Binomial Matrices and Discrete Taylor Series

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Abstract

Every $s \times s$ matrix A yields a composition map acting on polynomials on $\mathbb{R}^s$, mapping $p(x)$ to $p(Ax)$. For each n, the polynomials of degree n form an invariant subspace for this map. Its matrix representation on this subspace relative to the monomial basis gives a matrix that we denote by $A^{(n)}$ and call a *binomial matrix*. This paper deals with the asymptotic behavior of $A^{(n)}$ as $n \to \infty$. The special case of 2×2 matrices A with the property that $A^2 = I$ corresponds to *discrete Taylor series* and motivated our original interest in binomial matrices.

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§1. Introduction

This paper deals with the following matrix theoretic construction. We begin with an $s \times s$ matrix A. On the space $\Pi(\mathbb{R}^s)$ of all polynomials on $\mathbb{R}^s$, we consider the composition map C_A defined by

$$(C_A p)(x) = p(Ax), \qquad x \in \mathbb{R}^s, \qquad p \in \Pi(\mathbb{R}^s).$$

For each $n \in \mathbb{Z}_+$, the space $H_n(\mathbb{R}^s)$ of homogeneous polynomials of degree n on $\mathbb{R}^s$ is an invariant subspace of C_A. So there is a $d \times d$ matrix $A^{(n)}$ with

$$d = \dim H_n(\mathbb{R}^s) = \binom{n+s-1}{s-1}$$

which represents C_A relative to the basis for $H_n(\mathbb{R}^s)$ given by the monomials

$$m_\alpha(x) = x^\alpha, \qquad \alpha \in \Gamma_n, \qquad x \in \mathbb{R}^s.$$

As usual $x = (x_1, \ldots, x_s)$, $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{Z}_+^s$, $|\alpha| = \alpha_1 + \cdots + \alpha_s$, $x^\alpha = x_1^{\alpha_1} \cdots x_s^{\alpha_s}$ and $\Gamma_n := \{\alpha : \ |\alpha| = n\}$. We refer to this matrix $A^{(n)}$ of order d as a *binomial matrix.*

Our interest in this construction started with the special case $s = 2$, $A = B$ where

$$B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The matrices $B^{(n)}$ appeared in the work of Haddad [6]. They came from rescaling the rows and columns of *Hermite matrices*, which were the object of study in [1, 6, 7]. The columns of $B^{(n)}$ have the appearance of a *discrete Taylor series* and therefore we were led to study their asymptotic behavior as $n \to \infty$. It is precisely this issue that concerns us here. Our goal is to study the properties and obtain the asymptotic behavior of binomial matrices $A^{(n)}$ for an *arbitrary* $s \times s$ matrix A.

Binomial matrices also appeared in the study of N-widths of Hilbert spaces of holomorphic functions on unitarily invariant domains in $\mathbb{C}^s$. (We would not be surprised by earlier appearances.) The eigenvalues of a compact integral operator identify the N-widths, and invariance reduces the operator to the composition map C_A for an appropriate matrix A, [8]. In that context, it sufficed to know that when A is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_s$, the eigenvalues of $A^{(n)}$ are the products λ^α, $|\alpha| = n$, with $\lambda = (\lambda_1, \ldots, \lambda_s)$.

Later binomial matrices made their presence known in subdivision. A modification of the de Casteljau subdivision for Bernstein-Bézier curves led to the binomial matrices corresponding to

$$\begin{bmatrix} 1 & 0 \\ 1-x & x \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1-x & x \\ 0 & 1 \end{bmatrix}$$

$x \in [0,1]$. This is described in equation (1.71) of [9]. In a subsequent study of the algebraic properties of B-patches as well as the analysis of B-patch subdivision, binomial matrices played a central role, [4]. For these reasons they are prominent in [9] where their relationship to "blossoming" is explained. One attractive property provides a formula for the elements of $A^{(n)}$ as the *permanent* of a certain matrix. To obtain $A^{(n)}_{\alpha\beta}$ with $|\alpha| = |\beta| = n$, we form an $n \times n$ matrix M from A by first repeating its i-th row α_i times, $i = 1, 2, \ldots, s$, and then the j-th column of this $n \times s$ matrix β_j times, $j = 1, 2, \ldots, s$. We then have the formula

$$A^{(n)}_{\alpha\beta} = \frac{1}{\beta!} per\, M.$$

See p. 28 and Theorem 1.3 of [9]. Alternative formulas shall be provided here.

Quite recently binomial matrices were also shown to play a key role in questions related to multidimensional refinement equations as they arise in wavelet analysis, [2, 3].

