Assignment 8 (MATH 215, Q1)

1. Evaluate the surface integral $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ for the given vector field $\mathbf{F}$ and the oriented surface S. In other words, find the *flux* of $\mathbf{F}$ across S.

   (a) $\mathbf{F}(x,y,z) = xy\,\mathbf{i}+yz\,\mathbf{j}+zx\,\mathbf{k}$, S is the part of the paraboloid $z=4-x^2-y^2$ that lies above the square $-1\le x\le 1$, $-1\le y\le 1$, and has the upward orientation.

   Solution. The surface S can be represented by the vector form

$$\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + (4-x^2-y^2)\,\mathbf{k}, \quad -1\le x\le 1, -1\le y\le 1.$$

It follows that $\mathbf{r}_x = \mathbf{i} - 2x\,\mathbf{k}$ and $\mathbf{r}_y = \mathbf{j} - 2y\,\mathbf{k}$. Consequently,

$$\mathbf{r}_x \times \mathbf{r}_y = 2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k}.$$

Hence, with $Q := \{(x,y) : -1\le x\le 1, -1\le y\le 1\}$ we obtain

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_Q \mathbf{F}\cdot(\mathbf{r}_x\times\mathbf{r}_y)\,dA \\
&= \int_{-1}^{1}\int_{-1}^{1} \left[2x^2y + 2y^2(4-x^2-y^2) + x(4-x^2-y^2)\right]dx\,dy \\
&= \int_{-1}^{1}\left(\frac{4}{3}\,y + 16y^2 - \frac{4}{3}\,y^2 - 4y^4\right)dy \\
&= \left[\frac{2}{3}\,y^2 + \frac{44}{3}\,\frac{y^3}{3} - 4\,\frac{y^5}{5}\right]_{-1}^{1} = \frac{368}{45}.
\end{aligned}$$

   (b) $\mathbf{F}(x,y,z) = -y\,\mathbf{i}+x\,\mathbf{j}+3z\,\mathbf{k}$, S is the hemisphere $z=\sqrt{16-x^2-y^2}$ with upward orientation.

   Solution. The surface S has parametric equations

$$\mathbf{r}(\phi,\theta) = x(\phi,\theta)\,\mathbf{i} + y(\phi,\theta)\,\mathbf{j} + z(\phi,\theta)\,\mathbf{k} = 4\sin\phi\cos\theta\,\mathbf{i} + 4\sin\phi\sin\theta\,\mathbf{j} + 4\cos\phi\,\mathbf{k},$$

where $0\le\phi\le\pi/2$, $0\le\theta\le 2\pi$. We have

$$\mathbf{r}_\phi\times\mathbf{r}_\theta = 16\sin\phi(\sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}).$$

Moreover,

$$\mathbf{F} = -4\sin\phi\sin\theta\,\mathbf{i} + 4\sin\phi\cos\theta\,\mathbf{j} + 12\cos\phi\,\mathbf{k}.$$

Consequently,

$$\mathbf{F}\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta) = 192\sin\phi\cos^2\phi.$$

Therefore, with $Q := \{(\phi, \theta) : 0 \le \phi \le \pi/2, 0 \le \theta \le 2\pi\}$ we obtain

$$\begin{aligned}\iint_S \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_Q \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\, dA \\ &= \int_0^{2\pi} \int_0^{\pi/2} 192 \sin\phi \cos^2\phi \, d\phi\, d\theta \\ &= 2\pi \cdot 192 \Big[-\frac{\cos^3\phi}{3} \Big]_0^{\pi/2} = 128\pi.\end{aligned}$$

2. Let S be the conical surface $z = \sqrt{x^2 + y^2}$, $z \le 2$.

(a) Find the center of mass of S, if it has constant density.

Solution. The surface has parametric equations

$$x = z\cos t, \quad y = z \sin t, \quad z = z, \qquad (t, z) \in Q,$$

where $Q := \{(t, z) : 0 \le t \le 2\pi, 0 \le z \le 2\}$. Let $\mathbf{r}(t, z) := z\cos t\, \mathbf{i} + z\sin t\, \mathbf{j} + z\, \mathbf{k}$. Then

$$\mathbf{r}_t \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -z\sin t & z\cos t & 0 \\ \cos t & \sin t & 1 \end{vmatrix} = z\cos t\, \mathbf{i} + z\sin t\, \mathbf{j} - z\, \mathbf{k}.$$

