Chapter-4

Legendre's Polynomials

4.1 Introduction

The following second order linear differential equation with variable coefficients is known as **Legendre's differential equation**, named after **Adrien Marie Legendre** (1752-1833), a French mathematician, who is best known for his work in the field of elliptic integrals and theory of numbers :

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad \ldots(1)$$

where n is a non-negative integer.

Legendre's differential equation occurs in many physical and engineering problems involving spherical geometry and gravitation.

4.2 Legendre's Differential Equation

We know that the differential equation of the form

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad \ldots(1)$$

is called Legendre's differential equation (or simply Legendre's equation), where n is a non-negative integer.

This equation can also be put in the following form:

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0.$$

Clearly, the only singular points of (1) are $x = 1$, $x = -1$ and $x = \infty$, which are regular. Therefore, the Legendre's differential equation is a **Fuchsian differential equation**.

The points other than singular points e.g., $x = 0$, $x = 2$, etc. behave like ordinary points of (1).

Let the series solution of (1) be of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, C_0 \neq 0 \quad \ldots(2)$$

∴ $$y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} \quad \ldots(3)$$

and $$y'' = \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} \quad \ldots(4)$$

Putting the above values of y, y′and y" in (1), we have

$$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - x^2 \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2}$$

$$-2x \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} + n(n+1) \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2}$$

$$-\sum_{m=0}^{\infty} C_m \{(k+m)(k+m-1) + 2(k+m) - n(n+1)\} x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2}$$

$$-\sum_{m=0}^{\infty} C_m \{(k+m)^2 + (k+m) - n^2 - n\} x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2}$$

$$-\sum_{m=0}^{\infty} C_m \{(k+m+n)(k+m-n) + (k+m-n)\} x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2}$$

$$-\sum_{m=0}^{\infty} C_m (k+m-n)(k+m+n+1) x^{k+m} = 0 \quad \ldots(5)$$

which is an identity in x and, therefore, the coefficients of various powers of x in it should be zero.

Thus, equating to zero, the coefficient of the smallest power of x, namely x^{k-2} in (5), we get the following **indicial equation**

$$C_0 k(k-1) = 0 \quad \text{or} \quad k(k-1) = 0, \quad [\because C_0 \neq 0]$$

which gives two indicial roots $k = k_1 = 1$ and $k = k_2 = 0$.

Note that the roots of indicial equation are unequal and differ by an integer.

Now, to get the recurrence relation, we equate to zero, the coefficient of x^{k+m-2}in (5). Thus, we have

$$C_m(k+m)(k+m-1) - C_{m-1}(k+m-2-n)(k+m-2+n+1) = 0$$

or $$C_m = \frac{(k+m-2-n)(k+m-1+n)}{(k+m)(k+m-1)} C_{m-2} \qquad \ldots(6)$$

Next, equating to zero, the coefficient of x^{k-1}in (5), we obtain

$$C_1(k+1)k = 0 \qquad \ldots(7)$$

For $k = 0$, we note from (7) that C_1 is indeterminate.

Thus, putting $k = 0$ in (6), we get $C_m = \frac{(m-2-n)(m-1+n)}{m(m-1)} C_{m-2}$ ...(8)

We now express C_2, C_4, C_6…. in terms of C_0 and C_3, C_5 C_7…. in terms of C_1 by assuming that C_1 is finite.

Putting m = 2 in (8), we have $C_2 = \frac{(-n)(n+1)}{2.1} C_0 = -\frac{n(n+1)}{2!} C_0$ ...(9)

Putting m = 4 in (8) and using (9), we obtain

$$C_4 = \frac{(2-n)(3+n)}{4.3} C_2 = \frac{(n-2)n(n+1)(n+3)}{4!} C_0 \qquad \ldots(10)$$

and so on.

Next, putting m = 3 in (8), we obtain

$$C_3 = \frac{(1-n)(2+n)}{3.2} C_1 = -\frac{(n-1)(n+2)}{3!} C_1 \qquad \ldots(11)$$

Again, putting m = 5 in (8) and using (11), we have

$$C_5 = \frac{(3-n)(4+n)}{5.4} C_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} C_1 \quad \ldots(12)$$

and so on.

Now, the solution (2) can be re-written as:

$$y = x^k(C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + \cdots), \text{ where k = 0}$$

or $$y = (C_0 + C_2x^2 + C_4x^4 + \cdots) + (C_1x + C_3x^3 + C_5x^5 + \cdots) \quad \ldots(13)$$

Using the values of C_2, C_3, C_4, C_5,…... in the above equation, we get

$$y = C_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n-1)(n+3)}{4!}x^4 - \cdots\right]$$

$$+C_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 + \cdots\right] \quad \ldots(14)$$

which is the required general series solution, C_0 and C_1 being arbitrary constants.

4.3. Solution of Legendre's Differential Equation in Descending Powers

Consider Legendre's differential equation of the type

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0 \quad \ldots(1)$$

where n is a non-negative integer.

It is possible to obtain the solution of (1) in terms of descending powers of x. Due to its applications to physical problems, this form of solution of Legendre's differential equation is more important.

For such a solution, let us assume that the Legendre's differential equation (1) has a series solution of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k-m}, C_0 \neq 0 \quad \ldots(2)$$

Then, by Frobenius method, we can find two linearly independent solutions of (1) in descending powers of x as:

$$y_1 = a\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} - \cdots\right] \qquad \ldots(3)$$

and $$y_2 = b\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)}x^{-n-5} + \cdots\right] \qquad \ldots(4)$$

If we take $a = \frac{1.3.5\ldots(2n-1)}{n!}$, the solution (3) is denoted by P_n (x) and is called **Legendre's function of the first kind** or **Legendre's polynomial of degree n** [since (3) is a terminating series and so, it gives rise to a polynomial of degree n]. Again, if we take $b = \frac{n!}{1.3.5\ldots(2n+1)}$, the solution (4) is denoted by Q_n(x) and is **called Legendre's function of the second kind**. Since n is positive, (4) is an infinite or non-terminating series and hence Q_n(x) is not a polynomial. Thus, P_n(x) and Q_n(x) are two linearly independent solutions of (1). Hence, the most general solution of (1) is given by

$$y = AP_n(x) + BQ_n(x) \qquad \ldots(5)$$

where A and B are arbitrary constants.

Remarks: When there is no confusion regarding the variable x, we shall use a shorter notation P_n for $P_n(x)$, $P_n{}'$ for $\frac{d}{dx}P_n(x)$ and so on.

4.4 Legendre's Functions of First and Second Kinds

Legendre's function of the first kind or **Legendre's polynomial of degree n** is denoted by P_n(x) and is defined by

$$P_n(x) = \frac{1.3.5\ldots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} - \cdots\right] \qquad \ldots(1)$$

We can also write $P_n(x)$ in a compact from as :

$$P_n(x) = \sum_{r=0}^{[n/2]}(-1)^r \frac{(2n-2r)!}{2^n r!(n-2r)!(n-r)} x^{n-2r} \qquad \ldots(2)$$

where $[n/2] = \begin{cases} n/2, \text{if } n \text{ is even} \\ (n-1)/2, \text{if } n \text{ is odd} \end{cases}$.

