1 Boas, problem p.594, 12.16-8

Find the solutions of the followind differential equation in terms of Bessel functions:

$$y'' + xy = 0 : \tag{1}$$

This is an equation of the form

$$y'' + \frac{1-2a}{x}y' + \left[(bcx^{c-1})^2 + \frac{a^2 - p^2c^2}{x^2}\right] y = 0 \tag{2}$$

with

$$1 - 2a = 0, \qquad (bc)^2 = 1, \qquad 2(c-1) = 1, \qquad a^2 - p^2c^2 = 0 \tag{3}$$

or

$$a = \tfrac{1}{2}, \qquad c = \frac{3}{2}, \qquad b = \frac{2}{3}, \qquad p = \frac{1}{3} \tag{4}$$

Then the solution of (1) is

$$y = x^{1/2} Z_{1/3}(\tfrac{2}{3}x^{3/2}) \tag{5}$$

The general solution is then

$$y = x^{1/2}[J_{1/3}(\tfrac{2}{3}x^{3/2}) + BN_{1/3}(\tfrac{2}{3}x^{3/2})] \tag{6}$$

2 Boas, problem p.597, 12.17-2

From problem 12.9, $J_{1/2}(x) = \sqrt{2/\pi x}\sin x$. Use the recursion relation $\frac{d}{dx}[x^{-p}J_p(x)] = -x^{-p}J_{p+1}(x)$ find $J_{3/2}$ and $J_{5/2}$ and verify the formulas for the spherical Bessel functions in terms of $\sin x$ and $\cos x$:

We have

$$J_{3/2} = -x^{1/2}\frac{d}{dx}[x^{-1/2}J_{1/2}] = -\sqrt{\frac{2}{\pi}}x\frac{d}{dx}\left[\frac{\sin x}{x}\right] = -\sqrt{\frac{2}{\pi}}x\left[\frac{\cos x}{x} - \frac{\sin x}{x^2}\right] \tag{7}$$

$$J_{5/2} = -x^{3/2}\frac{d}{dx}[x^{-3/2}J_{3/2}] = \sqrt{\frac{2}{\pi}}x^3\frac{d}{dx}\left[\frac{\cos x}{x^2} - \frac{\sin x}{x^3}\right] = \sqrt{\frac{2}{\pi}}x^3\left[-\frac{\sin x}{x^2} - 2\frac{\cos x}{x^3} - \frac{\cos x}{x^3} + 3\frac{\sin x}{x^4}\right]$$

Now we want to verify that

$$j_n(x) = \sqrt{\pi/2x}J_{\frac{2n+1}{2}}(x) = x^n\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right) \tag{8}$$

So we write down

$$j_0(x) = \sqrt{\frac{\pi}{2x}}J_{1/2} = \sqrt{\frac{\pi}{2x}}\sqrt{\frac{2}{\pi x}}\sin x = \frac{\sin x}{x} \equiv x^0\left(-\frac{1}{x}\frac{d}{dx}\right)^0\left(\frac{\sin x}{x}\right); \tag{9}$$

$$j_1(x) = \sqrt{\frac{\pi}{2x}}J_{3/2} = -\sqrt{\frac{\pi}{2x}}\sqrt{\frac{2}{\pi}}x\left[\frac{\cos x}{x} - \frac{\sin x}{x^2}\right] = \left[\frac{\cos x}{x} - \frac{\sin x}{x^2}\right] \equiv -x\frac{1}{x}\frac{d}{dx}\left(\frac{\sin x}{x}\right);$$

$$j_2(x) = \sqrt{\frac{\pi}{2x}}J_{5/2} = -\sqrt{\frac{\pi}{2x}}\sqrt{\frac{2}{\pi}}x^3\left[-\frac{\sin x}{x^2} - 3\frac{\cos x}{x^3} + 3\frac{\sin x}{x^4}\right] = \left[-\frac{\sin x}{x} - 3\frac{\cos x}{x^2} + 3\frac{\sin x}{x^3}\right] = \tag{10}$$

$$\equiv x^2\left(-\frac{1}{x}\frac{d}{dx}\right)^2\left(\frac{\sin x}{x}\right) = \left(-\frac{1}{x}\frac{d}{dx} + \frac{d^2}{dx^2}\right)\left(\frac{\sin x}{x}\right) \tag{11}$$

Formula (8) is then verified.

