LEGENDRE POLYNOMIALS AND APPLICATIONS

We construct Legendre polynomials and apply them to solve Dirichlet problems in spherical coordinates.

1. Legendre equation: series solutions

The Legendre equation is the second order differential equation

$$(1-x^2)y'' - 2xy' + \lambda y = 0 \tag{1}$$

which can also be written in self-adjoint form as

$$\left[(1-x^2)y'\right]' + \lambda y = 0\ . \tag{2}$$

This equation has regular singular points at $x = -1$ and at $x = 1$ while $x = 0$ is an ordinary point. We can find power series solutions centered at $x = 0$ (the radius of convergence will be at least 1). Now we construct such series solutions.

Assume that

$$y = \sum_{j=0}^{\infty} c_j x^j$$

be a solution of (1). By substituting such a series in (1), we get

$$(1-x^2)\sum_{j=2}^{\infty} j(j-1)c_j x^{j-2} - 2x\sum_{j=1}^{\infty} jc_j x^{j-1} + \lambda\sum_{j=0}^{\infty} c_j x^j = 0$$

$$\sum_{j=2}^{\infty} j(j-1)c_j x^{j-2} - \sum_{j=2}^{\infty} j(j-1)c_j x^j - \sum_{j=1}^{\infty} 2jc_j x^j + \lambda\sum_{j=0}^{\infty} c_j x^j = 0$$

After re-indexing the first series and grouping the other series, we get

$$\sum_{j=0}^{\infty}(j+2)(j+1)c_{j+2}x^j - \sum_{j=0}^{\infty}(j^2+j-\lambda)c_j x^j = 0$$

and then

$$\sum_{j=0}^{\infty}\left[(j+2)(j+1)c_{j+2} - (j^2+j-\lambda)c_j\right]x^j = 0\ .$$

By equating each coefficient to 0, we obtain the recurrence relations

$$c_{j+2} = \frac{(j+1)j-\lambda}{(j+2)(j+1)}c_j\ , \qquad j = 0,\ 1,\ 2,\ \cdots$$

We can obtain two independent solutions as follows. For the first solution we make $c_0 \neq 0$ and $c_1 = 0$. In this case the recurrence relation gives

$$c_3 = \frac{2-\lambda}{6}c_1 = 0,\quad c_5 = \frac{12-\lambda}{20}c_3 = 0,\quad \cdots, c_{\text{odd}} = 0\ .$$

The coefficients with even index can be written in terms of c_0:

$$c_2 = \frac{-\lambda}{2}c_0,\quad c_4 = \frac{(2\cdot 3-\lambda)}{4\cdot 3}c_2 = \frac{(2\cdot 3-\lambda)(-\lambda)}{4!}c_0$$

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we prove by induction that

$$c_{2k} = \frac{[(2k-1)(2k-2)-\lambda][(2k-3)(2k-4)-\lambda]\cdots[3\cdot 2-\lambda][-\lambda]}{(2k)!}c_0$$

We can write this in compact form as

$$c_{2k} = \frac{c_0}{(2k)!}\left(\prod_{i=1}^{k-1}[(2i+1)(2i)-\lambda]\right)$$

This give a solution

$$y_1(x) = c_0\sum_{k=0}^{\infty}\left(\prod_{i=0}^{k-1}[(2i+1)(2i)-\lambda]\right)\frac{x^{2k}}{(2k)!}$$

A second series solution (independent from the first) can be obtained by making $c_0 = 0$ and $c_1 \neq 0$. In this case $c_{\text{even}} = 0$ and

$$c_{2k+1} = \frac{c_1}{(2k+1)!}\left(\prod_{i=1}^{k}[2i(2i-1)-\lambda]\right)$$

The corresponding solution is

$$y_2(x) = c_1\sum_{k=0}^{\infty}\left(\prod_{i=1}^{k}[(2i)(2i-1)-\lambda]\right)\frac{x^{2k+1}}{(2k+1)!}$$

Remark 1. It can be proved by using the ratio test that the series defining y_1 and y_2 converge on the interval $(-1,\ 1)$ (check this as an exercise).
2. It is also proved that for every λ either y_1 or y_2 is unbounded on $(-1,\ 1)$. That is, as $x \to 1$ or as $x \to -1$, one of the following holds, either $|y_1(x)| \to \infty$ or $|y_2(x)| \to \infty$.
3. The only case in which Legendre equation has a bounded solution on $[-1,\ 1]$ is when the parameter λ has the form $\lambda = n(n+1)$ with $n = 0$ or $n \in \mathbb{Z}^+$. In this case either y_1 or y_2 is a polynomial (the series terminates). This case is considered below.

2. Legendre polynomials

Consider the following problem
Problem. Find the parameters $\lambda \in \mathbb{R}$ so that the Legendre equation

$$\left[(1-x^2)y'\right]' + \lambda y = 0, \quad -1 \le x \le 1\ . \tag{3}$$

has a bounded solution.

This is a singular Sturm-Liouville problem. It is singular because the function $(1-x^2)$ equals 0 when $x = \pm 1$. For such a problem, we don't need boundary conditions. The boundary conditions are replaced by the boundedness of the solution. As was pointed out in the above remark, the only values of λ for which we have bounded solutions are $\lambda = n(n+1)$ with $n = 0,\ 1,\ 2, \cdots$. These values of λ are the eigenvalues of the SL-problem.

To understand why this is so, we go back to the construction of the series solutions and look again at the recurrence relations giving the coefficients

$$c_{j+2} = \frac{(j+1)j-\lambda}{(j+2)(j+1)}c_j\,, \qquad j = 0,\ 1,\ 2,\ \cdots$$

If $\lambda = n(n+1)$, then

$$c_{n+2} = \frac{(n+1)n - \lambda}{(n+2)(n+1)} c_n = 0.$$

By repeating the argument, we get $c_{n+4} = 0$ and in general $c_{n+2k} = 0$ for $k \geq 1$.

This means

- if $n = 2p$ (even), the series for y_1 terminates at c_{2p} and y_1 is a polynomial of degree $2p$. The series for y_2 is infinite and has radius of convergence equal to 1 and y_2 is unbounded.
- If $n = 2p+1$ (odd), then the series for y_2 terminates at c_{2p+1} and y_2 is a polynomial of degree $2p+1$ while the solution y_1 is unbounded.