Legendre's function of the second kind is denoted by Q_n(x) and is defined by

$$Q_n(x) = \frac{n!}{1.3.5\ldots(2n+1)}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)}x^{-n-5} + \cdots\right] \ldots(3)$$

4.5 First Few Legendre's Polynomials

Using the definition (1) or (2), the first few Legendre's polynomials are given by

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x), \text{etc.}$$

4.6. Generating Function for Legendre's Polynomial $P_n(x)$

The function $(1-2x\,h + h^2)^{-1/2}$ is called as the **generating function for $P_n(x)$** and, therefore, $P_n(x)$ is the coefficient of h^n in the expansion of $(1 - 2xh + h^2)^{-1/2}$ in ascending powers of h, i.e., $(1 - 2xh + h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x), |x| \leq 1$ and $|h| < 1$.

Proof: Since $|h| < 1$ and $|x| \leq 1$, therefore, we can write

$$(1 - 2xh + h^2)^{-1/2} = \{1 - h(2x - h)\}^{-1/2}$$

$$= 1 + \frac{1}{2}h(2x - h) + \frac{1.3}{2.4}h^2(2x - h)^2 + \cdots + \frac{1.3\ldots(2n-3)}{2.4\ldots(2n-2)}h^{n-1}(2x - h)^{n-1}$$

$$\frac{1.3\ldots(2n-3)}{2.4\ldots2}h^n(2x - h)^n + \cdots \qquad \ldots(1)$$

∴ Coefficient of h^n in R.H.S. of (1) is

$$= \frac{1.3\ldots(2n-1)}{2.4\ldots2n}.(2x)^n - \frac{1.3\ldots(2n-3)}{2.4\ldots(2n-2)}\, n - 1C_1(2x)^{n-2} + \frac{1.3\ldots(2n-5)}{2.4\ldots(2n-4)}\, {}^{n-2}C_2(2x)^{n-4} - \cdots$$

$$= \frac{1.3\ldots(2n-1)}{2.4\ldots2n}2^n\left[x^n - \frac{2n}{2n-1}(n-1)\frac{x^{n-2}}{2^2} + \frac{2n(2n-2)}{(2n-1)(2n-3)}.\frac{(n-2)(n-3)}{2!}.\frac{x^{n-4}}{2^2} - \cdots\right]$$

$$= \frac{1.3\ldots(2n-1)}{n!}\left[x^n - \frac{2n}{2n-1}.(n-1)\frac{x^{n-2}}{2^2} + \frac{2n(2n-2)}{(2n-1)(2n-3)}.\frac{(n-2)(n-3)}{2!}.\frac{x^{n-4}}{2^2} - \cdots\right]$$

$$= \frac{1.3\ldots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} - \cdots\right]$$

$$= P_n(x).$$

Thus, we can say that in the expansion of $(1 - 2xh + h^2)^{-1/2}$, in ascending powers of h, the Legendre's polynomials $P_0(x)$, $P_1(x)$,$P_2(x)$,…. respectively are the coefficients of h^0, h^1, h^2,…. in the expansion given by (1).

Hence, we have $(1-2hx+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n\, P_n(x)$, where $P_0(x) = 1$.

This shows that $P_n(x)$ is the coefficient of h^n in the expansion of $(1-2hx+h^2)^{-1/2}$. This is why $(1-2hx+h^2)^{-1/2}$ is called as the **generating function** of the Legendre's polynomial $P_n(x)$.

4.7 Murphy's Formula for Legendre's Polynomial $P_n(x)$

Consider the Legendre's differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad \ldots(1)$$

where n is a non-negative integer.

It has only three singular points namely x = 1, x = −1 and x = ∞ and all are regular. Therefore, Legendre 's differential equation is a **Fuchsian differential equation** with three regular singular points x = 1 , x = −1 and x = ∞ .

Let us find the solution of (1) about the singular point x = 1 as follows:

The substitution $t = \frac{1}{2}(1-x)$ transfers the singular point x = 1 to t = 0.

In this case, the Legendre's differential equation (1) is transformed to the following differential equation:

$$t(1-t)y'' + (1+2t)y' + n(n+1)y = 0 \qquad \ldots(2)$$

This transformed differential equation is in the hypergeometric form with $a = -n$, b = n + 1 and c = 1.

All solutions of the transformed differential equation (2) are represented by the P- symbol as follows:

$$y = P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & n+1 & t \\ 0 & 0 & -n & \end{Bmatrix} \qquad \ldots(3)$$

Hence, all solutions of the Legendre's differential equation (1) are represented by the following P- symbol :

$$y = P\begin{Bmatrix} 0 & -1 & \infty & \\ 0 & 0 & n+1 & x \\ 0 & 0 & -n & \end{Bmatrix} \qquad \ldots(4)$$

One of the solutions of the differential equation (2) is the polynomial F(− n; n + 1; 1;t).

Now, replacing $t \; by \, (1-x)/2$, we can have one of the solutions of Legendre's differential equation (1) as:

$$P_n(x) = F\left(-n, n+1; 1; \frac{1-x}{2}\right) \qquad \ldots(5)$$

which is the polynomial solution of (1). This relation (5) for $P_n(x)$ is known as the **Murphy's formula for Legendre's polynomial $P_n(x)$.**

4.8. Laplace's Definite Integrals for $P_n(x)$

(I) Laplace's First Integral for $P_n(x)$: When n is a positive integer , then Laplace's first integral for $P_n(x)$ is given by

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}[x \pm \sqrt{(x^2-1)}\cos\phi]^n \, d\phi \qquad \ldots(1)$$

Proof: From integral calculus, we have

$$\int_0^{\pi}\frac{d\phi}{a \pm b\cos\phi} = \frac{\pi}{\sqrt{a^2-b^2}}, \text{ where } a^2 > b^2. \qquad \ldots(A)$$

Putting $a = 1 - hx$ and $b = h\sqrt{(x^2-1)}$ so that $a^2 - b^2 = (1-hx)^2 - h^2(x^2-1) = 1 - 2hx + h^2$.

Using these values of a, b and $a^2 - b^2$ in (A), we have

$$\pi(1-2hx+h^2)^{-1/2} = \int_0^\pi [1-hx \pm h\sqrt{(x^2-1)}\cos\phi]^{-1}\,d\phi$$

$$= \int_0^\pi [1-h\{x \mp \sqrt{(x^2-1)}\cos\phi\}^{-1}\,d\phi$$

$$= \int_0^\pi [1-ht]^{-1}\,d\phi, \text{ where } t = x \mp \sqrt{(x^2-1)}\cos\phi$$

or
$$\pi\sum_{n=0}^{\infty} h^n P_n(x) = \int_0^\pi (1-ht+h^2t^2+\cdots+h^nt^n+\cdots)\,d\phi$$

$$= \int_0^\pi [\sum_{n=0}^{\infty}(ht)^n]d\phi = \sum_{n=0}^{\infty}[h^n \int_0^\pi t^n\,d\phi].$$

Equating the coefficients of h^n on both sides, we have

$$\pi\,P_n(x) = \int_0^\pi t^n\,d\phi = \int_0^\pi [x \pm \sqrt{(x^2-1)}\cos\phi]^n\,d\phi$$

or
$$P_n(x) = \frac{1}{\pi}\int_0^\pi [x \pm \sqrt{(x^2-1)}\cos\phi]^n\,d\phi.$$

(II) Laplace's Second Integral for $P_n(x)$: When n is a positive integer, then Laplace's second integral for $P_n(x)$ is given by

$$P_n(x) = \frac{1}{\pi}\int_0^\pi \frac{d\phi}{[x \pm \sqrt{(x^2-1)}\cos\phi]^{n+1}}. \quad \ldots(2)$$

Proof: From integral calculus, we have

$$\int_0^\pi \frac{d\phi}{a \pm b\cos\phi} = \frac{\pi}{\sqrt{a^2-b^2}}, \text{ where } a^2 > b^2. \quad \ldots(A)$$

Putting $a = hx - 1$ and $b = h\sqrt{(x^2-1)}$ so that $a^2 - b^2 = 1 - 2hx + h^2$.