3 Boas, problem p.597, 12.17-4

Using $I_p(x) = i^{-p} J_p(ix),\ K_p(x) = \frac{\pi}{2} i^{p+1} H_p^{(1)}(ix)$ and the results stated in problem 17.2 and 17.3 for $J_{1/2}$ and $Y_{1/2}$ show that

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh x \qquad \text{and} \qquad K_{1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \ : \tag{12}$$

We calculate the hyperbolic Bessel functions as

$$I_{1/2}(x) = i^{-1/2} J_{1/2}(ix) = i^{-1/2} \sqrt{\frac{2}{\pi i x}} \sin(ix) = \sqrt{\frac{2}{\pi x}} \sinh x \tag{13}$$

$$\begin{aligned} K_{1/2}(x) &= \frac{\pi}{2} i^{3/2} H_{1/2}^{(1)}(ix) = \frac{\pi}{2} i^{3/2} \left(\sqrt{\frac{2}{\pi i x}} \sin(ix) - i \sqrt{\frac{2}{\pi i x}} \cos(ix) \right) \\ &= -\sqrt{\frac{\pi}{2x}} (\sinh x - \cosh x) = \sqrt{\frac{\pi}{2x}} e^{-x} \end{aligned} \tag{14}$$

after using $\sinh x = -i \sin(ix)$ and $\cosh x = \cos(ix)$.

4 Boas, problem p.598, 12.17-12

Obtain the following recursion relation for the spherical Bessel function:

$$j_{n-1}(x) + j_{n+1}(x) = (2n+1) \frac{j_n(x)}{x} \ : \tag{15}$$

This follows directly from the definition of the spherical Bessel function and the recursion relation for the Bessel functions of the first kind:

$$J_{p-1} + J_{p+1} = \frac{2p}{x} J_p(x) \tag{16}$$

$$j_{n-1}(x) + j_{n+1}(x) = \sqrt{\frac{\pi}{2x}} \left(J_{\frac{2n-1}{2}}(x) + J_{\frac{2n+3}{2}}(x) \right) = \sqrt{\frac{\pi}{2x}} \frac{2n+1}{x} J_{2n+1/2}(x) = (2n+1) \frac{j_n(x)}{x} \tag{17}$$

5 Boas, problem p.600, 12.18-5

Use the recursion relation for J and N and Problem 18.4 to show that

$$J_n(x) N_{n+1}(x) - J_{n+1}(x) N_n(x) = -\frac{2}{\pi x} \ : \tag{18}$$

The quoted result from Problem 18.4 is

$$J_p N_p' - J_p' N_p = \frac{2}{\pi x}, \tag{19}$$

and we use the recursion relations (holding for both N and J)

$$J_{p-1} - J_{p+1} = 2 J_p' \tag{20}$$

We prove it by induction: first we show that it holds for $n = 0$, then we show that if it holds for $n - 1$ it also does for n, completing the proof:

- Starting from $n=0$, we want to prove

$$J_0 N_1 - J_1 N_0 = -\frac{2}{\pi x} \tag{21}$$

This is easy: we have $J_1 = J_{-1} - 2J_0' = -J_1 - 2J_0' \implies J_1 = -J_0'$ so that

$$J_0 N_1 - J_1 N_0 = -J_0 N_0' + J_0' N_0 = -\frac{2}{\pi x} \tag{22}$$

- We assume that

$$J_{n-1}(x) N_n(x) - J_n(x) N_{n-1}(x) = -\frac{2}{\pi x} \tag{23}$$

Then, using (20), we have

$$J_n(x)N_{n+1}(x) - J_{n+1}(x)N_n(x) = J_n(x)N_{n-1}(x) - 2J_n(x)N_n'(x) - J_{n-1}(x)N_n(x) + 2J_n'(x)N_n(x) = \\ = \frac{2}{\pi x} - 2\frac{2}{\pi x} = -\frac{2}{\pi x} \tag{24}$$

and the proof is complete.

6 Boas, problem p.600, 12.18-6

For the initial conditions $\theta = \theta_0$, $\dot{\theta} = 0$ show that the constants A, B of Boas, page 598, are given by

$$A = -\frac{\pi u_0^2}{2}\theta_0 N_2(u_0), \qquad B = \frac{\pi u_0^2}{2}\theta_0 J_2(u_0): \tag{25}$$

where $u = bl^{1/2} = 2\frac{\sqrt{g}}{v} l^{1/2}$.