For $\lambda = n(n+1)$, we can rewrite the recurrence relation for a polynomial solution in terms of c_n. We have,

$$c_j = \frac{(j+2)(j+1)}{j(j+1) - n(n+1)} c_{j+2} = -\frac{(j+2)(j+1)}{(n-j)(n+j+1)} c_{j+2},$$

for $j = n-2,\ n-4,\ \cdots 1$ or 0. Equivalently,

$$c_{n-2k} = -\frac{(n-2k+2)(n-2k+1)}{(2k)(2n-2k+1)} c_{n-2k+2}, \quad k = 1,\ 2,\ \cdots,\ [n/2].$$

I will leave it as an exercise to verify the following formula for c_{n-2k} in terms of c_n:

$$c_{n-2k} = \frac{(-1)^k}{2^k\, k!} \frac{n(n-1)\cdots(n-2k+1)}{(2n-1)(2n-3)\cdots(2n-2k+1)} c_n\ .$$

The polynomial solution is therefore

$$y(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k}{2^k\, k!} \frac{n(n-1)\cdots(n-2k+1)}{(2n-1)(2n-3)\cdots(2n-2k+1)} c_n x^{n-2k}$$

where c_n is an arbitrary constant. The n-th *Legendre polynomial* $P_n(x)$ is the above polynomial of degree n for the particular value of c_n

$$c_n = \frac{(2n)!}{2^n (n!)^2}\ .$$

This particular value of c_n is chosen to make $P_n(1) = 1$. We have then (after simplification)

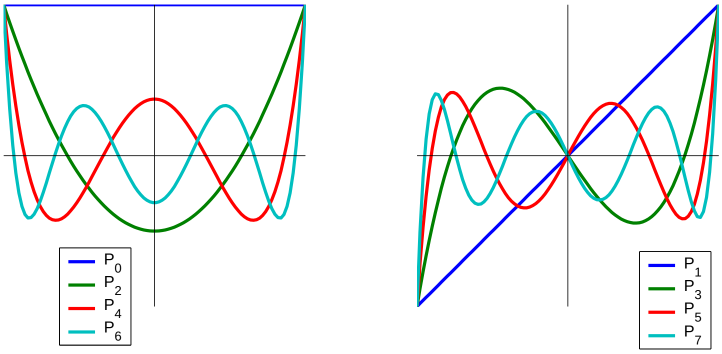
$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k}\ .$$

Note that if n is even (resp. odd), then the only powers of x involved in P_n are even (resp. odd) and so P_n is an even (resp. odd).

The first six Legendre polynomials are.

$$P_0(x) = 1 \qquad P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1) \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \qquad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

We have the following proposition.



Proposition. *If $y(x)$ is a bounded solution on the interval $(-1,\ 1)$ of the Legendre equation (1) with $\lambda = n(n+1)$, then there exists a constant K such that*

$$y(x) = KP_n(x)$$

where P_n is the n-th Legendre polynomial.

Remark. When $\lambda = n(n+1)$ a second solution of the Legendre equation, independent from P_n, can be found in the form

$$Q_n(x) = \frac{1}{2} P_n(x) \ln \frac{1+x}{1-x} + R_n(x)$$

where R_n is a polynomial of degree $n-1$. The construction of Q_n can be achieved by the method of reduction of order. Note that $|Q_n(x)| \to \infty$ as $x \to \pm 1$. The general solution of the Legendre equation is then

$$y(x) = AP_n(x) + B_n Q_n(x)$$

and such a function is bounded on the interval $(-1,\ 1)$ if and only if $B = 0$.

3. Rodrigues' formula

The Legendre polynomials can be expressed in a more compact form.

Theorem 1. (Rodrigues' Formula) *The n-th Legendre polynomial P_n is given by the following*

$$P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n} \left[(x^2-1)^n \right] \tag{4}$$

(thus expression (4) gives a solution of (3) with $\lambda = n(n+1)$).

Proof. Let $y = (x^2-1)^n$. We have following

Claim. The k-th derivative $y^{(k)}(x)$ of y satisfies the following:

$$(1-x^2)\frac{d^2 y^{(k)}}{dx^2} + 2(n-k-1)x\frac{dy^{(k)}}{dx} + (2n-k)(k+1)y^{(k)} = 0. \tag{5}$$

Proof of the Claim. By induction. For $k = 0$, $y = y^{(0)}$. We have

$$y' = 2nx(x^2-1)^{n-1} \ \Rightarrow\ (1-x^2)y' + 2nxy = 0$$

and after differentiation, we get

$$(1-x^2)y'' + 2(n-1)xy' + 2ny = 0$$

So formula (5) holds when $k = 0$. By induction suppose the (5) holds up to order $k-1$. We can rewrite (5) for $k-1$ as

$$(1-x^2)\frac{dy^{(k)}}{dx} + 2(n-k)xy^{(k)} + (2n-k+1)ky^{(k-1)} = 0.$$

We differentiate to obtain

$$(1-x^2)\frac{d^2y^{(k)}}{dx^2} + [2(n-k)-2]x\frac{dy^{(k)}}{dx} + [2(n-k)+k(2n-k+1)]y^{(k)} = 0.$$

which is precisely (5).

Now if we let $k = n$ in (5), we obtain

$$(1-x^2)\frac{d^2y^{(n)}}{dx^2} - 2x\frac{dy^{(n)}}{dx} + n(n+1)y^{(k)} = 0.$$

Hence $y^{(n)}$ solves the Legendre equation with $\lambda = n(n+1)$. Since $y^{(n)}$ is a polynomial of degree $2n$, then by Proposition 1, it is a multiple of P_n. There is a constant K such that $P_n(x) = Ky^{(n)}(x)$. To complete the proof, we need to find K. For this notice that the coefficient of x^n in P_n is $(2n)!/(2^n\,(n!)^2)$. The coefficient of x^n in $y^{(n)}$ is that of

$$\frac{d^n(x^{2n})}{dx^n} = (2n)(2n-1)\cdots(2n-n+1)x^n = \frac{(2n)!}{n!}x^n$$

Hence

$$K\frac{(2n)!}{n!} = \frac{(2n)!}{2^n\,(n!)^2} \Rightarrow K = \frac{1}{2^n\,n!}\ .$$

This completes the proof of the Rodrigues' formula.

A consequence of this formula is the following property between three consecutive Legendre polynomials.

Proposition. *The Legendre polynomials satisfy the following*

$$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x) \tag{6}$$

Proof. From Rodrigues' formula we have

$$\begin{aligned} P'_k(x) &= \frac{d}{dx}\left(\frac{1}{2^k\,k!}\frac{d^k}{dx^k}[(x^2-1)^k]\right) = \frac{2k}{2^k\,k!}\frac{d^k}{dx^k}[x(x^2-1)^{k-1}] \\ &= \frac{1}{2^{k-1}\,(k-1)!}\frac{d^{k-1}}{dx^{k-1}}\left[((2k-1)x^2-1)(x^2-1)^{k-2}\right] \end{aligned}$$

For $k = n+1$, we get

$$P'_{n+1}(x) = \frac{1}{2^n\,n!}\frac{d^n}{dx^n}\left[((2n+1)x^2-1)(x^2-1)^{n-1}\right]$$

From Rodrigues' formula at $n-1$, we get

$$P'_{n-1}(x) = \frac{d}{dx}\left(\frac{1}{2^{n-1}(n-1)!}\frac{d^{n-1}}{dx^{n-1}}[(x^2-1)^{n-1}]\right) = \frac{2n}{2^n\,n!}\frac{d^n}{dx^n}[(x^2-1)^{n-1}]$$

As a consequence, we have

$$P'_{n+1}(x) - P'_{n-1}(x) = \frac{2n+1}{2^n\,n!}\frac{d^n}{dx^n}[(x^2-1)^n] = (2n+1)P_n(x)\ .$$

Generating function. It can be shown that the Legendre polynomials are generated by the function

$$g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} \; .$$

More precisely, if we extend $g(x,t)$ as a Taylor series in t, then the coefficient of t^n is the polynomial $P_n(x)$:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n \; . \tag{7}$$

A consequence of (7) is the following relation between three consecutive Legendre polynomials.