Using these values of a, b and $a^2 - b^2$ in (A), we have

$$\pi(1-2hx+h^2)^{-1/2} = \int_0^\pi [1-hx \pm h\sqrt{(x^2-1)}\cos\phi]^{-1}\,d\phi$$

or
$$\frac{\pi}{h}\left[1-2\frac{1}{h}x+\frac{1}{h^2}\right]^{-1/2} = \int_0^\pi [h\{x \pm \sqrt{(x^2-1)}\cos\phi\}-1]^{-1}d\phi$$

or
$$\frac{\pi}{h}\sum_{n=0}^{\infty}\frac{1}{h^n}P_n(x) = \int_0^\pi (ht-1)^{-1}\,d\phi, \text{ where } t = x \pm \sqrt{(x^2-1)}\cos\phi$$

$$= \int_0^\pi \frac{1}{ht}\left[1-\frac{1}{ht}\right]^{-1}d\phi = \int_0^\pi \frac{1}{ht}\left[1+\frac{1}{ht}+\frac{1}{h^1t^2}+\cdots+\frac{1}{h^nt^n}+\cdots\right]d\phi$$

$$= \int_0^\pi \left[\frac{1}{ht}+\frac{1}{h^2t^2}+\frac{1}{h^3t^3}+\cdots+\frac{1}{h^{n+1}t^{n+1}}+\cdots\right]d\phi = \sum_{n=0}^{\infty}\left[\frac{1}{h^{n+1}}\int_0^\pi \frac{1}{t^{n.1}}\,d\phi\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{h^{n+1}}\int_0^{\pi}\frac{d\phi}{\left\{x\pm\sqrt{(x^2-1)}\cos\phi\right\}^{n+1}}\right]$$

Now, equating the coefficient of $\frac{1}{h^{n+1}}$ from both sides, we have

$$\pi\, P_n(x) = \int_0^{\pi}\frac{d\phi}{\left\{x\pm\sqrt{(x^2-1)}\cos\phi\right\}^{n+1}}$$

$$\therefore \qquad P_n(x) = \frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\left\{x\pm\sqrt{(x^2-1)}\cos\phi\right\}^{n+1}}$$

Remarks: Replacing n by – (n + 1) in Laplace's second integral, we have

$$P_{-(n+1)}(x) = \frac{1}{\pi}\int_0^{\pi}\{x\pm\sqrt{(x^2-1)}\cos\phi\}^n\, d\phi$$

$$= P_n(x), \qquad \text{[From Laplace's first integral.]}$$

Hence, we have $\boldsymbol{P_{-(n+1)}(x) = P_n(x)}$, which can also be obtained by using the Murphy's formula for $P_n(x)$.

4.9. Orthogonal Properties of Legendre's Polynomials

(I) $\boldsymbol{\int_{-1}^{+1} P_m(x)\, P_n(x) dx = 0}$ **if** $\boldsymbol{m \neq n}$.   **(II)** $\boldsymbol{\int_{-1}^{+1}[P_n(x)]^2\, dx = \frac{2}{2n+1}}$.

Proof: (I) Legendre's differential equation may be written as

$$\frac{d}{dx}\left\{(x^2-1)\frac{dy}{dx}\right\} + n(n+1)y = 0 \qquad \ldots(1)$$

Since $P_n(x)$ is a solution of Legendre's differential equation (1), therefore, we have

$$\therefore \qquad \frac{d}{dx}\left\{(x^2-1)\frac{dP_n}{dx}\right\} + n(n+1)P_n = 0. \qquad \ldots(2)$$

Similarly, if we consider the Legendre's differential equation

$$\frac{d}{dx}\left\{(x^2-1)\frac{dy}{dx}\right\} + m(m+1)y = 0 \qquad \ldots(3)$$

Then, we have $\frac{d}{dx}\left\{(x^2-1)\frac{dP_m}{dx}\right\} + m(m+1)P_m = 0.$ ...(4)

Multiplying (2) by P_m and (4) by P_n and then subtracting, we have

$$P_m \frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} - P_n \frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\} + \{n(n+1) - m(m+1)\}P_n P_m = 0$$

Integrating the above between the limits – 1 to 1, we have

$$\int_{-1}^{-1}\left[P_m \frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}\right]dx - \int_{-1}^{-1}\left[P_n \frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}\right]dx$$

$$+\{n(n+1) - m(m+1)\}\int_{-1}^{1} P_m P_n dx = 0$$

On integrating by parts, we obtain

$$\left[P_m(1-x^2)\frac{dP_n}{dx}\right]_{-1}^{+1} - \int_{-1}^{+1}\frac{dP_n}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}dx - \left[P_n(1-x^2)\frac{dP_m}{dx}\right]_{-1}^{+1}$$

$$+\int_{-1}^{+1}\frac{dP_n}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}dx + [n(n+1) - m(m+1)]\int_{-1}^{+1} P_m P_n dx = 0.$$

or $$\{n(n+1) - m(m+1)\}\int_{-1}^{+1} P_m P_n dx = 0$$

Hence, we have $\int_{-1}^{+1} P_m(x)P_n(x)dx = 0$, if $m \neq n$

(II) We know that the generating function for $P_n(x)$ is given by

$$(1-2hx+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x) \qquad \ldots(3)$$

Also, we have $$(1-2hx+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^m P_m(x) \qquad \ldots(4)$$

Multiplying the corresponding sides of (3) and (4), we get

$$(1-2hx+h^2)^{-1} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h^{m+n} P_m(x) P_n(x).$$

Integrating the above between the limits – 1 to + 1, we have

$$\int_{-1}^{+1}(1-2hx+h^2)^{-1}dx = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[h^{m+n}\left\{\int_{-1}^{+1} P_m(x)P_n(x)\right\}\right]$$

or $$\sum_{n=0}^{\infty}\int_{-1}^{+1} h^{2n}[P_n(x)]^2 + \sum_{\substack{m,n=0 \\ m\neq n}}^{\infty}\int_{-1}^{+1} h^{m+n} P_m(x) P_n(x)\, dx = \int_{-1}^{+1}\frac{dx}{(1-2hx+h^2)}$$

Now, since $\int_{-1}^{+1} P_m(x)P_n(x)\, dx = 0$, where $m \neq n$. Therefore, we have

$$\sum_{n=0}^{\infty} \int_{-1}^{+1} h^{2n} [P_n(x)]^2 dx = \int_{-1}^{+1} \frac{dx}{(1-2hx+h^2)} = -\frac{1}{2h} [\log(1-2hx+h^2)]_{-1}^{+1}$$

$$= -\frac{1}{2h} \{\log(1-h)^2 - \log(1+h)^2\},$$

$$= \frac{1}{h} [\log(1-h) \; - \log(1-h)\,]$$

$$= \frac{1}{h} \left[\left(h - \frac{h^2}{2} + \frac{h^3}{3} - \cdots\right) - \left(-h - \frac{h^2}{2} - \frac{h^3}{3} - \cdots\right)\right]$$

$$= \frac{2}{h} \left[h + \frac{h^3}{3} + \frac{h^5}{5} + \cdots\right] = \frac{2}{h} \sum_{n=0}^{\infty} \frac{h^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{2h^{2n}}{2n+1}$$