The solution to the pendulum differential equation is given as

$$\theta = Au^{-1}J_1(u) + Bu^{-1}N_1(u) \tag{26}$$

Differentiating and using the recursion relations, we also have

$$\frac{d\theta}{du} = -[Au^{-1}J_2(u) + Bu^{-1}N_2(u)] \tag{27}$$

But for $u = u_0$ we have $\theta = \theta_0$, $\dot{\theta} = 0$:

$$\frac{d\theta}{du} = \frac{d\theta}{dt}\frac{dt}{du} = \dot{\theta}\frac{1}{\frac{1}{2}bl^{-1/2}v} = 0 \text{ if } \dot{\theta} = 0 \tag{28}$$

Then, we have

$$\begin{cases} \theta_0 = Au_0^{-1}J_1(u_0) + Bu_0^{-1}N_1(u_0) \\ 0 = Au_0^{-1}J_2(u_0) + Bu_0^{-1}N_2(u_0) \end{cases} \implies \begin{cases} A = -BN_2(u_0)/J_2(u_0) \\ B = u_0\theta_0 J_2(u_0) 1/[N_1(u_0)J_2(u_0) - N_2(u_0)J_1(u_0)] \end{cases} \tag{29}$$

By using (18) for $n = 1$, we find the constants A and B

$$B = \frac{\pi u_0^2}{2}\theta_0 J_2(u_0), \qquad A = -\frac{\pi u_0^2}{2}\theta_0 N_2(u_0) \tag{30}$$

7 Boas, problem p.603, 12.19-1

We are going to prove eq. 19.10, p. 602; given the Bessel function of order p, $J_p(x)$ we have

$$x(xJ_p')' + (\alpha^2 x^2 - p^2)J_p(\alpha x) = 0 \tag{31}$$

$$x(xJ_p')' + (\beta^2 x^2 - p^2)J_p(\beta x) = 0 \tag{32}$$

for any value of α, β. Calling $u = J_p(\alpha x)$ and $v = J_p(\beta x)$, we get

$$(vxu' - uxv')\Big|_0^1 + (\alpha^2 - \beta^2)\int_0^1 xuv\,dx = 0 \tag{33}$$

Now, if we assume that α is a zero of the Bessel function, we have $u(1) = J_p(\alpha) = 0$ and

$$J_p(\beta)\alpha J_p'(\alpha) + (\alpha^2 - \beta^2)\int_0^1 xuv\,dx = 0 \implies \int_0^1 xuv\,dx = \frac{J_p(\beta)\alpha J_p'(\alpha)}{\beta^2 - \alpha^2} \tag{34}$$

For $\beta \to \alpha$, this is

$$\int_0^1 xuv\,dx = \lim_{\beta\to\alpha} \frac{J_p'(\beta)\alpha J_p'(\alpha)}{2\beta} = \tfrac{1}{2}{J_p'}^2(\alpha) \tag{35}$$

We can express this result in other forms using the recursion relation:

$$J_p'(x) = -\frac{p}{x}J_p(x) + J_{p-1}(x) = \frac{p}{x}J_p(x) - J_{p+1}(x) \tag{36}$$

As α is a zero, we have $J_p'(\alpha) = J_{p-1}(\alpha) = -J_{p+1}(\alpha)$, and

$$\int_0^1 xuv\,dx = \tfrac{1}{2}J_{p-1}^2(\alpha) = \tfrac{1}{2}J_{p+1}^2(\alpha) \tag{37}$$

8 Boas, problem p.603, 12.19-6

By problem 19.5, $\int_0^1 xN_{1/2}(\alpha x)N_{1/2}(\beta x)dx = 0$ if α, β are two different zeros of $N_{1/2}(x)$. We can write $N_{1/2}(x)$ as

$$\sqrt{\frac{\pi}{2x}}N_{1/2}(x) = \frac{\cos x}{x} \tag{38}$$

so that its zeros are $\alpha_n = \left(n + \frac{1}{2}\right)\pi$. So we have

$$\int_0^1 xN_{1/2}(\alpha_n x)N_{1/2}(\alpha_m x)\,dx = \int_0^1 \frac{2}{\pi}\frac{1}{\sqrt{\alpha_n}}\cos[(n+\tfrac{1}{2})\pi x]\frac{1}{\sqrt{\alpha_m}}\cos[(m+\tfrac{1}{2})\pi x]\,dx = 0 \qquad \text{for } n \neq m \tag{39}$$