Note that $\mathbf{r}_t \times \mathbf{r}_z$ gives the *downward* orientation. Moreover,

$$|\mathbf{r}_t \times \mathbf{r}_z| = \sqrt{(z\cos t)^2 + (z\sin t)^2 + (-z)^2} = \sqrt{2}\, z.$$

Suppose the density is k. Then $M = \iint_S k\, dS$ and $M_{xy} = \iint_S kz\, dS$. The center of mass is $(0, 0, \bar{z})$, where $\bar{z} = M_{xy}/M$. We have

$$M = \iint_S k\, dS = k \iint_Q |\mathbf{r}_t \times \mathbf{r}_z|\, dA = k \int_0^{2\pi} \int_0^2 \sqrt{2}\, z\, dz\, dt = 4\sqrt{2}\,\pi k.$$

Moreover,

$$M_{xy} = \iint_S kz\, dS = k \iint_Q z|\mathbf{r}_t \times \mathbf{r}_z|\, dA = k \int_0^{2\pi} \int_0^2 z\sqrt{2}\, z\, dz\, dt = \frac{16\sqrt{2}\,\pi k}{3}.$$

Therefore, the center of mass is $(0, 0, 4/3)$.

(b) A fluid has density 15 and velocity $\mathbf{v} = x\,\mathbf{i} + y\,\mathbf{j} + \mathbf{k}$. Find the rate of flow **downward** through S.

Solution. We have

$$\mathbf{F} = \rho\mathbf{v} = 15(x\,\mathbf{i} + y\,\mathbf{j} + \mathbf{k}) = 15(z\cos t\, \mathbf{i} + z\sin t\, \mathbf{j} + \mathbf{k}).$$

Consequently,

$$\mathbf{F}\cdot(\mathbf{r}_t\times\mathbf{r}_z)=15(z^2\cos^2 t+z^2\sin^2 t-z)=15(z^2-z).$$

Hence, the rate of flow downward through S is

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_Q \mathbf{F}\cdot(\mathbf{r}_t\times\mathbf{r}_z)\,dA\\ &= 15\int_0^{2\pi}\int_0^2 (z^2-z)\,dz\,dt\\ &= 30\pi\left[\frac{z^3}{3}-\frac{z^2}{2}\right]_0^2=20\pi.\end{aligned}$$

3. Use the divergence theorem to find $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$.
   (a) $\mathbf{F}(x,y,z)=x^3\,\mathbf{i}+2xz^2\,\mathbf{j}+3y^2z\,\mathbf{k}$; S is the surface of the solid bounded by the paraboloid $z=4-x^2-y^2$ and the xy-plane.

Solution. The divergence of $\mathbf{F}$ is

$$\operatorname{div}\mathbf{F}=\frac{\partial}{\partial x}(x^3)+\frac{\partial}{\partial y}(2xz^2)+\frac{\partial}{\partial z}(3y^2z)=3x^2+3y^2.$$

Let E be the region $\{(x,y,z):0\le z\le 4-x^2-y^2\}$. By the divergence theorem, we have

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iiint_E \operatorname{div}\mathbf{F}\,dV=\iiint_E (3x^2+3y^2)\,dV\\ &= \int_0^{2\pi}\int_0^2\int_0^{4-r^2} 3r^2r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^2 (4-r^2)3r^3\,dr\,d\theta\\ &= 2\pi\int_0^2 (12r^3-3r^5)\,dr=32\pi.\end{aligned}$$

   (b) $\mathbf{F}(x,y,z)=(x^2+\sin(yz))\,\mathbf{i}+(y-xe^{-z})\,\mathbf{j}+z^2\,\mathbf{k}$; S is the surface of the region bounded by the cylinder $x^2+y^2=4$ and the planes $x+z=2$ and $z=0$.

Solution. The divergence of $\mathbf{F}$ is

$$\operatorname{div}\mathbf{F}=\frac{\partial}{\partial x}(x^2+\sin(yz))+\frac{\partial}{\partial y}(y-xe^{-z})+\frac{\partial}{\partial z}(z^2)=2x+1+2z.$$

Let E be the region $\{(x,y,z):0\le z\le 2-x, x^2+y^2\le 4\}$. By the divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS=\iiint_E \operatorname{div}\mathbf{F}\,dV=\iiint_E (2x+1+2z)\,dV.$$

Converting to cylindrical coordinates, we obtain

$$\iiint_E (2x+1+2z)\,dV = \int_0^{2\pi}\int_0^2\int_0^{2-r\cos\theta} (2r\cos\theta+1+2z)r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 (-r^2\cos^2\theta - r\cos\theta + 6)r\,dr\,d\theta = \int_0^{2\pi} (-4\cos^2\theta - 8\cos\theta/3 + 12)\,d\theta$$
$$= 20\pi.$$