Now, equating the coefficients of h^{2n} from both sides, we have

$$\int_{-1}^{+1} [P_n(x)]^2 dx = \frac{2}{2n+1}$$

Remarks: Making use of the Kronecker delta, the results (I) and (II) can be written in compact from as

$$\int_{-1}^{+1} P_m(x) P_n(x)\, dx = \frac{2}{2n+1} \delta_{mn}$$

where Kronecker delta δ_{mn} is defined by $\delta_{mn} = \begin{cases} 0, \text{if } m \neq n \\ 1, \text{if } m = n. \end{cases}$

4.10 Recurrence Formulae for Legendre's Polynomials

(I) $\boldsymbol{(2n+1)x\, P_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)}$.

Proof : Generating function for $P_n(x)$ is given by

$$(1 - 2xh + h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x) \qquad \ldots(1)$$

Differentiating both sides of (1) w.r.to h, we have

$$-\frac{1}{2} (1 - 2xh + h^2)^{-3/2} (-2x + 2h) = \sum_{n=0}^{\infty} n h^{n-1} P_n(x)$$

Multiplying both sides by($1 - 2xh + h^2$) and simplifying, we have

$$(x - h)(1 - 2xh + h^2)^{-1/2} = (1 - 2xh + h^2) \sum_{n=0}^{\infty} n h^{n-1} P_n(x)$$

or $\quad (x - h) \sum_{n=0}^{\infty} h^n P_n(x) = (1 - 2xh + h^2) \sum_{n=0}^{\infty} n h^{n-1} P_n(x)$, on using (1) or $$x\sum_{n=0}^{\infty} h^n P_n(x) - \sum_{n=0}^{\infty} h^{n+1} P_n(x) = \sum_{n=0}^{\infty} n h^{n-1} P_n(x) - 2x\sum_{n=0}^{\infty} n h^n P_n(x) + \sum_{n=0}^{\infty} n h^{n+1} P_n(x) \qquad \ldots(2)$$

Equating the general coefficients of h^n on both sides of (2), we have

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)$$

or $$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x).$$

This recurrence relation is the classical three-term relation for $P_n(x)$ and it is a pure **recurrence relation for Legendre's polynomials**.

Remarks: Equating the general coefficients of h^{n-1} on both sides of (2), we get

$$xP_{n-1}(x) - P_{n-2}(x) = nP_n(x) - 2x(n-1)P_{n-1}(x) + (n-2)P_{n-2}(x)$$

or $$\boldsymbol{nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x).}$$

This is a substitute recurrence relation of (I) and may be directly obtained by replacing n by (n – 1) in (I).

(II) $\boldsymbol{nP_n(x) = xP'_n(x) - P'_{n-1}(x),}$

Proof: Generating function for $P_n(x)$ is given by

$$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x) \qquad \ldots(1)$$

Differentiating (1) w.r.to h, we have

$$(x-h)(1-2xh+h^2)^{-3/2} = \sum_{n=0}^{\infty} n h^{n-1} P_n(x) \qquad \ldots(2)$$

Again, differentiating (1) w.r.to x, we have

$$h(1-2xh+h^2)^{-3/2} = \sum_{n=0}^{\infty} h^n P'_n(x) \qquad \ldots(3)$$

Multiplying both sides of (3) by (x – h), we have

or $$h(x-h)(1-2xh+h^2)^{-3/2} = (x-h)\sum_{n=0}^{\infty} h^n P'_n(x) \qquad \ldots(4)$$

Now, from (2) and (4), we have

$$h\sum_{n=0}^{\infty} n h^{n-1} P_n(x) = (x-h)\sum_{n=0}^{\infty} h^n P'_n(x)$$

or $$\sum_{n=0}^{\infty} nh^n P_n(x) = x\sum_{n=0}^{\infty} h^n P'_n(x) - \sum_{n=0}^{\infty} h^{n+1} P'_n(x) \qquad \ldots(5)$$

Equating the general coefficients of h^n on both sides of (5), we have

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x)$$

This recurrence relation is a **differential recurrence relation.**

(III) $\boldsymbol{(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)}$**.**

Proof: From recurrence formula (I), we have

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x). \qquad \ldots(1)$$

Differentiating both sides of (1) w.r.t. x, we have

$$(2n+1)xP'_n(x) + (2n+1)P_n(x) = (n+1)P'_{n+1}(x) + nP'_{n-1}(x), \qquad \ldots(2)$$

From recurrence formula (II) we have

$$xP'_n(x) = nP_n(x) + P'_{n-1}(x) \qquad \ldots(3)$$

Eliminating xP'_n from (2) and (3), we have

$$(2n+1)[nP_n(x) + P'_{n-1}(x)] + (2n+1)P_n(x) = (n+1)P'_{n+1}(x) + nP'_{n-1}(x)$$

or $$(2n+1)(n+1)P_n(x) = (n+1)P'_{n+1}(x) + nP'_{n-1}(x) - (2n+1)P'_{n-1}(x)$$

or $$(2n+1)(n+1)P_n(x) = (n+1)P'_{n+1}(x) - (n+1)P'_{n-1}(x).$$

$\therefore$ $$\boldsymbol{(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)}$$

(IV) $\boldsymbol{(n+1)P_n(x) = P'_{n+1}(x) - xP'_n(x)}$

Proof: Writing recurrence formulae (II) and (III), we have

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x) \qquad \ldots(1)$$

and $$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x) \qquad \ldots(2)$$

Subtracting (1) from (2), we have

$$\boldsymbol{(n+1)P_n(x) = P'_{n+1}(x) - xP_n(x)}$$

(V) $\boldsymbol{(1-x^2)P'_n(x) = n[P_{n-1}(x) - xP_n(x)]}$**.**

Proof: Replacing n by (n – 1) is recurrence formula (IV), we have

$$nP_{n-1}(x) = P'_n(x) - xP'_{n-1}(x). \qquad \ldots(1)$$

Writing recurrence formula (II), we have $nP_n(x) = xP'_n(x) - P'_{n-1}(x)$…(2)

Multiplying (2) by x and then subtracting from (1), we have

$$n\{P_{n-1}(x) - xP_n(x)\} = (1 - x^2)P'_n(x)$$

or $$\boldsymbol{(1 - x^2)P'_n(x) = n(P_{n-1}(x) - xP_n(x))}.$$

(VI) $\boldsymbol{(1 - x^2)P'_n(x) = (n + 1)[xP_n(x) - P_{n+1}(x)]}$.