§2. A Motivating Example

To motivate our line of investigation we begin with the 2×2 matrix

$$B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{2.1}$$

The jth row of the corresponding binomial matrices $B^{(n)} = (B^{(n)}_{jk})$, $j, k = 0, 1, \ldots, n$, is defined by

$$\sum_{k=0}^{n} B^{(n)}_{jk} x_1^{n-k} x_2^k = (x_1 + x_2)^{n-j}(x_1 - x_2)^j$$

or equivalently by the equation

$$\sum_{k=0}^{n} B^{(n)}_{jk} x^k = (1 - x)^j (1 + x)^{n-j}. \tag{2.2}$$

These equations yield the special cases

$$B^{(1)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad B^{(2)} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}, \quad B^{(3)} = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{bmatrix}.$$

In general, we think of the j-th row of $B^{(n)}$ as a j-th order difference operator. For example,

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_0 + 2x_1 + x_2 \\ x_0 - x_2 \\ x_0 - 2x_1 + x_2 \end{bmatrix}.$$

Looking at the columns of the matrices we see *discrete polynomials.* The k-th column gives the values at the integers $0, 1, ..., n$ of a polynomial of degree k. For the matrix $B^{(2)}$, these polynomials are $Q_{02}(x) = 1$, $Q_{12}(x) = 2 - 2x$ and $Q_{22}(x) = 1 - 4x + 2x^2$. The fact that the j-th difference of a k-th degree polynomial vanishes for $j > k$ confirms that the squares of these matrices $B^{(n)}$ are zero below the main diagonal. They are also zero above the main diagonal! This follows from the fact that $B^2 = 2I$, and an exact form of the discrete polynomials Q_{kn} is given in (2.5).

The matrices $B^{(n)}$ have an especially attractive LDU factorization. For example, in the 3×3 case the factors are

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

In general, we consider the $(n+1) \times (n+1)$ lower triangular Pascal matrix L defined by

$$L_{jk} = \binom{j}{k}, \qquad j, k = 0, 1, \ldots, n,$$

and the upper triangular matrix U obtained by permuting the rows and columns of L:

$$U_{jk} = \binom{n-j}{n-k}, \qquad j, k = 0, 1, \ldots, n.$$

Theorem 2.1. *Let D be the $(n+1) \times (n+1)$ diagonal matrix diag$\{1, -2, \ldots, (-2)^n\}$. There holds the formula*

$$B^{(n)} = LDU. \tag{2.3}$$

Proof. For $j, k = 0, 1, \ldots, n$, we have that

$$\left(LDU\right)_{jk} = \sum_{l=0}^{n} \binom{j}{l} (-2)^l \binom{n-l}{n-k}$$

and thus

$$\sum_{k=0}^{n} \left(LDU\right)_{jk} x^k = \sum_{k=0}^{n} \sum_{l=0}^{n} \binom{j}{l} (-2)^l \binom{n-l}{n-k} x^k.$$

Interchanging the indices k and l establishes that

$$\begin{aligned} \sum_{k=0}^{n} \left(LDU\right)_{jk} x^k &= \sum_{l=0}^{j} \binom{j}{l} (-2)^l \{\sum_{k=l}^{n} \binom{n-l}{n-k} x^{k-l}\} x^l \\ &= \sum_{l=0}^{j} \binom{j}{l} (-2x)^l (1+x)^{n-l} \\ &= (1-x)^j (1+x)^{n-j}, \end{aligned}$$

thereby confirming Theorem 2.1. $\square$

We inject here some general remarks about binomial matrices. The mapping $A \longrightarrow A^{(n)}$ is a homomorphism, that is, for any $s \times s$ matrices A and N we have that

$$(AN)^{(n)} = A^{(n)} N^{(n)}.$$

This fact implies several results. For example, knowing the LDU factorization of a matrix A will yield the factorization for $A^{(n)}$ (with due attention to the partial order of Γ_n). Thus, noting that

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

will give (2.3). Also, the Jordan form of $A^{(n)}$ can be obtained from that for A, see [2, 3, 4, 9] for details. Furthermore,

$$(A^{(n)})^{-1} = (A^{-1})^{(n)}$$

if A is invertible. Whenever $A^2 = cI$ for some constant c then $(A^{(n)})^2 = c^n I$. We do not elaborate on these interesting issues as our main concern here is with the behavior of $A^{(n)}$ as $n \to \infty$.