Proposition. *The Legendre polynomials satisfy the following*

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \tag{8}$$

Proof. We differentiate (7) with respect to t:

$$\frac{(x-t)}{\sqrt{1-2xt+t^2}^{\,3}} = \sum_{n=1}^{\infty} nP_n(x)t^{n-1} \; .$$

We multiply by $1-2xt+t^2$ and use (7)

$$\sum_{n=0}^{\infty}(x-t)P_n(x)t^n = \sum_{n=1}^{\infty}(1-2xt+t^2)nP_n(x)t^{n-1} \; .$$

Equivalently,

$$\sum_{n=0}^{\infty} xP_n(x)t^n - \sum_{n=0}^{\infty} P_n(x)t^{n+1} = \sum_{n=1}^{\infty} nP_n(x)t^{n-1} - \sum_{n=1}^{\infty} 2nxP_n(x)t^n + \sum_{n=1}^{\infty} nP_n(x)t^{n+1}$$

and after grouping the series

$$xP_0(x) - P_1(x) + \sum_{n=1}^{\infty}\left[(2n+1)xP_n(x) - (n+1)P_{n+1}(x) - nP_{n-1}(x)\right] t^n$$

Property (8) is obtained by equating to 0 the coefficient of t^n.

4. Orthogonality of Legendre polynomials

When the Legendre equation is considered as a (singular) Sturm-Liouville problem on $[-1,\ 1]$, we get the following orthogonality theorem

Theorem 2. *Consider the singular SL-problem*

$$(1-x^2)y'' - 2xy' + \lambda y = 0 \qquad -1 < x < 1 \, ,$$

with y bounded on $(-1,\ 1)$. The eigenvalues are $\lambda_n = n(n+1)$ with corresponding eigenfunctions $P_n(x)$. Furthermore, the eigenfunctions corresponding to distinct eigenvalues are orthogonal. That is

$$< P_n(x), P_m(x) >= \int_{-1}^{1} P_n(x)P_m(x)dx = 0 \, , \qquad n \neq m \; . \tag{9}$$

Proof. Recall that the self-adjoint form of the Legendre equation is

$$[(1-x^2)y']' + \lambda y = 0 \ ,$$

(with $p(x) = 1 - x^2$, $r(x) = 1$, and $q(x) = 0$. The corresponding weight function is $r = 1$. We have already seen that the eigenvalues and eigenfunctions are given by $\lambda_n = n(n+1)$ and $P_n(x)$. We are left to verify the orthogonality. We write the Legendre equation for P_m and P_n:

$$\lambda_n P_n(x) = [(x^2-1)P_n'(x)]'$$
$$\lambda_m P_m(x) = [(x^2-1)P_m'(x)]'$$

Multiply the first equation by P_m, the second by P_n and subtract. We get,

$$\begin{aligned}(\lambda_n - \lambda_m)P_n(x)P_m(x) &= [(x^2-1)P_n'(x)]'P_m(x) - [(x^2-1)P_m'(x)]'P_n(x) \\ &= \left[(x^2-1)P_n'(x)P_m(x) - (x^2-1)P_m'(x)P_n(x)\right]' \\ &= \left[(x^2-1)(P_n'(x)P_m(x) - P_m'(x)P_n(x))\right]'\end{aligned}$$

Integrate from -1 to 1

$$(\lambda_n - \lambda_m)\int_{-1}^{1} P_n(x)P_m(x)dx = \left[(x^2-1)(P_n'(x)P_m(x) - P_m'(x)P_n(x))\right]_{-1}^{1} = 0.$$

The square norms of the Legendre polynomials are given below.

Theorem 3. *We have the following*

$$||P_n(x)||^2 = \int_{-1}^{1} P_n(x)^2 dx = \frac{2}{2n+1} \tag{10}$$

Proof. We use generating function (7) to get

$$\frac{1}{1-2xt+t^2} = \left(\sum_{n=0}^{\infty} P_n(x)t^n\right)^2 = \sum_{n,m\geq 0} P_n(x)P_m(x)t^{n+m}$$

Now we integrate from -1 to 1:

$$\int_{-1}^{1} \frac{dx}{1-2xt+t^2} = \sum_{n,m\geq 0}\left(\int_{-1}^{1} P_n(x)P_m(x)dx\right)t^{n+m}$$

By using the orthogonality of P_n and P_m (for $n \neq m$), we get

$$\frac{-1}{2t}\left[\ln|1-2xt+t^2|\right]_{x=-1}^{x=1} = \sum_{n=0}^{\infty} ||P_n(x)||^2 t^{2n} \ ,$$

and after simplifying the left side:

$$\frac{1}{t}\ln\left|\frac{1+t}{1-t}\right| = \sum_{n=0}^{\infty} ||P_n(x)||^2 t^{2n} \ .$$

Recall that for $|s| < 1$, the Taylor series of $\ln(1+s)$ is

$$\ln(1+s) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} s^j.$$

Hence for $|t| < 1$, we have

$$\begin{aligned}
\frac{1}{t}\ln\left|\frac{1+t}{1-t}\right| &= \frac{1}{t}\left(\ln(1+t) - \ln(1-t)\right) \\
&= \frac{1}{t}\left(\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}t^j - \sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}(-t)^j\right) \\
&= \frac{1}{t}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}+1}{j}t^j \\
&= \sum_{n=0}^{\infty}\frac{2}{2n+1}t^{2n}
\end{aligned}$$

It follows that

$$\sum_{n=0}^{\infty}\frac{2}{2n+1}t^{2n} = \sum_{n=0}^{\infty}||P_n(x)||^2 t^{2n} \ .$$

An identification of the coefficient of t^{2n} gives (10).

5. Legendre series

The collection of Legendre polynomials $\{P_n(x)\}_{n\geq 0}$ forms a complete family in the space $C_p^1[-1,\ 1]$ of piecewise smooth functions on the interval $[-1,\ 1]$. Any piecewise function f has then a generalized Fourier series representation in terms of these polynomials. The associated series is called the *Legendre series* of f. Hence,

$$f(x) \sim \sum_{n=0}^{\infty} c_n P_n(x)$$

where

$$c_n = \frac{< f(x), P_n(x) >}{||P_n(x)||^2} = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx$$

Theorem 4. *Let f be a piecewise smooth function on $[-1,\ 1]$. Then,*

$$f_{av}(x) = \frac{f(x^+) + f(x^-)}{2} = \sum_{n=0}^{\infty} c_n P_n(x) \ .$$

In particular at the points x, where f is continuous, we have

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x) \ .$$

Remark 1. For each m, the Legendre polynomials $P_0,\ P_1,\ \cdots, P_m$ forms a basis in the space of polynomials of degree m. Thus, if $R(x)$ is a polynomial of degree m, then the Legendre series of R terminates at the order m (i.e. $c_k = 0$ for $k > m$). In particular,

$$x^m = c_0P_0(x) + c_1P_1(x) + \cdots + c_mP_m(x) = c_0 + c_1x + \frac{c_2}{2}(3x^2-1) + \cdots$$

For example,

$$\begin{aligned}
x^2 &= \frac{2}{3}P_0(x) + 0P_1(x) + \frac{2}{3}P_2(x) \\
x^3 &= 0P_0(x) + \frac{3}{5}P_1(x) + 0P_2(x) + \frac{2}{5}P_3(x)
\end{aligned}$$

Remark 2. Suppose that f is an odd function. Since P_n is odd when n is odd and P_n is even when n is even, then the Legendre coefficients of f with even indices are all zero ($c_{2j} = 0$). The Legendre series of f contains only odd indexed polynomials. That is,

$$f_{av}(x) = \sum_{j=0}^{\infty} c_{2j+1} P_{2j+1}(x)$$

where

$$c_{2j+1} = (2(2j+1)+1) \int_0^1 f(x) P_{2j+1}(x) dx = (4j+3) \int_0^1 f(x) P_{2j+1}(x) dx.$$

Similarly, if f is an even function, then its Legendre series contains only even indexed polynomials.

$$f_{av}(x) = \sum_{j=0}^{\infty} c_{2j} P_{2j}(x)$$

where

$$c_{2j} = (2(2j)+1) \int_0^1 f(x) P_{2j}(x) dx = (4j+1) \int_0^1 f(x) P_{2j}(x) dx.$$

If a function f is defined on the interval $[0,\ 1]$, then we can extend it as an even function f_{even} to the interval $[-1,\ 1]$. The Legendre series of f_{even} contains only even-indexed polynomials. Similarly, if we extend f as an odd function f_{odd} to $[-1,\ 1]$, then the Legendre series contains only odd-indexed polynomials. We have the following theorem.