We have the proved that the functions $\cos(n + \frac{1}{2})\pi x$ are a set of orthogonal functions on $(0, 1)$. We can find the normalization constant using (35):

$$\int_0^1 xN_{1/2}(\alpha_n x)N_{1/2}(\alpha_n x)\,dx = \int_0^1 \frac{2}{\pi\alpha_n}\cos[(n+\tfrac{1}{2})\pi x]\cos[(n+\tfrac{1}{2})\pi x]\,dx = \frac{1}{2}\left[\frac{d}{dx}\left(\sqrt{\frac{2}{\pi x}}\cos x\right)\Big|_{x=\alpha_n}\right]^2 =$$

$$= \frac{1}{\pi}\left[\frac{1}{x}\left(\sin x\sqrt{x} - \cos x\frac{1}{2\sqrt{x}}\right)\Big|_{x=\alpha_n}\right]^2 = \frac{1}{\pi\alpha_n} \tag{40}$$

The orthonormal functions are then

$$\frac{1}{\sqrt{2}}\cos[(n + \tfrac{1}{2})\pi x]. \tag{41}$$

9 Boas, problem p.604, 12.20-6

Evaluate the following limit:

$$\lim_{x\to 0} x j_n(x) y_n(x) = \lim_{x\to 0} x \left(\frac{x^n}{(2n+1)!!} + \mathcal{O}(x^{n+2}) \right) \left(-\frac{(2n-1)!!}{x^{n+1}} + \mathcal{O}(x^{1-n}) \right) \tag{42}$$

$$= -\frac{(2n-1)!!}{(2n+1)!!} = -\frac{1}{2n+1} \tag{43}$$

10 Boas, problem p.606, 12.21-3

Find one solution of the differential equation by series and then find the second solution by the method of reduction of order :

$$x^2 y'' + x^2 y' - 2y = 0 \tag{44}$$

Inserting the series $y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$, we have

$$\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s} + \sum_{n=0}^{\infty} (n+s) a_n x^{n+s+1} - 2 \sum_{n=0}^{\infty} a_n x^{n+s} = 0\,. \tag{45}$$

Shifting the index on the second sum, we obtain:

$$\sum_{n=0}^{\infty} [(n+s)(n+s-1) - 2] a_n x^{n+s} + \sum_{n=1}^{\infty} (n+s-1) a_{n-1} x^{n+s} = 0\,. \tag{46}$$

For $n = 0$ we obtain the indicial equation, $s(s-1)a_0 - 2a_0 = (s+1)(s-2)a_0 = 0$, which yields the indicial indices $s = -1$ and $s = 2$.

For $s = -1$, the recurrence relation obtained by setting the coefficient of x^{n+s} to zero for $n = 1, 2, 3, \ldots$ is given by:

$$n(n-3)a_n = -(n-2)a_{n-1}\,, \qquad \text{for } n = 1, 2, 3, \ldots \tag{47}$$

It follows that $a_1 = -\frac{1}{2}a_0$ and $a_2 = 0$. When we put $n = 3$ in (47), we obtain the equation $0 = 0$. Thus, a_3 is a free parameter that is not determined by (47). However, it is a simple matter to check that all higher coefficients a_4, a_5, $a_6, \ldots$ can be determined from a_3. You can easily check that:

$$a_{n+3} = \frac{6a_3(-1)^n}{(n+3)(n+2)n!}\,, \qquad \text{for } n = 0, 1, 2, 3, \ldots \tag{48}$$

Thus, the series solution obtained is:

$$y(x) = -\tfrac{1}{2}a_0 \left(1 - \frac{2}{x}\right) + \frac{6a_3}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+3}}{(n+3)(n+2)n!}\,. \tag{49}$$

Thus, by employing $s = -1$, we have already obtained two linearly independent solutions—one proportional to a_0 and one proportional to a_3.