The eigenvalues of B are $-\sqrt{2}$ and $\sqrt{2}$. From our general remark about the eigenvalues of $A^{(n)}$, we conclude that the eigenvalues of $B^{(n)}$ are $(\sqrt{2})^k(-\sqrt{2})^{n-k}$, $k = 0, 1, \ldots, n$.

The asymptotic behavior of the binomial matrix $B^{(n)}$ as $n \to \infty$ is more challenging. Let us begin this analysis with some useful formulas. For $j, k = 0, 1, \ldots, n$, we see from equation (2.2) that

$$B_{jk}^{(n)} = \frac{1}{k!} p_{jn}^{(k)}(0) = \frac{(-2n)^k}{k!} Q_{kn}(j/n) \tag{2.4}$$

where we define polynomials p_{jn} and Q_{kn} by

$$p_{jn}(x) = (1-x)^j (1+x)^{n-j},$$

and

$$Q_{kn}(x) = (-2)^{-k} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \Pi_{p=0}^{i-1}(x - p/n) \Pi_{q=0}^{k-i-1}(1 - x - q/n). \tag{2.5}$$

It follows from equation (2.4) that

$$B_{kj}^{(n)} = \sum_{\max\{0, j+k-n\} \le i \le \min\{j,k\}} \binom{k}{i} (-1)^i \binom{n-k}{j-i}. \tag{2.6}$$

For a fixed $k \in \mathbb{Z}_+$, we derive from equation (2.5) that

$$\lim_{n \to \infty} Q_{kn}(x) = (x - 1/2)^k \tag{2.7}$$

uniformly in x on every bounded subinterval of $\mathbb{R}$. Therefore we conclude for any $k \in \mathbb{Z}_+$ that

$$\lim_{n \to \infty} \max_{0 \le j \le n} \left| \frac{k!}{(-2n)^k} B_{jk}^{(n)} - \left(\frac{j}{n} - \frac{1}{2}\right)^k \right| = 0. \tag{2.8}$$

Motivated by our viewpoint that the $B^{(n)}$ acts as a difference operator we will express in an alternate manner the asymptotic behavior of $B^{(n)}$. To this end, we define for $k = 0, 1, \ldots, n$, a linear functional on $C[0,1]$ by setting

$$T_{kn} f = \sum_{j=0}^{n} B_{kj}^{(n)} f\left(\frac{j}{n}\right). \tag{2.9}$$

Recall that the modulus of continuity $\omega(f,h)$ of a function $f \in C[0,1]$ is defined as

$$\omega(f,h) := \sup\{|f(x+t)-f(x)| : |t| \le h, x, x+t \in [0,1]\}.$$

Theorem 2.2. *Let $k \in \mathbb{Z}_+$, $f \in C^k[0,1]$ and $n \ge k$. We have that*

$$|(-1)^k n^k 2^{k-n} T_{kn}(f) - f^{(k)}(1/2)| \le \frac{3}{2}\omega(f^{(k)}, 1/\sqrt{n}).$$

Proof. The first step in the proof is to introduce the positive linear functional

$$L_{kn}(f) = \frac{n^k}{2^{n-k}} \sum_{j=0}^{n-k} \binom{n-k}{j} \int_0^{1/n} \cdots \int_0^{1/n} f(\frac{j}{n} + t_1 + \cdots + t_k) dt_1 \cdots dt_k.$$

We claim for $f \in C^k[0,1]$ that

$$(-1)^k n^k 2^{k-n} T_{kn} f = L_{kn} f^{(k)}. \tag{2.10}$$

We give two proofs for this equation. The first follows by substituting equation (2.6) directly into the definition of T_{kn}. Specifically, we have that

$$\begin{aligned} T_{kn}(f) &= \sum_{i=0}^{k} \binom{k}{i} (-1)^i \sum_{j=i}^{i+n-k} \binom{n-k}{j-i} f(\frac{j}{n}) \\ &= \sum_{j=0}^{n-k} \binom{n-k}{j} \Big\{ \sum_{i=0}^{k} \binom{k}{i} (-1)^i f(\frac{j}{n} + \frac{i}{n}) \Big\}. \end{aligned}$$

Next, we recall the formula for divided difference $\Delta_h^k f(x) = \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} f(x+ih)$. Writing this formula as a multidimensional integral (2.10) follows.