Theorem 5. *Let f be a piecewise smooth function on $[0,\ 1]$. Then, f has an expansion into even Legendre polynomials*

$$f_{av}(x) = \frac{f(x^+) + f(x^-)}{2} = \sum_{j=0}^{\infty} c_{2j} P_{2j}(x)\ .$$

Similarly, f has an expansion into odd Legendre polynomials

$$f_{av}(x) = \frac{f(x^+) + f(x^-)}{2} = \sum_{j=0}^{\infty} c_{2j+1} P_{2j+1}(x)\ .$$

The coefficients are given by

$$c_n = (2n+1) \int_0^1 f(x) P_n(x) dx\ .$$

Example 1. Consider the function $f(x) = \begin{cases} 1 & 0 < x < 1, \\ 0 & -1 < x < 0 \end{cases}$ The n-th Legendre coefficient of f is

$$c_n = \frac{2n+1}{2} \int_{-1}^1 f(x) P_n(x) dx = \frac{2n+1}{2} \int_0^1 P_n(x) dx\ .$$

The first four coefficients are

$$
\begin{aligned}
c_0 &= \frac{1}{2}\int_0^1 dx = \frac{1}{2}\\
c_1 &= \frac{3}{2}\int_0^1 x dx = \frac{3}{4}\\
c_2 &= \frac{5}{2}\int_0^1 \frac{1}{2}(3x^2-1)dx = 0\\
c_3 &= \frac{7}{2}\int_0^1 \frac{1}{2}(5x^3-3x)dx = -\frac{7}{16}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
1 &= \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \cdots \qquad 0 < x < 1,\\
0 &= \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \cdots \qquad -1 < x < 0
\end{aligned}
$$

Example 2. Let $f(x) = \begin{cases} 1 & 0 < x < 1, \\ -1 & -1 < x < 0 \end{cases}$. Since f is odd, its Legendre series contains only odd indexed polynomials. We have

$$c_{2n+1} = (4n+3)\int_0^1 P_{2n+1}(x)dx \ .$$

By using the recurrence relation (6) ($(2k+1)P_k = P'_{k+1} - P'_{k-1}$) with $k = 2n+1$, we get

$$c_{2n+1} = \int_0^1 (P'_{2n+2}(x) - P'_{2n}(x))dx = P_{2n+2}(1) - P_{2n+2}(0) - P_{2n}(1) + P_{2n}(0) \ .$$

Since $P_{2j}(1) = 1$ that $P_{2j}(0) = (-1)^j\,(2j)!/2^{2j}(j!)^2$ (see exercise 1), then it follows that

$$c_{2n+1} = P_{2n}(0) - P_{2n+2}(0) = \left(\frac{(-1)^n(4n+3)}{2^{n+1}(n+1)}\right)\frac{(2n)!}{(n!)^2} \ .$$

We have then the expansion

$$1 = \sum_{n=0}^{\infty}\left(\frac{(-1)^n(4n+3)}{2^{n+1}(n+1)}\right)\frac{(2n)!}{(n!)^2}P_{2n+1}(x)\,, \quad 0 < x < 1 \ .$$

6. Separation of variables for $\Delta u = 0$ in spherical coordinates

Recall that if (x, y, z) and (ρ, θ, ϕ) denote, respectively, the cartesian and the spherical coordinates in $\mathbb{R}^3$:

$$x = \rho\cos\theta\sin\phi, \quad y = \rho\sin\theta\sin\phi, \quad z = \rho\cos\phi,$$

then the Laplace operator has expression

$$\Delta = \frac{\partial^2}{\partial\rho^2} + \frac{2}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\cot\phi}{\rho^2}\frac{\partial}{\partial\phi}$$

with $\rho > 0$, $\theta \in \mathbb{R}$, and $\phi \in (0,\ \pi)$.

Consider the problem of finding bounded solutions $u = u(\rho, \theta, \phi)$ of the Laplace equation $\Delta u = 0$. That is, $u(\rho, \theta, \phi)$ bounded inside the sphere $\rho < A$ and satisfies

$$\frac{\partial^2 u}{\partial\rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial\rho} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2} + \frac{\cot\phi}{\rho^2}\frac{\partial u}{\partial\phi} = 0 \tag{11}$$

The method at our disposal is that of separation of variables. Suppose that

$$u(\rho, \theta, \phi) = R(\rho)\Theta(\theta)\Phi(\phi)$$

solves the Laplace equation (we are assuming that Θ and Θ' are 2π-periodic). After replacing u and its derivatives in terms of R, Θ, Φ and their derivatives, we can rewrite (11) as

$$\rho^2 \frac{R''(\rho)}{R(\rho)} + 2\rho \frac{R'(\rho)}{R(\rho)} + \frac{\Theta''(\theta)}{\sin^2\phi\,\Theta(\theta)} + \frac{\Phi''(\phi)}{\Phi(\phi)} + \frac{\cot\phi\,\Phi'(\phi)}{\Phi(\phi)} = 0 \tag{12}$$

Separating the variable ρ from θ and ϕ leads to

$$\rho^2 R'' + 2\rho R' - \lambda R = 0, \qquad \frac{\Theta''(\theta)}{\sin^2\phi\,\Theta(\theta)} + \frac{\Phi''(\phi)}{\Phi(\phi)} + \frac{\cot\phi\,\Phi'(\phi)}{\Phi(\phi)} = -\lambda$$

where λ is the separation constant. A further separation of the second equation leads to the three ODEs

$$\rho^2 R''(\rho) + 2\rho R'(\rho) - \lambda R(\rho) = 0 \tag{13}$$

$$\Theta''(\theta) + \alpha\Theta(\theta) = 0 \tag{14}$$

$$\Phi''(\phi) + \frac{\cos\phi}{\sin\phi}\Phi'(\phi) + \left(\lambda - \frac{\alpha}{\sin^2\phi}\right)\Phi(\phi) = 0 \tag{15}$$

with α and λ constants. The R-equation and Θ-equation are familiar and we know how to solve them: (13) is a Cauchy-Euler equation and (14) has constant coefficients. Furthermore, the periodicity of Θ implies that $\alpha = m^2$ with m nonnegative integer and the eigenfunctions are $\cos(m\theta)$ and $\sin(m\theta)$.