4. Use the divergence theorem to calculate $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, where

$$\mathbf{F}(x,y,z) = z^2x\,\mathbf{i} + (y^3/3+\tan z)\,\mathbf{j} + (x^2z+y^2)\,\mathbf{k}$$

and S is the top half of the sphere $x^2+y^2+z^2=1$ oriented upward. (Hint: Note that S is not a closed surface. Let S_1 be the disk $\{(x,y,0) : x^2+y^2\le 1\}$ oriented downward and let $S_2 = S\cup S_1$. The surface integral over S can be derived from integrals over S_1 and S_2.)

Solution. Let E be the semi-ball $\{(x,y,z) : x^2+y^2+z^2\le 1, z\ge 0\}$. Then S_2 is the boundary of E. Hence, the divergence theorem applies to the surface integral $\iint_{S_2}\mathbf{F}\cdot\mathbf{n}\,dS$:

$$\iint_{S_2}\mathbf{F}\cdot\mathbf{n}\,dS = \iiint_E \operatorname{div}\mathbf{F}\,dV.$$

The divergence of $\mathbf{F}$ is given by

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(z^2x) + \frac{\partial}{\partial y}(y^3/3+\tan z) + \frac{\partial}{\partial z}(x^2z+y^2) = z^2+y^2+x^2.$$

Hence, we obtain

$$\iint_{S_2}\mathbf{F}\cdot\mathbf{n}\,dS = \iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E (x^2+y^2+z^2)\,dV.$$

The triple integral can be calculated by using the spherical coordinates:

$$\iiint_E (x^2+y^2+z^2)\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta = \frac{2\pi}{5}.$$

For the surface integral $\iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS$ we note that $\mathbf{n}=-\mathbf{k}$ and $z=0$ on S_1. It follows that $\mathbf{F}\cdot\mathbf{n} = -y^2$. Consequently,

$$\iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{x^2+y^2\le 1} -y^2\,dA = \int_0^{2\pi}\int_0^1 -(r^2\sin^2\theta)r\,dr\,d\theta = -\frac{\pi}{4}.$$

Therefore,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS - \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS = \frac{2\pi}{5} - \frac{-\pi}{4} = \frac{13\pi}{20}.$$

5. Use Stokes' theorem to evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$. In each case C is oriented counterclockwise as viewed from above.
   (a) $\mathbf{F}(x,y,z) = z^2\,\mathbf{i} + y^2\,\mathbf{j} + xy\,\mathbf{k}$, C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,2)$.

Solution. The curl of $\mathbf{F}$ is

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & xy \end{vmatrix} = x\,\mathbf{i} + (2z - y)\,\mathbf{j}.$$

The plane that passes through the points $(1,0,0)$, $(0,1,0)$, and $(0,0,2)$ has an equation $z = 2 - 2x - 2y$. Hence, $\mathbf{r}_x \times \mathbf{r}_y = 2\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$. By Stokes' theorem we obtain

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_Q \operatorname{curl}\mathbf{F}\cdot(\mathbf{r}_x \times \mathbf{r}_y)\,dA,$$

where

$$Q = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1 - x\}.$$

Consequently,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1\int_0^{1-x} \big(2x + 4(2 - 2x - 2y) - 2y\big)\,dy\,dx \\ &= \int_0^1\int_0^{1-x} (-6x - 10y + 8)\,dy\,dx = \int_0^1 (x^2 - 4x + 3)\,dx = 4/3.\end{aligned}$$

   (b) $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + (x^2 + y^2)\,\mathbf{k}$, C is the boundary of the part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant.

Solution. The curl of $\mathbf{F}$ is

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & x^2 + y^2 \end{vmatrix} = 2y\,\mathbf{i} - 2x\,\mathbf{j}.$$

The surface S can be represented as $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + (1 - x^2 - y^2)\,\mathbf{k}$, $x \ge 0$, $y \ge 0$, $x^2 + y^2 \le 1$. It follows that

$$\mathbf{r}_x \times \mathbf{r}_y = 2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k}.$$

Consequently,

$$\operatorname{curl}\mathbf{F}\cdot(\mathbf{r}_x \times \mathbf{r}_y) = 4xy - 4xy = 0.$$

Therefore, by Stokes's theorem, we have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\operatorname{curl}\mathbf{F}\cdot\mathbf{n})\,dS = 0.$$