The second proof of the equation (2.10) proceeds differently. For every real ρ, we define the function $f_\rho(x) = e^{\rho x}$. Their linear combinations are dense in $C^k[0,1]$ as ρ varies over $\mathbb{R}$. Thus it suffices to verify (2.10) for $f = f_\rho$. A direct computation will confirm this fact, starting from

$$L_{kn}(f_\rho) = \frac{n^k}{2^{n-k}} (1+e^{\rho/n})^{n-k} \left(\frac{e^{\rho/n}-1}{\rho} \right)^k. \tag{2.11}$$

To prove the theorem, we consider the monomials $m_0 = 1, m_1 = x$, and $m_2 = x^2$. Differentiating and evaluating (2.11) at $\rho = 0$ gives the equations

$$L_{kn} m_0 = 1, \qquad L_{kn} m_1 = m_1(1/2) \tag{2.12}$$

and

$$L_{kn} m_2 = m_2(1/2) + \frac{3n-2k}{12n^2}. \tag{2.13}$$

It is a standard fact that for any nonnegative linear functional L on $C[0,1]$, whose value is $f(1/2)$ for *every* linear function f, there holds for any $\delta > 0$

$$|Lf - f(1/2)| \leq \left(1 + \frac{\sqrt{\rho}}{\delta}\right) \omega(f;\delta) \tag{2.14}$$

where

$$\rho = Lm_2 - m_2(1/2).$$

According to (2.13) for $L = L_{kn}$ we have $4\rho \leq n^{-2}$ and so choosing $\delta = n^{-1/2}$ in (2.14) proves Theorem 2.2. $\square$

§3. Asymptotic Behavior of Binomial Matrices

In this section we study the asymptotic behavior of the matrices $A^{(n)}$ as $n \to \infty$ for an $s \times s$ matrix A. Specifically, we consider the sequence of linear functionals

$$T_{\alpha n} f = \sum_{\beta \in \Gamma_n} A^{(n)}_{\alpha\beta} f(\beta/n)$$

for any $f \in C(\Delta_s)$ where Δ_s is the simplex

$$\Delta_s = \{x : x \cdot e = 1,\ x \in \mathbb{R}^s_+\}$$

and $e = (1,1,\ldots,1) \in \mathbb{R}^s$.

Definition 3.1. *Let $x = (x_1, x_2, \ldots, x_s) \in \mathbb{R}^s$, $s > 1$ and $k \in \mathbb{Z}_+$. The* **moments** *of x are defined by*

$$I_{k\alpha}(x) := \sum_{\beta \in \Gamma_k} \binom{k}{\beta} x^\beta \beta^\alpha, \qquad \alpha \in \mathbb{Z}^s_+,$$

where for $\beta = (\beta_1, \beta_2, \ldots, \beta_s)$, the multinomials with $s > 1$ are given by

$$\binom{k}{\beta} = \frac{k!}{\beta!},$$

with $\beta! := \beta_1!\beta_2! \cdots \beta_s!$. (When $s = 2$, $\binom{k}{\beta}$ is denoted by $\binom{k}{\beta_1}$ or $\binom{k}{\beta_2}$).

Note by the multinomial theorem that $I_{k0}(x) = (x \cdot e)^k$ which is nonzero for $x \cdot e \neq 0$. On the contrary, when $x \cdot e = 0$ we have that $I_{k\alpha}(x) = 0$ whenever $|\alpha| < k$ and $I_{k\alpha}(x) = k! x^\alpha$ for $|\alpha| = k$. To see this we compute for $y \in \mathbb{R}^s$

$$\sum_{\alpha \in \mathbb{Z}^s_+} \frac{I_{k\alpha}(x)}{\alpha!} y^\alpha = (x \cdot e^y)^k = (x \cdot y + o(y))^k, \qquad y \to 0.$$

Recall the standard notation D for the *vector* of first derivatives and the notation $D^\alpha = \Pi(\partial/\partial x_i)^{\alpha_i}$ for the α_i-derivative in the x_i-variable of a function, $i = 1, 2, \ldots, s$. Also, $D_y = y \cdot D$ is the derivative in the "direction" y and for any finite subset K of $\mathbb{R}^s$ we let

$$D_K = \Pi_{y \in K} D_y$$

(empty products are set equal to I).

We denote the row vectors of A as $a^1, a^2, \ldots, a^s$. In what follows it will be important to identify the indices in $G = \{j : 1 < j \leq s\}$ for which $a^j \cdot e = 0$:

$$J = \{j : a^j \cdot e = 0,\ 1 < j \leq s\}.$$

We use the following notational convention. When $s > 1$ and $\gamma = (\gamma_2, \ldots, \gamma_s) \in \mathbb{Z}_+^{s-1}$ with $|\gamma| \leq n$ we define $\alpha \in \Gamma_s$ uniquely by setting $\alpha_i = \gamma_i$, $1 < i \leq s$, denote $T_{\alpha n} f$ by $T_{\gamma n} f$ and set $|\gamma|_A := \sum_{j \in J} \gamma_j$ (empty sums are set equal to zero). From the vector γ and the matrix A we form the *set* H^γ consisting of the vectors a^ℓ, $\ell \in J$ each *repeated* γ_ℓ-times and the *constant*