What would have happened if we had obtained the recurrence relation corresponding to $s = 2$? It is straightforward to check that the resulting recurrence relation is:

$$n(n+3)a_n = -(n+1)a_{n-1}\,, \qquad \text{for } n = 1, 2, 3, \ldots \tag{50}$$

which yields

$$a_n = \frac{6a_0(-1)^n}{(n+3)(n+2)n!}\,, \qquad \text{for } n = 0, 1, 2, 3, \ldots \tag{51}$$

Thus, for $s = 2$ one obtains the series solution

$$6a_0x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+3)(n+2)n!}\,, \tag{52}$$

which simply reproduces the second linearly independent solution obtained in (49). This is not surprising, since the recurrence relation given by (47) starting from $n = 4$ is precisely the same relation as (47) starting from $n = 1$ [check this by letting $n \to n + 3$ in (47)]. In the class handout on series solutions to differential equations, this corresponds to case 3 in which the indicial indices differ by an integer, but neither series solution involves a logarithm.

Boas instructs us to complete this problem as follows. First identify the simpler solution as the one proportional to a_0 in (49),

$$y_1(x) = 1 - \frac{2}{x}\,. \tag{53}$$

Instead of finding the second solution by explicitly evaluating the sum that multiplies a_3 in (49) [or equivalently, evaluating the sum in (52)], Boas suggests that we make use of the "reduction of order" method for obtaining the second linearly independent solution. In the class handout on the Wronskian, I showed that for the differential equation,

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0\,, \tag{54}$$

the reduction of order method yields

$$y_2(x) = y_1(x) \int \frac{W(x)}{[y_1(x)]^2}\, dx\,, \tag{55}$$

where the Wronskian is given by Abel's formula,

$$W(x) = c \exp\left\{-\int \frac{a_1(x)}{a_0(x)}\, dx\right\} \tag{56}$$

and c is a constant, which we can ignore in this calculation since it can be reabsorbed into the definition of $y_2(x)$. In the present problem, $a_0(x) = a_1(x) = x^2$, and we immediately obtain $W(x) = e^{-x}$. Hence,

$$\begin{aligned} y_2(x) &= \left(1 - \frac{2}{x}\right) \int e^{-x} \left((1 - \frac{2}{x}\right)^{-2} = \left(1 - \frac{2}{x}\right) \int \frac{x^2 e^{-x}}{(x-2)^2} \\ &= \left(1 - \frac{2}{x}\right)\left(\frac{2+x}{2-x}\right) e^{-x} = -\left(1 + \frac{2}{x}\right) e^{-x}\,. \end{aligned}$$

The indefinite integral above can be computed by the substitution $y = x - 2$ followed by an integration by parts. Luckily, the indefinite integral $\int dy(e^{-y}/y)$ drops out of the final answer, and the end result is rather simple.

For fun, let us check that the same result can be obtained by evaluating the sum proportional to a_3 in (49). Define the function:

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+3}}{(n+3)(n+2)n!}\,. \tag{57}$$

Then, taking two derivatives, we obtain:

$$f''(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n!} = xe^{-x}\,, \tag{58}$$

after recognizing the power series for e^{-x}. We now integrate twice to get back to $f(x)$,

$$f'(x) = \int f''(x)dx = -(x+1)e^{-x} + C_1\,,$$

$$f(x) = \int f'(x)dx = (x+2)e^{-x} + C_1 x + C_2\,.$$

The constants of integration C_1 and C_2 can be determined by noting that the first term in the power series of $f(x)$ is $\mathcal{O}(x^3)$. Noting that

$$(x+2)e^{-x} = (x+2)\left[1 - x + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \mathcal{O}(x^4)\right] = 2 - x + \tfrac{1}{6}x^3 + \mathcal{O}(x^4)\,, \tag{59}$$

it follows that we must take $C_1 = 1$ and $C_2 = -2$. Thus, we conclude that:

$$f(x) = (x+2)e^{-x} + x - 2\,. \tag{60}$$

Indeed the first term of the power series given by (57) is $\frac{1}{6}x^3$ as required. Using this result in (49), we see that the term proportional to $6a_3$ is

$$\left(1 + \frac{2}{x}\right)e^{-x} + 1 - \frac{2}{x}\,. \tag{61}$$

Hence, (49) can be rewritten as:

$$y(x) = \left(6a_3 - \tfrac{1}{2}a_0\right)\left(1 - \frac{2}{x}\right) + 6a_3\left(1 + \frac{2}{x}\right)e^{-x}\,. \tag{62}$$