Proof: Writing recurrence formula (I), we have

$$(2n + 1)xP_n(x) = (n + 1)P_{n+1}(x) + nP_{n-1}(x) \qquad \ldots(1)$$

which may also be written as

$$(n + 1)xP_n(x) + nxP_n(x) = (n + 1)P_{n+1}(x) + nP_{n-1}(x)$$

or $$(n + 1)[xP_n(x) - P_{n+1}(x)] = n[P_{n-1}(x) - xP_n(x)].$$

Writing recurrence formula (V), we have

$$(1 - x^2)P'_n(x) = n[P_{n-1}(x) - xP_n(x)]. \qquad \ldots(2)$$

Now, from (1) and (2), we have

$$\boldsymbol{(1 - x^2)P'_n(x) = (n + 1)[xP_n(x) - P_{n+1}(x)]}. \qquad \ldots(3)$$

4.11 Beltrami's Result

The following relation is known as **Beltrami's Result:**

$$(2n + 1)(x^2 - 1)P'_n(x) = n(n + 1)[P_{n+1}(x) - P_{n-1}(x)].$$

Proof : From recurrence formulae (V) and (VI), we have

$$(1 - x^2)P'_n(x) = n[P_{n-1}(x) - xP_n(x)] \qquad \ldots(1)$$

and $$(1 - x^2)P'_n(x) = (n + 1)[xP_n(x) - P_{n+1}(x)] \qquad \ldots(2)$$

Multiplying (1) by (n +1) and (2) by n and then adding, we get

$$[(n+1)+n](1-x^2)P'_n(x) = n(n+1)[P_{n-1}(x) - P_{n+1}(x)]$$

or $$(2n+1)(x^2-1)P'_n(x) = n(n+1)[P_{n+1}(x) - P_{n-1}(x)].$$

4.12 Christoffel's Expansion

The following relation is known as **Christoffel's Expansion:**

$P'_n(x) = (2n-1)P_{n-1}(x) + (2n-5)P_{n-1}(x) + (2n-9)P_{n-1}(x) + \cdots,$ the last term of the series being $3P_1(x)$ or $P_0(x)$ according as n is even or odd.

Proof: From recurrence formula (III), we have

$$P'_{n+1}(x) = (2n+1)P_n(x) + P'_{n-1}(x) \qquad \ldots(A)$$

Replacing n by (n – 1), we have $P'_n(x) = (2n-1)P_{n-1}(x) + P'_{n-2}(x) \quad \ldots(1)$

Replacing n by (n – 2), (n – 4),….4,2 in (1), we have

$$P'_{n-2}(x) = (2n-5)P_{n-3}(x) + P'_{n-4}(x) \qquad \ldots(2)$$

$$P'_{n-4}(x) = (2n-9)P_{n-5}(x) + P'_{n-6}(x) \qquad \ldots(3)$$

… …. …. …. …. …. ….

$$P'_2(x) = 3P_1(x) + P'_0(x) = 3P_1(x) \quad [\because P'_0(x) = 0]$$

Adding (1),(2),(3), ……, we have (when n is even):

$$P'_n(x) = (2n-1)P_{n-1}(x) + (2n-5)P_{n-3}(x) + (2n-9)P_{n-5}(x) + \cdots + 3P_1(x)$$

Again, when n is odd, the last of the above relation is

$$P'_3(x) = 5P_2(x) + P'_1(x) = 5P_2(x) + P_0(x). \qquad [\because P'_1(x) = 1 = P_0(x)]$$

Adding as before, we have (when n is odd):

$$P'_n(x) = (2n-1)P_{n-1}(x) + (2n-5)P_{n-3}(x) + \cdots + 5P_2(x) + P_0(x).$$

Hence $$P'_n(x) = (2n-1)P_{n-1}(x) + (2n-5)P_{n-3}(x) + (2n-9)P_{n-5}(x) + \cdots$$

The last term of the series being $3P_1(x)$ or $P_0(x)$ according as n is even or odd.

4.13 Christoffel's Summation Formula

The following formula is known as **Christoffel's Summation Formula:**

$$\sum_{r=0}^{n}(2r+1)\,P_r(x)P_r(y) = (n+1)\left[\frac{P_{n+1}(x)P_n(y)-P_{n+1}(y)P_n(x)}{(x-y)}\right]$$

Proof: From recurrence formula (I), we have

$$(2r+1)xP_r(x) = (r+1)P_{r+1}(x) + rP_{r-1}(x) \qquad \ldots(1)$$

and $$(2r+1)yP_r(y) = (r+1)P_{r+1}(y) + rP_{r-1}(y) \qquad \ldots(2)$$

Multiplying (1) by $P_r(y)$ and (2) by $P_r(x)$ and then subtracting, we have

$$(2r+1)(x-y)P_r(x)P_r(y) = (r+1)[P_{r+1}(x)P_r(y) - P_{r+1}(y)P_r(x)]$$
$$-r[P_{r-1}(y)P_r(x) - P_{r-1}(x)P_r(y)] \qquad \ldots(3)$$

Now, putting r = 0 in (3), we have

$$(x-y)P_0(x)P_0(y) = P_1(x)P_0(y) - P_1(y)P_0(x) \qquad \ldots(A_0)$$

$$[\because P_0(x) = 1 = P_0(y) and\ P_1(x) = x = P_1(y)]$$

Again, putting r = 1, 2, 3,….., (n – 1), n in (3), we have

$$3(x-y)P_1(x)P_1(y) = 2[P_2(x)P_1(y) - P_2(y)P_1(x)]$$
$$-[P_0(y)P_1(x) - P_0(x)P_1(y)] \qquad \ldots(A_1)$$

$$5(x-y)P_2(x)P_2(y) = 3[P_3(x)P_2(y) - P_3(y)P_2(x)]$$
$$-2[P_1(y)P_2(x) - P_1(x)P_2(y)] \qquad \ldots(A_1)$$

…. …. ….. ….. …. …. …. …… …..

…. …. ….. ….. …. …. …. …… …..

$$(2n-1)(x-y)P_{n-1}(x)P_{n-1}(y) = n.[P_n(x).P_{n-1}(y) - P_n(y)P_{n-1}(x)]$$
$$-n(n-1)[P_{n-2}(y)P_{n-1}(x) - P_{n-2}(x)P_{n-1}(y)] \qquad \ldots(A_{n-1})$$

$$(2n-1)(x-y)P(x)P(y) = (n+1)[P_{n+1}(x)P(y) - P_{n+1}(y)P(x)]$$

$$-n\left[P_{n-1}(y)P_n(x) - P_{n-1}(x)P_n(y)\right] \qquad \ldots(A_n)$$

Adding (A_0), (A_1), (A_2),….(A_{n-1}) and (A_n) together, we have

$$(x-y)\sum_{r=0}^{n}(2r+1)\,P_r(x)P_r(y) = (n+1)[P_{n+1}(x)P_n(y) - P_{n+1}P_n(x)]$$

or $$\sum_{r=0}^{n}(2r+1)\,P_r(x)P_r(y) = (n+1)\left[\frac{P_{n+1}(x)P_n(y) - P_{n+1}(y)P_n(x)}{(x-y)}\right]$$