$$\mu_\gamma = \Pi_{\ell \in G \setminus J} (a^\ell \cdot e)^{\gamma_\ell}.$$

Theorem 3.2. *If A is an $s \times s$ matrix, $s > 1$, such that $a^1 \cdot e \neq 0$, $\gamma \in \mathbb{Z}_+^{s-1}$ and $f \in C^{|\gamma|}(\Delta_s)$, then*

$$\lim_{n \to \infty} \frac{n^{|\gamma|_A}}{(a^1 \cdot e)^{n-|\gamma|}} T_{\gamma n} f = \mu_\gamma (D_{H^\gamma} f) \left(\frac{a^1}{a^1 \cdot e} \right).$$

For the proof of this result we first recall from the definition of $A^{(n)}$ that

$$A^{(n)}_{\alpha\beta} = \frac{1}{\beta!} D^\beta \left(C_A(m_\alpha) \right)(0).$$

To evaluate this derivative we use the following fact

Lemma 3.3 [5]. *Let p be a polynomial and $f \in C^\infty(\mathbb{R}^s)$. For vectors $x^1, \ldots, x^s \in \mathbb{R}^s$ we have that*

$$p(D)\{f(x^1 \cdot x, \ldots, x^s \cdot x)\} = \sum_{\alpha \in \mathbb{Z}_+^s} \frac{(D_{K^\alpha} p)(0)}{\alpha!} (D^\alpha f)(x^1 \cdot x, \ldots, x^s \cdot x), \tag{3.1}$$

where K^α is the set

$$K^\alpha := \{\underbrace{x^1, \ldots, x^1}_{\alpha_1}, \ldots, \underbrace{x^s, \ldots, x^s}_{\alpha_s}\}, \qquad \alpha = (\alpha_1, \ldots, \alpha_s).$$

Proof of Theorem 3.2. We always choose n such that $|\gamma| < n$ and apply Lemma 3.3 to the functions $f = m_\alpha$ and $p = m_\beta$ to obtain that

$$A^{(n)}_{\alpha\beta} = \frac{1}{\beta!} D^\beta \left(C_A(m_\alpha) \right)(0) = \frac{1}{\beta!} D_{J^\alpha}(m_\beta)(0)$$

where J^α is the set

$$J^\alpha := \{\underbrace{a^1,\ldots,a^1}_{\alpha_1},\ldots,\underbrace{a^s,\ldots,a^s}_{\alpha_s}\}.$$

Since

$$\begin{aligned} D_{J^\alpha} &= (a^1\cdot D)^{\alpha_1}\cdots(a^s\cdot D)^{\alpha_s} \\ &= \sum_{|\sigma^1|=\alpha_1}\cdots\sum_{|\sigma^s|=\alpha_s}\Pi_{j=1}^s\left[\binom{\alpha_j}{\sigma^j}(a^j)^{\sigma_j}\right]D^{\sigma^1+\cdots+\sigma^s}\end{aligned}$$

we have that

$$A_{\alpha\beta}^{(n)} = \sum_{|\sigma^2|=\alpha_2}\cdots\sum_{|\sigma^s|=\alpha_s}\binom{\alpha_1}{\beta-\sum_{j=2}^s\sigma^j}\Pi_{j=2}^s\left[\binom{\alpha_j}{\sigma^j}(a^j)^{\sigma^j}\right](a^1)^{\beta-\sum_{j=2}^s\sigma^j}$$

and so

$$\begin{aligned} T_{\gamma n}f = &\sum_{|\sigma^1|=n-|\gamma|}\binom{n-|\gamma|}{\sigma^1}(a^1)^{\sigma^1} \\ &\sum_{|\sigma^2|=\gamma_2}\cdots\sum_{|\sigma^s|=\gamma_s}\Pi_{j=2}^s\left[\binom{\gamma_j}{\sigma^j}(a^j)^{\sigma^j}\right]f\left(\frac{\sum_{j=1}^s\sigma^j}{n}\right).\end{aligned}$$

Our hypothesis on f ensures that

$$f\left(\frac{\sum_{j=1}^s\sigma^j}{n}\right) = \sum_{p=0}^{|\gamma|}\frac{1}{p!}\left[\left(\frac{\sum_{j=2}^s\sigma^j}{n}\cdot D\right)^p f\right]\left(\frac{\sigma^1}{n}\right)+o(n^{-|\gamma|}), \qquad n\to\infty \tag{3.2}$$

uniformly for $\alpha\in\Gamma_n$, $\sigma^j\in\Gamma_{\alpha_j}$, $j=1,2,\ldots,s$.