A class of solutions of $\Delta u = 0$. Consider the case $\alpha = m = 0$. In this case $\Theta(\theta) = 1$ and the function u is independent on θ. The R-equation and Φ-equations are

$$\rho^2 R''(\rho) + 2\rho R'(\rho) - \lambda R(\rho) = 0 \tag{16}$$

$$\Phi''(\phi) + \frac{\cos\phi}{\sin\phi}\Phi'(\phi) + \lambda\Phi(\phi) = 0 \tag{17}$$

Equation (16) is a Cauchy-Euler equation with solutions

$$R_1(\rho) = \rho^{p_1}, \quad R_2(\rho) = \rho^{p_2}$$

where $p_{1,2}$ are the roots of $p^2 + p - \lambda = 0$.

To understand the Φ-equation, we need to make a change of variable. For $\phi \in (0,\ \pi)$, consider the change of variable

$$t = \cos\phi \qquad t \in (-1,\ 1)\ ,$$

and let

$$w(t) = \Phi(\phi) = \Phi(\arccos t) \ \Leftrightarrow\ \Phi(\phi) = w(\cos\phi)\ .$$

We use equation (17) to write an ODE for w. We have

$$\begin{aligned} \Phi'(\phi) \quad &= -\sin\phi\, w'(\cos\phi) = -\sqrt{1-t^2}\, w'(t)\,, \\ \Phi''(\phi) \quad &= \sin^2\phi\, w''(\cos\phi) - \cos\phi\, w'(\cos\phi) = (1-t^2)w''(t) - tw'(t) \end{aligned}$$

By replacing these expressions of Φ' and Φ'' in (17), we get

$$(1-t^2)w''(t) - 2tw'(t) + \lambda w(t) = 0\ . \tag{18}$$

This is the Legendre equation. To get w a bounded solution, we need to have $\lambda = n(n+1)$ with n a nonnegative integer. In this case w is a multiple of the

Legendre polynomial $P_n(t)$. By going back to the function Φ and variable ϕ, we get the following lemma.

Lemma 1. *The eigenvalues and eigenfunctions of equation* (17) *are* $\lambda = n(n+1)$ *with* n *a nonnegative integer and*

$$\Phi_n(\phi) = P_n(\cos\phi) \ ,$$

where P_n *is Legendre polynomial of degree* n.

For $\lambda = n(n+1)$ the exponents $p_{1,2}$ for the R-equation are $p_1 = n$ and $p_2 = -(n+1)$ and the independent solution of (16) are

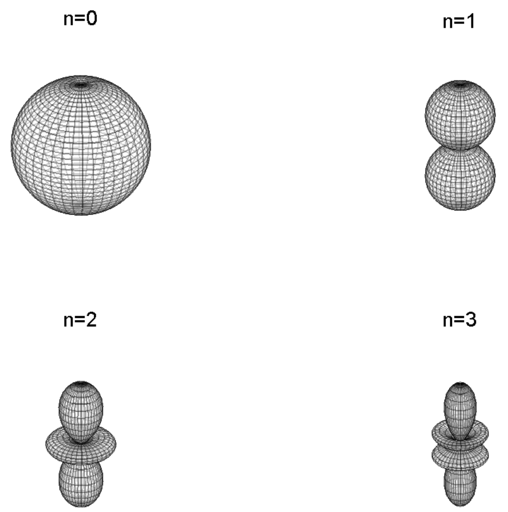
$$\rho^n \ , \qquad \rho^{-(n+1)} = \frac{1}{\rho^{n+1}} \ .$$

Note that the second solution is unbounded. We have established the following.

Proposition 1. *If* $u(\rho,\phi) = R(\rho)\Phi(\phi)$ *is a bounded solution of the Laplace equation* $\Delta u = 0$ *inside a sphere, then there exists a nonnegative integer* n *so that*

$$u(\rho,\phi) = C\rho^n P_n(\cos(\phi)) \, , \quad (C \text{ constant}).$$

The following figure illustrates the graphs of the spherical polynomials $P_n(\cos\phi)$ for $n = 0,\ 1,\ 2,$ and 3. The surface has parametric equation $(|P_n(\cos\phi)|, \theta, \phi)$.



7. Dirichlet problem in the sphere with longitudinal symmetry

Consider the following Dirichlet problem in the sphere of radius L:

$$\begin{cases} \Delta u(\rho,\theta,\phi) = 0 \\ u(L,\theta,\phi) = f(\theta,\phi) \end{cases} \tag{19}$$

Such a problem models the steady state heat distribution inside the sphere when the temperature on the surface is given by f. The case with longitudinal symmetry means that the problem is independent on the angle θ. Thus $f = f(\phi)$ and the temperature u depends only on the radius ρ and the altitude ϕ ($u = u(\rho, \phi)$). Problem (19) can be written as

$$\left\{ \begin{array}{l} u_{\rho\rho} + \dfrac{2}{\rho} u_\rho + \dfrac{1}{\rho^2} u_{\phi\phi} + \dfrac{\cos\phi}{\rho^2 \sin\phi} u_\phi = 0 \\ u(L, \phi) = f(\phi) \end{array} \right. \tag{20}$$

Since u is assumed to be a bounded function, then the method of separation of variables for the PDE in (20) leads (as we have seen above) to the following solutions with separated variables

$$u_n(\rho, \phi) = \rho^n P_n(\cos\phi)\,, \qquad n = 0,\ 1,\ 2,\ \cdots$$

The general series solution of the PDE in (20) is therefore

$$u(r, \phi) = \sum_{n=0}^{\infty} C_n \rho^n P_n(\cos\phi)\ .$$

In order for such a solution to satisfy the nonhomogeneous condition we need to have

$$u(L, \phi) = f(\phi) = \sum_{n=0}^{\infty} C_n L^n P_n(\cos\phi)\ .$$

The last series is a Legendre series but it is expressed in terms of $\cos\phi$. To find the coefficients C_n, we need to express the last series as a Legendre series in standard form . We resort again to the substitution $t = \cos\phi$ with $-1 < t < 1$. We rewrite f in terms of the variable t (for instance $g(t) = f(\phi)$). The last series is

$$g(t) = \sum_{n=0}^{\infty} C_n L^n P_n(t)\ .$$

Now this is the usual Legendre series of the function $g(t)$. Its n-th Legendre coefficients $C_n L^n$ is given by

$$C_n L^n = \frac{2n+1}{2} \int_{-1}^{1} g(t) P_n(t) dt\ .$$

We can rewrite C_n this in terms of the variable ϕ as

$$C_n = \frac{2n+1}{2L^n} \int_0^{\pi} \sin\phi\, f(\phi) P_n(\cos\phi)\, d\phi\ . \tag{21}$$

The solution of problem (20) is therefore

$$u(\rho, \phi) = \sum_{n=0}^{\infty} C_n \rho^n P_n(\cos\phi)\ ,$$

where C_n is given by formula (21).