4.14. Rodrigue's Formula for $P_n(x)$

The following is known as the **Rodrigue's formula for $P_n(x)$:**

$$\boldsymbol{P_n(x) = \frac{1}{n!2^n}.\frac{d^n}{dx^n}(x^2-1)^n}$$

Proof: Let us take $$y = (x^2-1)^n. \qquad \ldots(1)$$

Differentiating it w.r.t x, we have $y' = 2nx(x^2-1)^{n-1}$

Multiplying both sides by (x^2-1) and using (1), we have

$$(x^2-1)y' = 2nxy \qquad \ldots(2)$$

Differentiating it (2), (n + 1) times by Lebnitz's theorem, we have

$$(x^2-1)\frac{d^{n+2}y}{dx^{n+2}} + (n+1)\frac{d^{n+1}y}{dx^{n+1}}.2x + \frac{(n+1)n}{2!}\frac{d^ny}{dx^n}.2 = 2n\left[x.\frac{d^{n+1}y}{dx^{n+1}} + (n+1)\frac{d^ny}{dx^n}.1\right]$$

or $$(x^2-1)\frac{d^{n+2}y}{dx^{n+2}} + 2x\frac{d^{n+1}y}{dx^{n+1}} - n(n+1)\frac{d^ny}{dx^n} = 0$$

or $$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - 2x\frac{d^{n+1}y}{dx^{n+1}} + n(n+1)\frac{d^ny}{dx^n} = 0. \qquad \ldots(3)$$

Putting $\frac{d^ny}{dx^n} = z$ in (3), it becomes

$$(1-x^2)\frac{d^2z}{dx^2} - 2x\frac{dz}{dx} + n(n+1)z = 0 \qquad \ldots(4)$$

which is the Legendre's differential equation whose solution is given by z = C $P_n(x)$, where C is a constant.

∴ $$\frac{d^ny}{dx^n} = CP_n(x) \qquad \ldots(5)$$

Putting $x = 1$ in (4), and then using $P_n(1) = 1$, we get

$$\left(\frac{d^n y}{dx^n}\right)_{x=1} = C \quad , \text{ since } P_n(1) = 1. \qquad \ldots(6)$$

Now, from (1), we have

$$y = (1 - x^2)^n = (x+1)^n.(x-1)^n \qquad \ldots(7)$$

Differentiating (7) w.r.t. x, n times by Leibnitz's theorem, we have

$$\frac{d^n y}{dx^n} = (x-1)^n.\frac{d^n}{dx^n}(x+1)^n + n.\left\{\frac{d^{n-1}}{dx^{n-1}}(x+1)^n\right\}.n(x+1)^{n-1} + \cdots$$

$$+n\left\{\frac{d}{dx}(x+1)^n\right\}\frac{d^{n-1}}{dx^{n-1}}(x-1)^n + (x+1)^n\frac{d^n}{dx^n}(x-1)^n$$

$$= (x-1)^n n! + n\frac{n!}{1!}(x+1)n(x-1)^n + \cdots + n.n(x+1)^{n-1}\frac{n!}{1!}(x-1) + (x+1)^n.n!$$

Putting $x = 1$ in it, we have $\left(\frac{d^n y}{dx^n}\right)_{x=1} = (1+1)^n.n!$

Using (6) in it, we find $\quad C = 2^n.n! \qquad \ldots(8)$

Therefore, by putting the value of C from (8) in (5), we get

$$P_n(x) = \frac{1}{n!2^n.}.\frac{d^n}{dx^n}(x^2-1)^n. \qquad \ldots(9)$$

which is the required Rodrigue's formula for $P_n(x)$.

Illustrative Examples

Example 1. Using generating function for $P_n(x)$, prove the following:

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \tfrac{1}{2}(3x^2-1), P_3(x) = \tfrac{1}{3}(5x^3-3x)$$

and $\qquad P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3).$

Solution: Generating function for $P_n(x)$ is given by

$$\textstyle\sum_{n=0}^{\infty} h^n P_n(x) = (1 - 2xh + h^2)^{-1/2} = [1 - h(2x-h)]^{-1/2}$$

$$= 1 + \frac{h}{2}(2x-h) + \frac{1.3}{2.4}h^2(2x-h)^2 + \frac{1.3.5}{2.4.6}h^3(2x-h)^2 + \frac{1.3.5.7}{2.4.6.8}h^4(2x-h)^4 + \cdots$$

or $\quad P_0(x) + hP_1(x) + h^2P_2(x) + h^3P_3(x) + h^4P_4(x) + \cdots$

$$= 1 + x.h + \tfrac{1}{2}(3x^2-1)h^2\frac{1}{2}(5x^3-3x)h^3 + \tfrac{1}{8}(35x^2 + 30^2 + 3)h^4 + \cdots$$

Equating the coefficients of like powers of h on both sides, we have

$$P_0(x) = 1, P_1(x) = x,$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 + 30x^2 + 3)$$

Example 2. Express $f(x) = x^4 + 2x^3 + 2x^2 - x - 3$ in terms of Legendre's polynomials.

Solution: We know that

$$P_0(x) = 1, P_1(x) = x, \quad \ldots(1)$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \quad \ldots(2)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \quad \ldots(3)$$

$$P_4(x) = \frac{1}{8}(35x^4 + 30x^2 + 3) \quad \ldots(4)$$

Now, from (4), we have $x^4 = \frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35}$.

Again, from (3), we have $x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x$.

Next, from (2), we have $x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}$.

Also, from (1), we have $x = P_1(x), 1 = P_0(x)$.

Substituting in succession the values of x^4, x^3,….in the given polynomial, we have

$$f(x) = \frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35} + 2x^3 + 2x^2 - x - 3$$

$$= \frac{8}{35}P_4(x) + 2x^3 + \frac{20}{7}x^2 - x - \frac{108}{35}$$

$$= \frac{8}{35}P_4(x) + 2\left[\frac{2}{5}P_3(x) + \frac{3}{5}x\right] + \frac{20}{7}x^2 - x - \frac{108}{35}$$

$$= \frac{8}{35}P_4(x) + \frac{4}{5}P_3(x) + \frac{20}{7}\left[\frac{2}{3}P_2(x) + \frac{1}{3}\right] + \frac{1}{5}x - \frac{108}{35}$$

$$= \tfrac{8}{35}P_4(x) + \tfrac{4}{5}P_3(x) + \tfrac{40}{41}P_2(x) + \tfrac{1}{5}x - \tfrac{224}{105}$$

$$= \tfrac{8}{35}P_4(x) + \tfrac{4}{5}P_3(x) + \tfrac{40}{41}P_2(x) + \tfrac{1}{5}P_1x - \tfrac{224}{105}P_0(x).$$

Example 3. Prove that $P_n(0) = \begin{cases} 0 & \text{, if } n\ is\ odd \\ \frac{(-n)^{n/2}n!}{2^n\{(n/2)!\}^2}, & \text{if } n\ is\ even \end{cases}$

Solution: We know that the generating function for $P_n(x)$ is

$$\sum_{n=0}^{\infty} h^n\, P_n(x) = (1 - 2xh + h^2)^{-1/2} \qquad \ldots(1)$$

Putting $x = 0$ in both sides of (1), we have

$$\sum_{n=0}^{\infty} h^n\, P_n(0) = (1 + h^2)^{-1/2} = \{(1 - (-h^2)\}^{-1/2}$$

$$= 1 + \tfrac{1}{2}(-h^2) + \tfrac{1.3}{2.4}(-h^2)^2 + \tfrac{1.3.5}{2.4.6}(-h^2)^3 + \cdots + \tfrac{1.3.5\ldots(2r-1)}{2.4\ldots2r}(-h^2)^r + \cdots \qquad \ldots(2)$$

We observe that all the powers of h in R.H.S. of (2) are even.