Let us first consider the case that $J=\emptyset$. In this case we conclude that

$$(a^1\cdot e)^{|\gamma|-n}T_{\gamma n}f = \mu_\gamma(B_{n-|\gamma|}f)\left(\frac{a^1}{a^1\cdot e}\right)+o(1), \qquad n\to\infty$$

uniformly on the simplex Δ_s where

$$B_nf := \sum_{\beta\in\Gamma_n} f\left(\frac{\beta}{n}\right)\binom{n}{\beta}m_\beta$$

is the n-th degree Bernstein polynomial of f on Δ_s. We know for any $f\in C(\Delta_s)$ that

$$B_nf = f+o(1)$$

uniformly on Δ_s. This fact allows us to conclude that the result is valid when $J=\emptyset$.

For $J\neq\emptyset$, we have to deal with the fact that

$$\Pi_{j\in G}(a^j\cdot e)^{\gamma_j}=0.$$

To this end, using (3.2) and the equation

$$\frac{1}{p!}\left(\sum_{j=2}^{s}\sigma^{j}\cdot D\right)^{p}=\sum_{|\theta^{2}|+\cdots+|\theta^{s}|=p}\Pi_{j=2}^{s}\frac{\left(\sigma^{j}\right)^{\theta^{j}}D^{\theta^{j}}}{\theta^{j}!},$$

with care, we conclude that

$$T_{\gamma n}f=\sum_{|\sigma|=n-|\gamma|}\binom{n-|\gamma|}{\sigma}(a^{1})^{\sigma}$$
$$\left\{\sum_{p=1}^{|\gamma|}n^{-p}\sum_{|\theta^{2}|+\cdots+|\theta^{s}|=p}\Pi_{j=2}^{s}\frac{I_{\gamma_{j}\theta^{j}}(a^{j})}{\theta^{j}!}D^{\theta^{j}}f\left(\frac{\sigma}{n}\right)\right. \tag{3.3}$$
$$\left.+\,o(n^{-|\gamma|})\right\},\qquad n\to\infty.$$

To simplify the right-hand side of (3.3) we observe, if $\ell\in J$ and $|\theta^{\ell}|<\gamma_{\ell}$ then $I_{\gamma_{\ell}\theta^{\ell}}(a^{\ell})=0$. Therefore, if

$$\Pi_{j=2}^{s}\frac{I_{\gamma_{j}\theta^{j}}(a^{j})}{\theta^{j}!}\neq 0 \tag{3.4}$$

then for *all* $\ell\in J$ we obtain $|\theta^{\ell}|\geq\gamma_{\ell}$. In particular, (3.4) implies that

$$\sum_{\ell\in J}|\theta^{\ell}|\geq|\gamma|_{A}.$$

But the range of summation in (3.3) requires that

$$|\gamma|\geq p\geq\sum_{\ell=2}^{p}|\theta^{\ell}|\geq|\gamma|_{A}.$$

Thus the summands in (3.3) corresponding to $p=|\gamma|_{A}$ require $\theta^{\ell}=0$ for $\ell\notin J$ and $|\theta^{\ell}|=\gamma_{\ell}$ for $\ell\in J$. These facts give the asymptotic formula

$$T_{\gamma n}f=\frac{(a^{1}\cdot e)^{n-|\gamma|}}{n^{|\gamma|_{A}}}\mu_{\gamma}\left\{(B_{n-\gamma}D_{H^{\gamma}}f)\left(\frac{a^{1}}{a^{1}\cdot e}\right)+o(1)\right\},\qquad n\to\infty$$

which proves the result. □

We specialize this result to the case considered in Theorem 2.2. For the matrix B in (2.1) we have that $e\cdot b^{1}=2$, $e\cdot b^{2}=0$. Therefore, we conclude that $G=J=\{2\}$ and $\mu_{\gamma}=1$. Following our notational convention $A_{\alpha\beta}^{(n)}$ in this case is identified with $B_{kj}^{(n)}$ where $\alpha=(n-k,k)$ and $\beta=(n-j,j)$. Thus, for the function f defined for $(x_{1},x_{2})\in\Delta_{2}$ by $f(x_{1},x_{2})=g(x_{2})$ where g is defined on $[0,1]$ we see that $T_{\alpha n}f=T_{kn}g$. Hence, Theorem 3.2 says, when $g\in C^{k}[0,1]$ that

$$\lim_{n\to\infty}\frac{n^{k}}{2^{n-k}}T_{kn}g=\sum_{p=0}^{k}\frac{(-1)^{k-p}k!}{p!(k-p)!}\left(\frac{\partial^{k}}{\partial x_{1}^{p}\partial x_{2}^{k-p}}f\right)\left(\frac{1}{2},\frac{1}{2}\right)=(-1)^{k}g^{(k)}\left(\frac{1}{2}\right).$$

This limit also follows directly from Theorem 2.2.