Example 1. The following Dirichlet problem represents the steady-state temperature distribution inside a ball of radius 10 assuming that the upper hemisphere is kept at constant temperature 100 and the lower hemisphere is kept at temperature 0.

$$u_{\rho\rho} + \frac{2}{\rho} u_\rho + \frac{1}{\rho^2} u_{\phi\phi} + \frac{\cos\phi}{\rho^2 \sin\phi} u_\phi = 0, \quad u(10, \phi) = f(\phi)$$

$$f(\phi) = \left\{ \begin{array}{ll} 100 & \text{if } \ 0 < \phi < \pi/2, \\ 0 & \text{if } \ (\pi/2) < \phi < \pi, \end{array} \right.$$

The solution to this problem is

$$u(\rho,\phi)=\sum_{n=0}^{\infty} C_n\rho^n P_n(\cos\phi)$$

where

$$\begin{aligned} C_n &= \frac{2n+1}{2(10^n)}\int_0^{\pi} f(\phi)\sin\phi\, P_n(\cos\phi)\,d\phi \\ &= \frac{50(2n+1)}{10^n}\int_0^{\pi/2} \sin\phi\, P_n(\cos\phi)\,d\phi \\ &= \frac{50(2n+1)}{10^n}\int_0^{1} P_n(t)\,dt \end{aligned}$$

We can find a closed expression for C_n. You will be asked in the exercises to establish the formula

$$\int_0^1 P_n(x) = \frac{1}{2n+1}[P_{n-1}(0)-P_{n+1}(0)]$$

Thus, it follows from the fact that $P_{\text{odd}}(0)=0$ that $C_{\text{even}}=0$ and from

$$P_{2j}(0)=(-1)^j\frac{(2j)!}{2^{2j}(j!)^2}$$

that

$$C_{2j+1}=\frac{25(-1)^j(2j)!}{10^{2j+1}2^{2j}(j+1)(j!)^2}$$

The solution to the problem

$$u(\rho,\phi)=25\sum_{j=0}^{\infty}\frac{(-1)^j(2j)!}{2^{2j}(j+1)(j!)^2}\left(\frac{\rho}{10}\right)^{2j+1}P_{2j+1}(\cos\phi)$$

Example 2. (Dirichlet problem in a spherical shell). Consider the BVP

$$\begin{array}{ll} u_{\rho\rho}+\dfrac{2}{\rho}u_\rho+\dfrac{1}{\rho^2}u_{\phi\phi}+\dfrac{\cos\phi}{\rho^2\sin\phi}u_\phi=0, & 1<\rho<2\,,\quad 0<\phi<(\pi/2)\,, \\ u(1,\phi)=50,\quad u(2,\phi)=100, & 0<\phi<(\pi/2)\,, \\ u(\rho,\pi/2)=0 & 1<\rho<2\,. \end{array}$$

Such a problem models the steady-state temperature distribution inside in a spherical shell when the temperature on the outer hemisphere is kept at 100 degrees, that in the inner hemisphere is kept at temperature 50 degrees and the temperature at the base is kept at 0 degree.

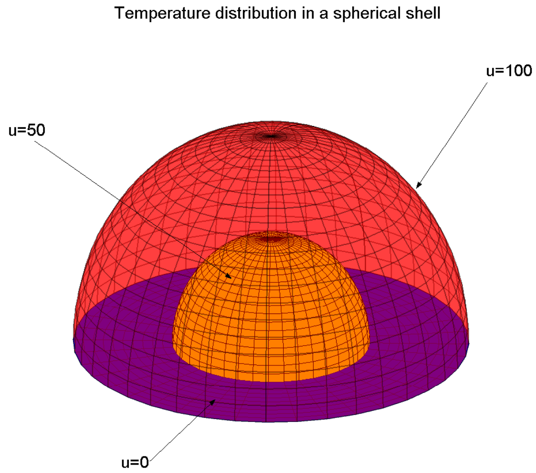
The separation of variables for the homogeneous part leads to the ODE problems

$$\begin{gathered} \rho^2R''+2\rho R'-\lambda R=0, \\ \sin\phi\,\Phi''+\cos\phi\,\Phi'+\lambda\sin\phi\,\Phi=0,\quad \Phi(\pi/2)=0 \end{gathered}$$

As we have seen above, the eigenvalues of the Φ-equation are $\lambda=n(n+1)$ and corresponding eigenfunctions $P_n(\cos\phi)$. This time however, we also need to have

$$\Phi_n(\pi/2)=P_n(0)=0\ .$$

Therefore n must be odd. The solutions with separated variables of the homogeneous part are $\rho^nP_n(\cos\phi)$ and $\rho^{-(n+1)}P_n(\cos\phi)$ with n odd (note that since in this



problem $\rho > 1$, the second solution $\rho^{-(n+1)}$ of the R-equation must be considered). The series solution is

$$u(\rho,\phi) = \sum_{j=0}^{\infty} \left[A_{2j+1}\rho^{2j+1} + \frac{B_{2j+1}}{\rho^{2j+2}} \right] P_{2j+1}(\cos\phi) \ .$$

Now we use the nonhomogeneous boundary condition to find the coefficients.

$$100 = \sum_{j=0}^{\infty} \left[A_{2j+1}2^{2j+1} + \frac{B_{2j+1}}{2^{2j+2}} \right] P_{2j+1}(\cos\phi)$$
$$50 = \sum_{j=0}^{\infty} \left[A_{2j+1} + B_{2j+1} \right] P_{2j+1}(\cos\phi)$$

for $0 < \phi < (\pi/2)$. Equivalently,

$$100 = \sum_{j=0}^{\infty} \left[A_{2j+1}2^{2j+1} + \frac{B_{2j+1}}{2^{2j+2}} \right] P_{2j+1}(x) \,, \quad 0 < x < 1 \,,$$
$$50 = \sum_{j=0}^{\infty} \left[A_{2j+1} + B_{2j+1} \right] P_{2j+1}(x) \,, \quad 0 < x < 1 \ .$$

By using the above series and the series expansion of 1 over $[0,\ 1]$ into odd Legendre polynomials (see previous examples)

$$1 = \sum_{j=0}^{\infty} \frac{(-1)^j (2j)!}{2^{2j+2}(j+1)(j!)^2} P_{2j+1}(x) \qquad 0 < x < 1 \,,$$

we get

$$A_{2j+1} + B_{2j+1} = \frac{50(-1)^j(2j)!}{2^{2j+2}(j+1)(j!)^2}$$
$$2^{2j+1}A_{2j+1} + B_{2j+1}\frac{1}{2^{2j+2}} = \frac{100(-1)^j(2j)!}{2^{2j+2}(j+1)(j!)^2}$$

From these equations A_{2j+1} and B_{2j+1} can be explicitly found.

8. More solutions of $\Delta u = 0$

Recall, from the previous section, that if $u = R(\rho)\Theta(\theta)\Phi(\phi)$ satisfies $\Delta u = 0$, then the functions R, Θ, and Φ satisfy the ODEs

$$\rho^2 R''(\rho) + 2\rho R'(\rho) - \lambda R(\rho) = 0 \tag{22}$$

$$\Theta''(\theta) + \alpha\Theta(\theta) = 0 \tag{23}$$

$$\Phi''(\phi) + \frac{\cos\phi}{\sin\phi}\Phi'(\phi) + \left(\lambda - \frac{\alpha}{\sin^2\phi}\right)\Phi(\phi) = 0 \tag{24}$$

The function Θ is 2π-periodic, R and Φ bounded. In the previous section we assumed u independent on θ (this is the case corresponding to the eigenvalue $\alpha = 0$.) Now suppose that u depends effectively on θ. The eigenvalues and eigenfunctions of the Θ-problem are

$$\alpha = m^2, \qquad \begin{cases} \Theta^1(\theta) = \cos(m\theta) \\ \Theta^2(\theta) = \sin(m\theta) \end{cases}, \qquad m \in \mathbb{Z}^+$$

For $\alpha = m^2$, the Φ-equation becomes

$$\Phi''(\phi) + \frac{\cos\phi}{\sin\phi}\Phi'(\phi) + \left(\lambda - \frac{m^2}{\sin^2\phi}\right)\Phi(\phi) = 0 \tag{25}$$

If we use the variable $t = \cos\phi$ and let $w(t) = \Phi(\cos\phi)$, then w solves

$$(1-t^2)w''(t) - 2tw'(t) + \left(\lambda - \frac{m^2}{1-t^2}\right)w(t) = 0\ . \tag{26}$$

or equivalently in self-adjoint form as

$$\left[(1-t^2)w'(t)\right]' + \left(\lambda - \frac{m^2}{1-t^2}\right)w(t) = 0\ . \tag{27}$$

Equation (26) (or (27)) is called the *generalized Legendre equation.* Its solutions are related to those of the Legendre equation and are given by the following lemma.