Therefore, equating the coefficients of h^n from both sides of (2), we have

$$P_n(0) = 0, \text{ if n is odd.} \qquad \ldots(3)$$

Again, equating the coefficients of h^{2m} from both sides of (2), we have

$$P_{2m}(0) = \frac{1.3.5\ldots(2m-1)}{2.4.6\ldots.2m}(-1)^m = (-1)^m \frac{(2m)!}{2^{2m}(m!)^2} \qquad \ldots(4)$$

Putting 2m = n in above, we have

$$P_n(0) = \frac{(-1)^{n/2}\, n!}{2^n\{(n/2)!\}^2} \qquad \ldots(5)$$

Example 4. Prove that $(1 - 2xz + z^2)^{-1/2}$ is a solution of the equation

$$z\frac{\partial^2(zv)}{\partial z^2} + \frac{\partial}{\partial x}\left\{(1 - x^2)\frac{\partial v}{\partial x}\right\} = 0$$

Solution. Let $v = (1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n\, P_n$, where $P_n = P_n(x)$

Then, we have $\quad zv = z\sum_{n=0}^{\infty} z^n\, P_n = \sum_{n=0}^{\infty} z^{n+1}\, P_n$

$$\therefore \quad \frac{\partial^2(zv)}{\partial z^2} = \frac{\partial^2}{\partial z^2}\left[\sum_{n=0}^{\infty} z^{n+1}\, P_n\right] = \sum_{n=0}^{\infty}(n+1)z^{n+1}\, P_n$$

or $$z\frac{\partial^2}{\partial z^2}(zv) = \sum_{n=0}^{\infty}(n+1)nz^n\,P_n$$

Also, we have $$\frac{\partial v}{\partial x} = \sum_{n=0}^{\infty} z^n\,P_n,$$

∴ $$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial v}{\partial x}\right\} = \frac{\partial}{\partial x}\{(1-x^2)\sum_{n=0}^{\infty} z^n\,P'_n\}$$

$$= (1-x^2)\sum_{n=0}^{\infty} z^n\,P'_n - 2x\sum_{n=0}^{\infty} z^n\,P'_n$$

Substituting the values from (1) and (2) in the L.H.S. of the given equation, we have

$$\frac{\partial^2(zv)}{\partial z^2} + \frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial v}{\partial x}\right\} = \sum_{n=0}^{\infty}[n(n+1)z^n\,P_n + (1-x^2)z^nP'_n - 2xz^nP'_n$$

$$= \sum_{n=0}^{\infty} z^n\,[(1-x^2)P''_n - 2xP'_n + n(n+1)P_n]$$

$= \sum_{n=0}^{\infty} z^n\,.\,0$ [since P_n is a solution of Legendre’s equation]

= 0.

Example 5. Prove that $|P_n(x)| \leq 1$, when $-1 \leq x \leq 1$.

Solution: From Laplace’s first integral for $P_n(x)$, we have

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\left[x \pm i\sqrt{(1-x^2)}\cos\phi\right]^n d\phi \qquad \ldots(1)$$

If $-1 \leq x \leq 1$, then putting $x = \cos\theta$ in (1), we get

$$P_n(\cos\theta) = \frac{1}{\pi}\int_0^{\pi}(\cos\theta \pm i\sin\theta\cos\phi)^n\,d\phi.$$

∴ $$|P_n(x)| = |P_n(\cos\theta)| = \left|\frac{1}{\pi}\int_0^{\pi}(\cos\theta \pm i\sin\theta\cos\phi)^n\,d\phi\right|$$

$$= \frac{1}{\pi}\left|\int_0^{\pi}(\cos\theta \pm i\sin\theta\cos\phi)^n\,d\phi\right|$$

$$\leq \frac{1}{\pi}\int_0^{\pi}|(\cos\theta \pm i\sin\theta\cos\phi)^n|\,d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\sqrt{(\cos\theta \pm i\sin\theta\cos\phi)^2}\right]^n d\phi$$

$$\leq \frac{1}{\pi}\int_0^{\pi}\left[\sqrt{(\cos^2\theta + \sin^2\theta\cos^2\phi)}\right]^n d\phi$$

$$\leq \frac{1}{\pi}\int_0^{\pi}\left[\sqrt{(\cos^2\theta + \sin^2\theta)}\right]^n d\phi \quad \text{, since } cos^2\phi \leq 1$$

$$= \frac{1}{\pi}\int_0^{\pi} d\phi = \frac{1}{\pi}[\phi]_0^{\pi} = \frac{1}{\pi}.\pi = 1.$$

Hence, we have $|P_n(x)| \leq 1$, when $-1 \leq x \leq 1$.

Example 6. Prove $P_n(x) = P_{-(n+1)}(x)$ by using Murphy's Formula for $P_n(x)$.

Solution: Murphy's formula for $P_n(x)$ is given by

$$P_n(x) = F\left(-n, n+1; 1; \frac{1-x}{2}\right) \qquad \ldots(1)$$

From symmetric property of hypergeometric function, we have

$$F\left(-n; n+1; 1; \frac{1-x}{2}\right) = F\left(n+1; -n; 1; \frac{1-x}{2}\right) = P_{-(n+1)}(x) \qquad \ldots(2)$$

Thus, from (5) and (6), we get

$$P_n(x) = P_{-(n+1)}(x) \qquad \ldots(3)$$

EXERCISE 4

Using Rodrigue's formula for $P_n(x)$, prove the following:

1. $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^3 - 3x)$

and $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$.

2. Show that $P_n(1) = (1)$.

3. (i) Show that $P_n(-x) = (-1)^n P_n(x)$. Hence, deduce that $P_n(-1) = (-1)^n$.

(ii) Prove that $P_n(x)$ is an even or odd function of x according as n is even or odd respectively.

4. Prove that $P'_n(x) - P'_{n-2}(x) = (2n-1)P_{n-1}(x)$

5. Prove that $xP'_9(x) = P'_8(x) + 9P_9(x)$.

6. Show that $11(x^2 - 1)P'_5(x) = 30[P_6(x) - P_4(x)]$.

7. Prove that $\frac{(1+z)}{z(1-2xz+z^2)^{1/2}} - \frac{1}{z} = \sum_{n=0}^{\infty}[P_n(x) + P_{n+1}(x)]\, z^n$.

8. Show that $\frac{(1-z^2)}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)\, P_n(x)z^n$.

9. Prove that

(i) $\int P_n(x)dx = \frac{P_{n+1}(x)-P_{n-1}(x)}{(2n+1)} + C$ and (ii) $\int_x^1 P_n(x)dx = \frac{P_{n-1}(x)-P_{n+1}(x)}{(2n+1)}$

10. Show that $\int_{-1}^{+1} xP_n(x)P_{n-1}(x)dx = \frac{2n}{4n^2-1}$.

11. Prove that $\int_{-1}^{+1}(1-x^2)\,[P'_n(x)]^2dx = \frac{2n(n+1)}{2n+1}$.

12. Prove that (i) $\int_{-1}^{+1} P_n(x)dx = 0, n \neq 0$ and (ii) $\int_{-1}^{+1} P_0(x)dx = 2$.

13. Evaluate (i) $\int_{-1}^{+1} x^3 P_4(x)dx$ (ii) $\int_{-1}^{+1} x^{99} P_{100}(x)dx$ and (iii) $\int_{-1}^{+1} x^2 P_2(x)dx$.