Thus, we again confirm that the two linearly independent solutions are:

$$y_1(x) = 1 - \frac{2}{x} \quad \text{and} \quad y_2(x) = \left(1 + \frac{2}{x}\right)e^{-x}\,. \tag{63}$$

11 Boas, problem p.606, 12.21-9

Solve the differential equation by Frobenius' method and then find the second solution using Fuchs's theorem:

$$x^2y'' + (x^2 - 3x)y' + (4 - 2x)y = 0. \tag{64}$$

We write the solution as a series $\sum_{n=0} a_n x^{n+s}$; the equation becomes

$$\sum (n+s)(n+s-1)a_n x^{n+s} + \sum (n+s)(a_n x^{n+s+1} - 3a_n x^{n+s}) + \sum (4a_n x^{n+s} - 2a_n x^{n+s+1}) = 0 \tag{65}$$

$$\sum x^{n+s}\Big[(n+s)(n+s-1)a_n + (n-1+s)a_{n-1} - 3(n+s)a_n + 4a_n - 2a_{n-1}\Big] = 0 \tag{66}$$

for $n = 0$ we find the indicial equation

$$s(s-1) - 3s + 4 = 0 \implies s^2 - 4s + 4 = 0 \implies s = 2 \tag{67}$$

The series starts with a x^2 term; for $s = 2$, the relation between the a_n's becomes

$$[(n+2)(n+1) - 3(n+2) + 4]a_n + [(n+1) - 2]a_{n-1} = 0 \implies n^2a_n + (n-1)a_n = 0 \tag{68}$$

for $n = 1$, we have $a_1 = 0$, implying $a_n = 0\ \forall n \geq 1$. The first solution is then

$$S_1(x) = a_0x^2 \tag{69}$$

Because the roots of the indicial equation coincide, by Fuchs's theorem we expect the other solution to be of the form

$$y(x) = S_1(x) \ln x + S_2(x). \tag{70}$$

We now want to find the unknown series $S_2(x) = \sum b_n x^n$: plugging (70) in (64),

$$y' = S_1' \ln x + S_1 \frac{1}{x} + S_2', \qquad y'' = S_1'' \ln x + \frac{2}{x} S_1' - S_1 \frac{1}{x^2} + S_2'' \tag{71}$$

$$2xS_1' - S_1 + (x-3)S_1 + x^2 S_2'' + (x^2 - 3x)S_2' + (4-2x)S_2 = 0. \tag{72}$$

$$\sum x^n \left[n(n-1)b_n + (n-1)b_{n-1} - 3nb_n + 4b_n - 2b_{n-1} \right] = a_0 x^2 - 4a_0 x^2 - a_0(x^3 - 3x^2) = -a_0 x^3$$

Here we find the relation

$$(n^2 - 4n + 4)b_n + (n-3)b_{n-1} = \begin{cases} 0, \ n > 3 \\ -a_0, \ n = 3 \end{cases}, \tag{73}$$

We can rewrite the relation as $b_n = -\frac{n-3}{(n-2)^2} b_{n-1} = \frac{(n-3)(n-4)}{(n-2)^2(n-3)^2} b_{n-2} = \ldots$. The first terms of the series are

$$S_2(x) = -a_0 x^3 + a_0 \frac{1}{4} x^4 - a_0 \frac{1}{3 \cdot 3 \cdot 2} x^5 + a_0 \frac{1}{4 \cdot 4 \cdot 3 \cdot 2} x^6 + \cdots \tag{74}$$

Then the general solution to (64) is a linear combination of the two particular solutions:

$$y(x) = Ax^2 + B \left[x^2 \ln x - x^3 + \frac{1}{2 \cdot 2!} x^4 - \frac{1}{3 \cdot 3!} x^5 + \frac{1}{4 \cdot 4!} x^6 + \ldots \right] \tag{75}$$

12 Boas, problem p.618, 12.23-26

Verify *Bauer's formula* $e^{ixw} = \sum_0^\infty (2l+1) i^l j_l(x) P_l(w)$.