Lemma. *Let $y(t)$ be a solution of the legendre equation*

$$(1-t^2)\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + \lambda y = 0\ .$$

Then the function

$$w(t) = (1-t^2)^{m/2}\frac{d^m y(t)}{dt^m}$$

solves the generalized Legendre equation

$$(1-t^2)w''(t) - 2tw'(t) + \left(\lambda - \frac{m^2}{1-t^2}\right)w(t) = 0\ .$$

For $\lambda = n(n+1)$, the bounded solutions of (26) are

$$P_n^m(t) = (1-t^2)^{m/2}\frac{d^m P_n(t)}{dt^m}$$

these are called the *associated Legendre polynomials* (of degree n and order m). Note that since P_n is a polynomial of degree n, then if $m > n$, the function P_n^m is identically zero ($P_n^m(t) = 0$). Note also that $P_n^0(x) = P_n(x)$.

Example.

$$\begin{aligned} P_2^1(x) &= 3x\sqrt{1-x^2}\,, & P_3^1(x) &= \frac{3}{2}(5x^2-1)\sqrt{1-x^2} \\ P_3^2 &= 15x(1-x^2) & P_4^2(x) &= \frac{15}{2}(7x^2-1)(1-x^2) \\ P_4^3(x) &= 105x(1-x^2)^{3/2} & P_5^3(x) &= \frac{105}{2}(9x^2-1)(1-x^2)^{3/2} \end{aligned}$$

In summary we have constructed solutions of $\Delta u = 0$ of the form

$$\rho^n \cos(m\theta) P_n^m(\cos\phi) \qquad (m \le n)$$

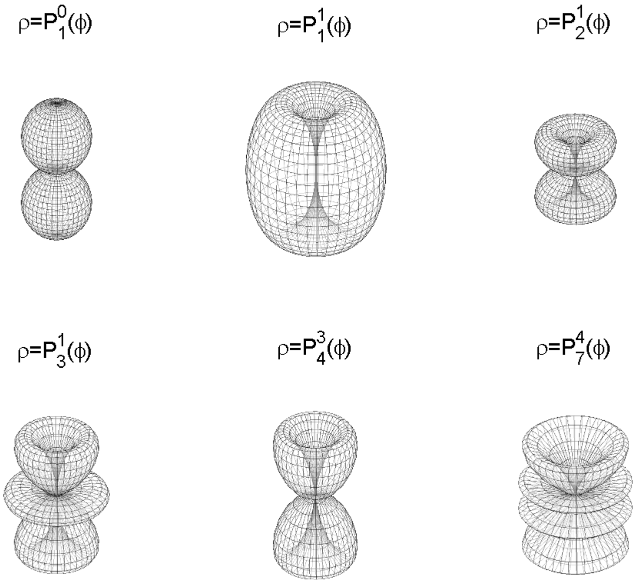
The functions

$$Y_{mn}(\theta,\phi) = P_n^m(\cos\phi)\cos(m\theta)$$

are called *spherical harmonics*. Some of the surfaces given in spherical coordinates by

$$\rho = P_n^m(\theta)$$

are plotted in the figure



Example: small vibrations of a spherical membrane. Consider the radial vibrations of a spherical membrane of radius L. Let $u(\theta,\phi,t)$ denotes the radial displacement at time t, from equilibrium, of the point on the L-sphere with coordinates (θ,ϕ). The function u satisfies the wave equation

$$u_{tt} = c^2\left(u_{\phi\phi} + \cot\phi\, u_\phi + \frac{1}{\sin\phi}u_{\theta\theta}\right)$$

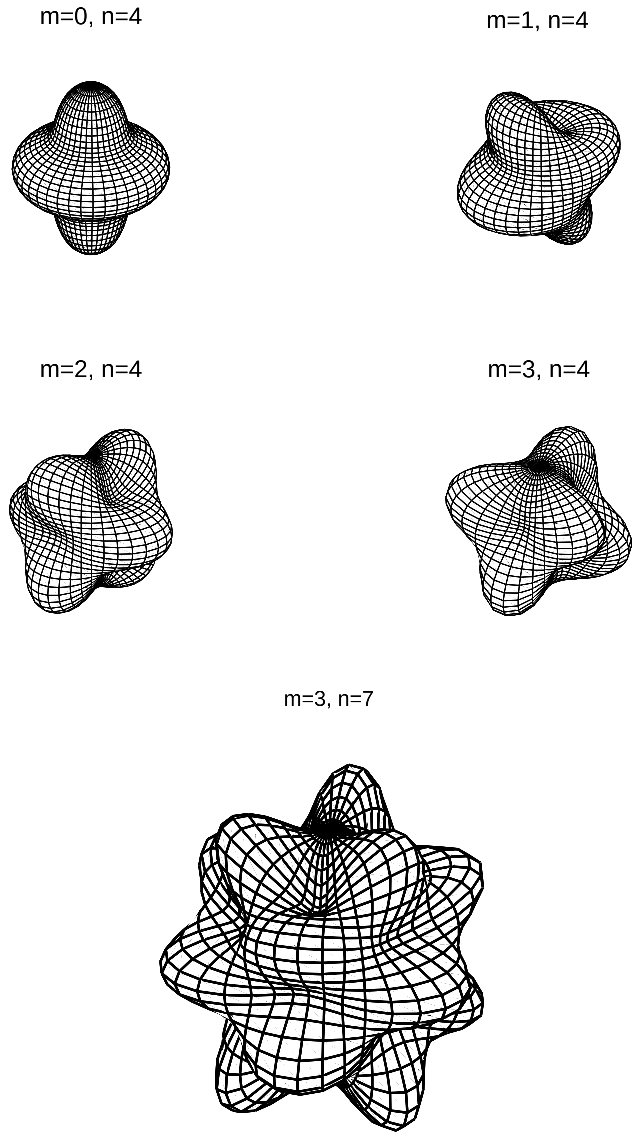
The method of separation of variables for the PDE leads to the following solutions

$$P_n^m(\cos\phi)\cos(m\theta)\cos(\omega_{mn}t),,$$

with $m \le n$ and $\omega_{mn} = c\sqrt{n(n+1)}$.The solution

$$u_{mn}(\theta,\phi,t) = L + aP_n^m(\cos\phi)\cos(m\theta)\cos(\omega_{mn}t),,$$

are the (m, n)-modes of vibration of the spherical membrane. Some of the profiles are illustrated in the following figure.