14. If $P_n(x)$ is defined by the relation $(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$, then, show that $(1-x^2)P''_n(x) - 2xP'_n(x) + n(n+1)P_n(x) = 0$

15. Prove that $\int_{-1}^{+1}[P'_n(x)]^2\,dx = n(n+1)$.

16. Express x^8 as series in Legendre's polynomials of various degrees.

17. Express the following in terms of Legendre's polynomials:

(i) $x^2 - 5x^2 + 6x + 1$ and (ii) $5x^3 + x$

18. Prove that

(i) $x^2 + \frac{1}{2}P_0(x) + \frac{2}{3}P_2(x)$, (ii) $x^3 = \frac{3}{5}P_1(x) + \frac{2}{3}P_3(x)$.

19. If $f(x) = \begin{cases} 0, -1 < x < 0 \\ x, \ \ 0 < x < 1, \end{cases}$ then show that

$$f(x) = \tfrac{1}{4}P_0(x) + \tfrac{1}{2}P_1(x) + \tfrac{5}{16}P_2(x) - \tfrac{3}{32}P_4(x) + \cdots$$

20. Prove that $x^4 = \frac{1}{35}[8P_4(x) + 20P_2(x) + 7P_0(x)]$.

21. Solve the Legendre's differential equation $(1-x^2)y'' - 2x\,y' + n(n+1)y = 0$

about its ordinary point $x = 0$ by assuming a solution of the form $y = \sum_{m=0}^{\infty} C_m x^m$ and show that the general solution of it is given by $y = au + bv$, where

$$u = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \cdots$$

and $$v = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \cdots$$

22. Prove that :

(i) $(2n+1)x\,P_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$.

(ii) $nP_n(x) = xP'_n(x) - P'_{n-1}(x)$

(iii) $(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

(iv) $(n+1)P_n(x) = P'_{n+1}(x) - xP'_n(x)$

23. Prove that

(i) $(1-x^2)P'_n(x) = n[P_{n-1}(x) - xP_n(x)]$

(ii) $(1-x^2)P'_n(x) = (n+1)[xP_n(x) - P_{n+1}(x)]$

24. Prove that

$(2n+1)(x^2-1)P'_n(x) = n(n+1)[P_{n+1}(x) - P_{n-1}(x)]$..

ANSWERS

13.(i) 0 (ii) 0 (iii) 4/15

16. $\frac{128}{6435}P_8(x) + \frac{64}{495}P_6(x) + \frac{48}{143}P_4(x) + \frac{40}{99}P_2(x) + \frac{1}{9}P_0(x)$

17. (i) $\frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \frac{35}{5}P_1(x) - \frac{2}{3}P_0(x)$, (ii) $2P_3(x) + 4P_1(x)$.

OBJECTIVE TYPE QUESTIONS

Choose the correct alternative in the following questions:

1. Legendre's differential equation is :

(A) $(1-x^2)y'' - 2x\,y' + n(n+1)y = 0$

(B) $(1-x^2)y'' + 2x\,y' + n(n+1)y = 0$

(C) $(1-x^2)y'' - 2x\,y' + n(n+1)y = 0$

(D) $(1-x^2)y'' - 2x\,y' - n(n+1)y = 0$.

2. The value of P_n (1) is :

(A) 0 (B) 1

(C) n (D) $\frac{1}{n}$

3. Rodrigue's formula for $P_n(x)$ is :

(A) $P_n(x) = \frac{1}{2^n \lfloor n}\frac{d^n}{dx^n}(x^2-1)^n$ (B) $P_n(x) = \frac{1}{2^n \lfloor n}\frac{d^n}{dx^n}(x^2-1)^n$

(C) $P_n(x) = \frac{1}{2^n}\frac{d^n}{dx^n}(x^2-1)^n$ (D) $P_n(x) = \frac{1}{2^n}\frac{d^n}{dx^n}(x^2-1)^n$.

4. The value of $\int_{-1}^{1} P_n(x)$ dx when n $\neq$ 0 is :

(A) 0 (B) 2

(C) 1 (D) – 1.

5. The value of $\int_{-1}^{1} P_{2n}(x)$ dx is :

(A) 1 (B) 0

(C) x (D) 2n.

6. Let $f_1(x) = u, f_2(x) = x^3, f_3(x) = 1 + Ax + Bx^2$. If $f_3(x)$ is orthogonal to $f_1(x)$ and $f_2(x)$ on the interval (- 2, 2), then

(A) A = 0, B = 1 (B) A = 0, B = 0

(C) A = 0, B = $-\frac{3}{4}$ (D) None of these

7. The value of $P_n(-1)$ is :

(A) 1 (B) 0

(C) – 1 (D) $(-1)^n$

8. The following differential equation is known as:

$$(1 - x^2)y'' - 2x\,y' + n(n + 1)y = 0$$

(A) Hermite's equation (B) Legendre's equation

(C) Chebyshev equation (D) Bessel's equation

9. All roots of $P_n(x) = 0$ are :

(A) Real (B) Some real and some complex

(C) 0 (D) Complex

10. If $P_n(x)$ is Legendre's polynomial, then

(A) $P_n(-x) = P_n(x)$ (B) $P_n(-x) = (-1)^n P_n(x)$

(C) $P_n(-1) = 1$ (D) None of these

11. Two real function $f_1(x)$ and $f_2(x)$ are said to be orthogonal functions on interval $a \le x \le b$, if

(A) $\int_a^b f_1(x)\, f_2(x) dx < 0$ (B) $\int_a^b f_1(x)\, f_2(x) dx > 0$

(C) $\int_a^b f_1(x)\, f_2(x) dx = 0$ (D) None of these

12. $(1 - x^2) P'_n(x) - n P_{n-1}(x) + \cdots = 0$

(A) $P_n(x)$ (B) $2x P_n(x)$

(C) $x P_n(x)$ (D) $-nx P_n(x)$

13. The value of P_0 (x) is :

(A) 0 (B) ∞

(C) 1 (D) None of these

14. $\int P_n(x) dx = \cdots$

(A) $\frac{1}{2n+1}\{P_{n+1}(x) - P_{n-1}(x)\}$ (B) $P_{n+1}(x) - P_{n-1}(x)$

(C) $\frac{1}{2n}\{P_{n+1}(x) - P_{n-1}(x)\}$ (D) $P_{n+1}(x) + P_{n-1}(x)$.

15. The value of $\frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^2$ is :

(A) 0 (B) 1

(C) $P_n(x)$ (D) None of these

16. The value of $\int_{-1}^{+1} x^{99}\, P_{100}(x) dx$ is

(A) 1 (B) – 1

(C) 0 (D) None of these

17. $P_n(-x) = (-1)^n$ ....

(A) $P_n(x)$ (B) $P_{-n}(x)$

(C) $Q_n(x)$ (D) $Q_{-n}(x)$

18. $P_{2n+1}(0) = \cdots$

(A) 0 (B) 1

(C) n (D) 2n + 1

19. $P_n(-1) = \cdots$

(A) 0 (B) 1

(C) − 1 (D) $(-1)^n$

20. All roots of $P_n(x) = 0$ lie between

(A) -1 and +1 (B) 0 and 1

(C) 0 and n (D) – n and n

ANSWERS

1. (A) 2. (B) 3. (A) 4. (A) 5. (B) 6. (C) 7. (D) 8. (B) 9. (A) 10.(B)
11.(C) 12. (D) 13.(C) 14. (A) 15. (C) 16. (C) 17. (A) 18. (A) 19. (D) 20.(A)