We can write the Legendre series, $e^{ixw} = \sum_{l=0}^\infty c_l P_l(w)$. The coefficients c_l are given by

$$\int_{-1}^1 dw\, e^{ixw} P_l(w) = \sum_m c_m \int_{-1}^1 dw P_l(w) P_m(w) = \frac{2}{2l+1} c_l \implies c_l = \frac{2l+1}{2} \int_{-1}^1 dw\, e^{ixw} P_l(w) \tag{76}$$

We can see c_l as a function of x, $y(x) = c_l$. Then we can find y' and y'':

$$y'(x) = \frac{2l+1}{2} \int_{-1}^1 dw\, (iw) e^{ixw} P_l(w), \qquad y''(x) = \frac{2l+1}{2} \int_{-1}^1 dw\, (iw)^2 e^{ixw} P_l(w) \tag{77}$$

so that they satisfy spherical Bessel's equation

$$x^2 y'' + 2xy' + [x^2 - l(l+1)]y = 0 : \tag{78}$$

$$\int_{-1}^1 dw\, e^{ixw} P_l(w)[-x^2 w^2 + 2ixw + (x^2 - l(l+1))] = \int_{-1}^1 dw\, e^{ixw} P_l(w)[x^2(1-w^2) + 2ixw - l(l+1)]$$

We look at the first two terms in this expression:

$$\int_{-1}^1 dw\, [x^2(1-w^2) + 2ixw] P_l e^{ixw} = -\int_{-1}^1 dw\, (1-w^2) P_l(w) \frac{d^2}{dw^2} e^{ixw} + \int_{-1}^1 dw\, 2ixw P_l e^{ixw} =$$

$$= -(1-w^2) P_l(w) \frac{d}{dw} e^{ixw} \Big|_{-1}^1 + \int_{-1}^1 dw\, (-2wP_l + (1-w^2)P_l') \frac{d}{dw} e^{ixw} + \int_{-1}^1 dw\, 2ixw P_l e^{ixw} = \tag{79}$$

$$= \int_{-1}^1 dw\, (-2ixw) P_l(w) e^{ixw} + \int_{-1}^1 dw\, (1-w^2) P_l' \frac{d}{dw} e^{ixw} + \int_{-1}^1 dw\, 2ixw P_l e^{ixw} = \tag{80}$$

$$= (1-w^2) P_l(w) e^{ixw} \Big|_{-1}^1 - \int_{-1}^1 dw\, [(1-w^2)P_l']' e^{ixw} \tag{81}$$

So that now equation (78) reads

$$\int_{-1}^{1} dw\, e^{ixw}\left[\frac{d}{dw}[(1-w^2)P_l'] + l(l+1)P_l(w)\right] = 0 \tag{82}$$

where the expression is zero because we have Legendre's equation in the brackets. Because c_l satisfies the spherical Bessel equation, it is given by a linear combination of the Bessel functions:

$$c_l = Aj_l(x) + Bn_l(x) \tag{83}$$

We can compute the $c_l(x)$ integral for small values of x by expanding e^{ixw}:

$$c_l(x) = \frac{2l+1}{2}\int_{-1}^{1}\left(\sum_{n=0}^{\infty}\frac{(ixw)^n}{n!}P_l(w)\right)dw = \frac{2l+1}{2}\frac{i^l x^l}{l!}\int_{-1}^{1} w^l P_l(w) + \mathcal{O}(x^{l+2}) \tag{84}$$

To calculate the integral, we use Rodrigues' formula for the Legendre polynomials:

$$\int_{-1}^{1} dw\, w^l P_l(w) = \int_{-1}^{1} dw\, w^l \frac{1}{2^l l!}\frac{d^l}{dw^l}(w^2-1)^l = (-1)^l\frac{1}{2^l}\int_{-1}^{1} dw (w^2-1)^l = \frac{1}{2^l}\frac{\sqrt{\pi}\Gamma(l+1)}{\Gamma(l+\frac{3}{2})} = \frac{2\cdot l!}{(2l+1)!!} \tag{85}$$

where we integrated by parts as we did in Homework set #1, Problem 9. So we found that

$$c_l(x) = (2l+1)i^l\frac{x^l}{(2l+1)!!} + \mathcal{O}(x^{l+2}) \tag{86}$$

but this is also the expansion of $j_l(x)$ for small x; thus, c_l is not singular at the origin and can be written in terms of $j_l(x)$. Putting everything together, we have found that

$$e^{ixw} = \sum_{l=0}^{\infty} c_l P_l(w) = \sum_{l=0}^{\infty}(2l+1)i^l j_l(x)P_l(w). \tag{87}$$