9. Exercises.

Exercise 1. Use the recurrence relation that gives the coefficients of the Legendre polynomials to show that

$$P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} \ .$$

Exercise 2. Use exercise 1 to verify that

$$P_{2n}(0) - P_{2n+2}(0) = (-1)^n \left(\frac{4n+3}{2n+2}\right) \frac{(2n)!}{2^{2n}(n!)^2} \ .$$

Exercise 3. Use $P_0(x) = 1$, $P_1(x) = x$ and the recurrence relation (8):

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

to find $P_2(x)$, $P_3(x)$, $P_4(x)$, and $P_5(x)$.

Exercise 4. Use Rodrigues' formula to find the Legendre polynomials $P_0(x)$ to $P_5(x)$.

Exercise 5. Use Rodrigues' formula to establish

$$xP'_n(x) = nP_n(x) + P'_{n-1}(x) \ .$$

Exercise 6. Write x^2 as a linear combination of $P_0(x)$, $P_1(x)$, and $P_2(x)$. That is, find constants A, B, and C so that

$$x^2 = AP_0(x) + BP_1(x) + CP_2(x) \ .$$

Exercise 7. Write x^3 as a linear combination of P_0, P_1, P_2, and P_3.

Exercise 8. Write x^4 as a linear combination of P_0, P_1, P_2, P_3 and P_4.

Exercise 9. Use the results from exercises 6, 7, and 8 to find the integrals.

$$\int_{-1}^{1} x^2 P_2(x)dx, \qquad \int_{-1}^{1} x^2 P_{31}(x)dx$$
$$\int_{-1}^{1} x^3 P_1(x)dx, \qquad \int_{-1}^{1} x^3 P_4(x)dx$$
$$\int_{-1}^{1} x^4 P_2(x)dx, \qquad \int_{-1}^{1} x^4 P_4(x)dx$$

Exercise 10. Use the fact that for $m \in \mathbb{Z}^+$, the function x^m can written as a linear combination of $P_0(x), \cdots, P_m(x)$ to show that

$$\int_{-1}^{1} x^m P_n(x) = 0, \qquad \text{for } n > m \ .$$

Exercise 11. Use formula (8) and a property (even/odd) of the Legendre polynomials to verify that

$$\int_{-1}^{h} P_n(x)dx = \frac{1}{2n+1}[P_{n+1}(h) - P_{n-1}(h)]$$
$$\int_{h}^{1} P_n(x)dx = \frac{1}{2n+1}[P_{n-1}(h) - P_{n+1}(h)]$$

Exercise 12. Find the Legendre series of the functions

$$f(x) = -3, \quad g(x) = x^3, \quad h(x) = x^4, \quad m(x) = |x|.$$

Exercise 13. Find the Legendre series of the function

$$f(x) = \begin{cases} 0 & \text{for } -1 < x < 0 \\ x & \text{for } 0 < x < 1 \end{cases}$$

Exercise 14. Find the Legendre series of the function

$$f(x) = \begin{cases} 0 & \text{for } -1 < x < h \\ 1 & \text{for } h < x < 1 \end{cases}$$

(Use exercise 11.)

Exercise 15. Find the first three nonzero terms of the Legendre series of the functions $f(x) = \sin x$ and $g(x) = \cos x$.

In exercises 16 to 19 solve the following Dirichlet problem inside the sphere

$$\begin{cases} u_{\rho\rho} + \dfrac{2}{\rho}u_\rho + \dfrac{1}{\rho^2}u_{\phi\phi} + \dfrac{\cos\phi}{\rho^2\sin\phi}u_\phi = 0, & 0 < \rho < L, \quad 0 < \phi < \pi \\ u(L,\phi) = f(\phi) & 0 < \phi < \pi \end{cases}$$

Assume $u(\rho,\phi)$ is bounded.

Exercise 16. $L = 10$, and $f(\phi) = \begin{cases} 50 & \text{for } 0 < \phi < (\pi/2), \\ 100 & \text{for } (\pi/2) < \phi < \pi. \end{cases}$

Exercise 17. $L = 1$ and $f(\phi) = \cos\phi$.

Exercise 18. $L = 5$ and $f(\phi) = \begin{cases} 50 & \text{for } 0 < \phi < (\pi/4), \\ 0 & \text{for } (\pi/4) < \phi < \pi. \end{cases}$

Exercise 19. $L = 2$ and $f(\phi) = \sin^2\phi = 1 - \cos^2\phi$.

Exercise 20. Solve the following Dirichlet problem in a hemisphere

$$\begin{cases} u_{\rho\rho} + \dfrac{2}{\rho}u_\rho + \dfrac{1}{\rho^2}u_{\phi\phi} + \dfrac{\cos\phi}{\rho^2\sin\phi}u_\phi = 0, & 0 < \rho < 1, \quad 0 < \phi < (\pi/2) \\ u(1,\phi) = 100 & 0 < \phi < (\pi/2) \\ u(\rho,\pi/2) = 0 & 0 < \rho < 1\,. \end{cases}$$

Exercise 21. Solve the following Dirichlet problem in a hemisphere

$$\begin{cases} u_{\rho\rho} + \dfrac{2}{\rho}u_\rho + \dfrac{1}{\rho^2}u_{\phi\phi} + \dfrac{\cos\phi}{\rho^2\sin\phi}u_\phi = 0, & 0 < \rho < 1, \quad 0 < \phi < (\pi/2) \\ u(1,\phi) = \cos\phi & 0 < \phi < (\pi/2) \\ u(\rho,\pi/2) = 0 & 0 < \rho < 1\,. \end{cases}$$

Exercise 22. Solve the following Dirichlet problem in a spherical shell

$$\begin{cases} u_{\rho\rho} + \dfrac{2}{\rho}u_\rho + \dfrac{1}{\rho^2}u_{\phi\phi} + \dfrac{\cos\phi}{\rho^2\sin\phi}u_\phi = 0, & 1 < \rho < 2, \quad 0 < \phi < \pi \\ u(1,\phi) = 50 & 0 < \phi < \pi \\ u(2,\phi) = 100 & 0 < \phi < \pi\,. \end{cases}$$

Exercise 23. Solve the following Dirichlet problem in a spherical shell

$$\begin{cases} u_{\rho\rho} + \dfrac{2}{\rho}u_\rho + \dfrac{1}{\rho^2}u_{\phi\phi} + \dfrac{\cos\phi}{\rho^2\sin\phi}u_\phi = 0, & 1 < \rho < 2, \quad 0 < \phi < \pi \\ u(1,\phi) = \cos\phi & 0 < \phi < \pi \\ u(2,\phi) = \sin^2\phi & 0 < \phi < \pi\,. \end{cases}$$

Exercise 24. Find the gravitational potential at any point outside the surface of the earth knowing that the radius of the earth is 6400 km and that the gravitational potential on the earth surface is given by

$$f(\phi) = \begin{cases} 200 - \cos\phi & \text{for } \; 0 < \phi < (\pi/2), \\ 200 & \text{for } \; (\pi/2) < \phi < \pi. \end{cases}$$

(This is an exterior Dirichlet problem)

Exercise 25. The sun has a diameter of 1.4×10^6 km. If the temperature on the sun's surface is $20,000^0$ C, find the approximate temperature on the following planets.

Planet	Mean distance from sun (millions of kilometers)
Mercury	57.9
Venus	108.2
Earth	149.7
Mars	228.1
Jupiter	778.6
Saturn	1429.0
Uranus	2839.6
Neptune	4491.6
Pluto	5880.2