Corollary 3.4. *If $A = (A_{ij})$, $i, j = 1, 2, \dots, s$, is a matrix such that for some constant $c \neq 0$, $A^2 = cI$, $A_{i1} = 1$, $1 \leq i \leq s$ and $f \in C^{|\gamma|}(\Delta_s)$ where $\gamma = (\gamma_2, \dots, \gamma_s) \in \mathbb{Z}_+^{s-1}$, then*

$$\lim_{n\to\infty} c^{-n+|\gamma|} n^{|\gamma|} T_{\gamma n} f = D_{H^\gamma} f\left(\frac{a^1}{c}\right)$$

where

$$H^\gamma := \{\underbrace{a^2, \dots, a^2}_{\gamma_2}, \dots, \underbrace{a^s, \dots, a^s}_{\gamma_s}\}.$$

Proof. Our hypothesis implies that $a^i \cdot e = c\delta_{1j}$, $j = 1, 2, \dots, s$. Thus $G = J$, $\mu_\gamma = 1$ and the result follows from Theorem 3.2. $\square$

The additional complexity of the multivariate case conceals from us an error bound of the type presented in Theorem 2.2. An improvement of Theorem 3.2 would be desirable.

Our intention now is to provide a multivariate version of equations (2.7) and (2.8). First, we define the limit polynomial. Starting with an $s \times s$ matrix A such that $\Pi_{j=1}^s A_{j1} \neq 0$ and a $\gamma \in \mathbb{Z}_+^{s-1}$ we introduce the vectors

$$\widehat{a}^j = A_{j1}^{-1}(A_{j2}, \dots, A_{js}), \qquad j = 1, 2, \dots, s$$

in $\mathbb{R}^{s-1}$ and define a polynomial Q_γ in $H_{|\gamma|}(\mathbb{R}^s)$ by setting for any $x = (x_1, x_2, \dots, x_s) \in \mathbb{R}^s$

$$Q_\gamma(x) = \frac{1}{\gamma!}\left(\sum_{j=1}^n x_j \widehat{a}^j\right)^\gamma .$$

Analogous to our notational convention in Theorem 3.2 we extend every $\gamma \in \mathbb{Z}_+^{s-1}$, $|\gamma| \leq n$ uniquely to a $\beta \in \Gamma_n$ and then denote $A_{\alpha\beta}^{(n)}$ by $A_{\alpha\gamma}^{(n)}$.

Theorem 3.5. *Let A be an $s \times s$ matrix, $s > 1$, such that $\Pi_{j=1}^s A_{j1} \neq 0$. For every $\gamma \in \mathbb{Z}_+^{s-1}$ we have that*

$$\lim_{n\to\infty} \max_{\alpha\in\Gamma_n} \left| n^{-|\gamma|} \rho_\alpha A_{\alpha\gamma}^{(n)} - Q_\gamma(\alpha/n) \right| = 0$$

where

$$\rho_\alpha := \Pi_{j=1}^s A_{j1}^{-\alpha_j}.$$

Proof. For the proof of this result we develop yet another formula for the entries of the matrix $A^{(n)}$. Let $\delta = (1, 0, \dots, 0) \in \mathbb{Z}_+^s$ and $\alpha, \beta \in \Gamma_n$. Use the definition of $A^{(n)}$ and the product rule for differentiation to obtain

$$\begin{aligned} \rho_\alpha A_{\alpha\beta}^{(n)} &= \frac{\rho_\alpha}{\beta!} D^\beta \{C_A(m_\alpha)\}(\delta) \\ &= \sum_{\sigma^1+\dots+\sigma^s=\beta} \Pi_{j=1}^s \frac{1}{\sigma^j!}\left(\frac{a^j}{A_{j1}}\right)^{\sigma^j} \Pi_{k=0}^{|\sigma^j|-1}(\alpha_j - k). \end{aligned} \tag{3.5}$$

