts:

$$\int_0^\infty J_1(x)dx = -\int_0^\infty dx\,\frac{d}{dx}[x^0 J_0(x)] = -J_0(x)\Big|_0^\infty = 1. \tag{54}$$

(b) Use $F(p) = \int_0^\infty e^{-pt}J_0(at) = (p^2+a^2)^{-1/2}$ to show that

$$\int_0^\infty J_0(t) = 1 \tag{55}$$

This is also immediate, as $\int_0^\infty J_0(t)$ is the Laplace transform of the Bessel function calculated in $p=0$ (with $a=1$).

$$\int_0^\infty J_0(t) = F(0) = (1)^{-1/2} = 1 \tag{56}$$

12 Boas, problem p.616, 12.23-19

(a) The generating function of the Bessel functions of integral order $p=n$ is

$$\Phi(x,h) = \exp\left[\frac{1}{2}x\left(h-\frac{1}{h}\right)\right] = \sum_{n=-\infty}^{+\infty} h^n J_n(x). \tag{57}$$

By expanding the exponential, show that the $n=0$ term is $J_0(x)$:

$$\Phi(x,h) = \sum_{n=0}^{\infty}\frac{[\frac{1}{2}x(h-1/h)]^n}{n!} = \tag{58}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{x}{2h}\right)^n(h^2-1)^n = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{x}{2h}\right)^n\sum_{k=0}^{n}\binom{n}{k}h^{2(n-k)}(-1)^k = \tag{59}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{n!}\left(\frac{x}{2}\right)^n\frac{n!}{k!(n-k)!}(-1)^k h^{n-2k} \tag{60}$$

We are looking for a term of the form $h^0 J_0(x)$; the power of h in (60) is $n-2k$, so that happens only for even n; then we change the sum variable to $n=2l$, and J_0 will come from a single term ($k=l$) in the sum over k: we have

$$J_0(x) = \sum_l \frac{1}{(2l)!}\left(\frac{x}{2}\right)^{2l}\frac{(2l)!}{l!l!}(-1)^l = \sum_l \frac{(-1)^l}{\Gamma(l+1)\Gamma(l+1)}\left(\frac{x}{2}\right)^{2l} \tag{61}$$

which is the definition of the Bessel function $J_0(x)$.

(b) Show that $\Phi(x,h)$ is a solution of the differential equation

$$x^2\frac{d^2\Phi}{dx^2} + x\frac{d\Phi}{dx} + x^2\Phi - \left(h\frac{d}{dh}\right)^2\Phi = 0: \tag{62}$$

We write down the derivatives of Φ:

$$\frac{d\Phi}{dx} = \frac{1}{2}\left(h - \frac{1}{h}\right)\Phi, \qquad \frac{d^2\Phi}{dx^2} = \frac{1}{4}\left(h - \frac{1}{h}\right)^2\Phi, \qquad \frac{d\Phi}{dh} = \frac{x}{2}\left(1 + \frac{1}{h^2}\right)\Phi \tag{63}$$

$$\frac{d^2\Phi}{dh^2} = \frac{x}{2}\left(-\frac{2}{h^3}\right)\Phi + \frac{x^2}{4}\left(1 + \frac{1}{h^2}\right)^2\Phi, \qquad \left(h\frac{d}{dh}\right)^2\Phi = h\frac{d\Phi}{dh} + h^2\frac{d^2\Phi}{dh^2} \tag{64}$$

Equation (62) is then verified:

$$\frac{x^2}{4}\left(h - \frac{1}{h}\right)^2 + \frac{x}{2}\left(h - \frac{1}{h}\right) + x^2 - \frac{x}{2}\left(h + \frac{1}{h}\right) + x\frac{1}{h} - \frac{x^2}{4}\left(h + \frac{1}{h}\right)^2 = 0 \tag{65}$$

Now, one can verify that this implies that the functions $J_n(x)$ in the series $\Phi = \sum_n h^n J_n(x)$ satisfy Bessel's equation; substituting the series in (62) we have

$$x^2 \sum_n h^n J_n''(x) + x \sum_n h^n J_n'(x) + x^2 \sum_n h^n J_n(x) - \sum_n n^2 h^n J_n(x) = 0 \tag{66}$$

from which we find Bessel's equation

$$x^2 J_n''(x) + x J_n'(x) + (x^2 - n^2) J_n(x) = 0. \tag{67}$$

Finally, we want to verify that the function J_n is a series that starts with the coefficient $\frac{1}{n!}\left(\frac{x}{2}\right)^n$, that is, that J_n is the solution to Bessel's equation that is regular in the origin (so that Φ is indeed the generating functional of the Bessel functions). We look at the expression (60):

$$\Phi(x,h) = \sum_n h^n J_n(x) = \sum_{m=0}^{\infty}\sum_{k=0}^{m} \frac{1}{m!}\left(\frac{x}{2}\right)^m \frac{m!}{k!(m-k)!}(-1)^k h^{m-2k} \tag{68}$$

Again, we have multiple terms in the sum contributing to the n-th power of h: $m - 2k = n \implies k = (m-n)/2$

- n even, $n = 2N$: since k is an integer, this receives contributions only from even values of m, $m = 2M$, so that $k = M - N$

$$\begin{aligned} J_{2N}(x) &= \sum_M \left(\frac{x}{2}\right)^{2M} \frac{1}{(M-N)!(M+N)!}(-1)^{M-N}, \qquad \text{changing variable } L = M - N \\ &= \sum_L \frac{(-1)^L}{L!(L+2N)!}\left(\frac{x}{2}\right)^{2L+2N} \end{aligned} \tag{69}$$

  One recognizes that this is the series form of the Bessel function J_n for $n = 2N$.

- n odd, $n = 2N+1$: since k is an integer, this receives contributions only from odd values of m, $m = 2M+1$, so that $k = M - N$

$$\begin{aligned} J_{2N+1}(x) &= \sum_M \left(\frac{x}{2}\right)^{2M+1} \frac{1}{(M-N)!(M+N+1)!}(-1)^{M-N}, \qquad \text{changing variable } L = M - N \\ &= \sum_L \frac{(-1)^L}{L!(L+2N+1)!}\left(\frac{x}{2}\right)^{2L+2N+1} \end{aligned} \tag{70}$$

  Again, one sees that this is the expression of the Bessel function J_n for $n = 2N+1$.

Finally, we have found that $\Phi(x,h)$ is indeed the generating functional of the Bessel functions.