To write this sum in the desired form we note the following identity. If $i_1, i_2, \ldots, i_s$, i, j are nonnegative integers with $i = i_1 + \cdots + i_s$, then

$$\sum_{j_1+\cdots+j_s=j} \binom{i_1}{j_1} \cdots \binom{i_s}{j_s} = \binom{i}{j}. \tag{3.6}$$

One way to prove this is to check that the sequences appearing in the left and right sides of (3.6) have the same generating function $(1+t)^i$. Now, with (3.6) in hand we return to (3.5) and sum over the *first* coordinates of the vectors $\sigma^1, \ldots, \sigma^s$ and then over the remaining $s-1$ coordinates. Since the first coordinate of the vector $A_{j1}^{-1} a^j$ is 1, the sum over the first coordinates of $\sigma^1, \ldots, \sigma^s$ is, by specializing (3.6), seen to be 1. Hence we conclude that

$$\rho_\alpha n^{-|\gamma|} A_{\alpha\beta}^{(n)} = \sum_{\mu^1+\cdots+\mu^s=\gamma} \Pi_{j=1}^s \frac{1}{\mu^j!} \left(\widehat{a}^j\right)^{\mu^j} \Pi_{k=0}^{|\mu^j|-1} \left(\frac{\alpha_j}{n} - \frac{k}{n}\right).$$

Since we have that

$$Q_\gamma(x) = \sum_{\mu^1+\cdots+\mu^s=\gamma} \Pi_{j=1}^s \frac{1}{\mu^j!} \left(\widehat{a}^j\right)^{\mu^j} x_j^{|\mu^j|}$$

the result follows. $\square$

Comparing Theorem 3.5 to (2.7) and (2.8) we see, at least in this case, that the polynomial Q_γ is a *shifted* monomial. It turns out for $s > 2$ that such a matrix A is unique.

Proposition 3.6. *If A is an $s \times s$ matrix, $s > 1$, such that $A^2 = cI$ for some constant $c \neq 0$, $A_{j1} = 1$, $j = 1, 2, \ldots, s$ and there is an $a \in \mathbb{R}^{s-1}$ such that for any $\gamma \in \mathbb{Z}_+^{s-1}$ there is a $c_\gamma \neq 0$ such that $Q_\gamma = c_\gamma m_\gamma(\cdot - a)$ on Δ_s then either $s = 2$ or*

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & -1 \end{bmatrix}. \tag{3.7}$$

Proof. Our requirement on A means for every $k = 1, 2, \ldots, s$ that there are nonzero constants c_k such that for all $(x_1, x_2, \ldots, x_s) \in \mathbb{R}^s$,

$$\sum_{j=1}^s A_{jk} x_j = c_k \left(x_k - a_k \sum_{j=1}^s x_j \right) \qquad \text{for} \quad k = 2, \ldots, s.$$

Thus we conclude for $j = 1, 2, \ldots, s$ and $k = 2, \ldots, s$ that

$$A_{kk} = c_k(1 - a_k), \qquad A_{jk} = -c_k a_k, \qquad j \neq k.$$

We now use the fact that the matrix $E := A^2 - cI$ is zero. The entry $E_{11} = 0$ tells us that $1 - \sum_{k=2}^{s} c_k a_k = c$, while for $j = 1, 2, \ldots, s$, the entry $E_{j1} = 0$ implies that $1 - \sum_{k=2}^{s} c_k a_k + c_j = 0$. Hence for $j = 2, \ldots, s$, $c_j = -c$ and $\sum_{j=2}^{s} a_j = (c-1)/c$. Likewise for $j = 2, \ldots s$ the entry $E_{jj} = 0$ gives the equation $c_j a_j = 1 - c$ which implies $a_j = (c-1)/c$. Therefore we conclude that either $s = 2$ or $c = 1$. In the latter case, A takes the form (3.7). This proves the result. $\square$

Specializing Corollary 3.4 to the matrix appearing in (3.7) we see that $J = G$ and $\mu_\gamma = 1$. Given the function f defined on Δ_s we define F by setting, for $(x_1, x_2, \ldots, x_s) \in \Delta_s$

$$F(x_2, \ldots, x_s) = f(x_1, x_2, \ldots, x_s).$$

Therefore

$$D_{H^\gamma} f(\delta) = (-1)^{|\gamma|}(D^\gamma F)(0)$$

and whenever $f \in C^{|\gamma|}(\Delta_s)$ we have (for the matrix in (3.7)) that

$$\lim_{n\to\infty} n^{|\gamma|}(-1)^{|\gamma|} T_{\gamma n} f = (D^\gamma f)(0)$$

which, in the spirit of the paper, is a "discrete Taylor series".

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