5.5 Series Solutions Near a regular Singular point, part I.

Our Aim. We need to solve the general second order lin eq. $P(x)y'' + Q(x)y' + R(x)y = 0$ in the neighborhood of a regular singular point $x = x_0$ as follows

ex. Find the first three nonzero terms of the series ~~sol~~ Solution of the eq.

(1) $\boxed{2x^2 y'' - xy' + (1+x)y = 0}$ which corresponds to the larger indicial root of the D.E around $x = 0$.

Sol. $y'' - \frac{1}{2x} y + \left(\frac{1+x}{2x^2}\right) y$

step 1

$$p(x) = (x-0)\left(\frac{1}{-2x}\right) = -\frac{1}{2}$$

$$q(x) = (x-0)^2\left(\frac{1+x}{2x^2}\right) = \frac{1}{2} + \frac{1}{2}x$$

analytic at $x = 0$

$\Rightarrow$ $x = 0$ is regular sing. pt.

step 2 Indicial Equation $r(r-1) + a_0 r + b_0 = 0$

where $a_0 =$ constant term in p.

$b_0 =$ " " " q.

$\Rightarrow a_0 = -\frac{1}{2}, \quad b_0 = \frac{1}{2}$.

$\therefore$ Indicial eq. $r(r-1) - \frac{1}{2}r + \frac{1}{2} = 0$

$$r^2 - r - \frac{1}{2}r + \frac{1}{2} = 0$$

$$\Rightarrow r^2 - \frac{3}{2}r + \frac{1}{2} = 0 \Rightarrow \boxed{2r^2 - 3r + 1 = 0}$$

Step 3 Indicial roots $2r^2-3r+1=0$

$$\Rightarrow (2r-1)(r-1)=0 \Rightarrow \boxed{r_1=1}, \boxed{r_2=\tfrac{1}{2}}$$

Step 4 For $r_1=1$ let $y=\sum_{n=0}^{\infty} a_n x^{n+1}$

$$y' = \sum_{n=0}^{\infty}(n+1)a_n x^n, \quad y'' = \sum_{n=0}^{\infty} n(n+1)a_n x^{n-1}$$

Substitute y, y', y'' into Eq (1),

$$2x^2\sum_{n=0}^{\infty} n(n+1)a_n x^{n-1} - \cancel{8}\sum_{n=0}^{\infty}(n+1)a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\Rightarrow x^1\left[2\sum_{n=0}^{\infty} n(n+1)a_n x^n - \sum_{n=0}^{\infty}(n+1)a_n x^n + \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1}\right] = 0$$

$$\Rightarrow x\left[\sum_{n=0}^{\infty}\left[2n(n+1)a_n - (n+1)a_n + a_n\right]x^n + \sum_{n=1}^{\infty} a_{n-1}x^n\right] = 0$$

$$\Rightarrow x\left[(0 - \cancel{a_0} + \cancel{a_0}) + \sum_{n=1}^{\infty}\left(2n(n+1)a_n - (n+1)a_n + a_n + a_{n-1}\right)x^n\right] = 0$$

$$\Rightarrow 2n(n+1)a_n - (n+1)a_n + a_n + a_{n-1} = 0, \quad n = 1, 2, \ldots$$

or $$(2n^2+2n-n-\cancel{1}+\cancel{1})a_n = -a_{n-1}$$

$$n(2n+1)a_n = -a_{n-1}$$

$$\Rightarrow a_n = \frac{-1}{n(2n+1)}a_{n-1}, \quad n = 1, 2, \ldots$$

thus, $a_1 = \dfrac{-a_0}{3\cdot 1}$, $a_2 = -\dfrac{a_1}{5\cdot 2} = \dfrac{a_0}{(3\cdot 5)(1\cdot 2)}$

$$a_3 = -\frac{a_2}{7\cdot 3} = -\frac{a_0}{(3\cdot 5\cdot 7)(1\cdot 2\cdot 3)} \quad \text{-----}$$

In general, $a_n = \dfrac{(-1)^n}{[3\cdot5\cdot7\cdot\,---\,(2n+1)]\,n!}\,a_0,\quad n\geq 4.$

$$\therefore y = \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$= x\left[a_0 + a_1 x + a_2 x^2 + \cdots\cdots\right.$$

$$= x\left[a_0 - \frac{a_0}{3\cdot1}x + \frac{a_0}{1\cdot2\cdot3\cdot5}x^2 + \cdots\right.$$

$$= a_0\left[\underbrace{x - \frac{x^2}{3} + \frac{x^2}{30} + \cdots}_{y_1}\right.$$

<u>steps</u> For $\boxed{r_2 = \frac{1}{2}}$

let $y = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$, $y' = \sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)a_n x^{n-\frac{1}{2}}$

$$y'' = \sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)a_n x^{n-\frac{3}{2}}.$$

<u>Subst.</u>

$$2x^2\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)a_n x^{n-\frac{3}{2}} - \sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)a_n x^{n+\frac{1}{2}}$$

$$+\sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} + \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}} = 0.$$

$$\Rightarrow \sum_{n=0}^{\infty}(2n+1)\left(n-\frac{1}{2}\right)a_n x^{n+\frac{1}{2}} - \sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)a_n x^{n+\frac{1}{2}} + \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} + \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[(2n+1)\left(n-\frac{1}{2}\right) - \left(n+\frac{1}{2}\right) + 1\right]a_n x^{n+\frac{1}{2}} + \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}} = 0.$$

$$\Rightarrow \sum_{n=0}^{\infty}\left(2n^2 - n + \cancel{n} - \cancel{\tfrac{1}{2}} - \cancel{n} - \cancel{\tfrac{1}{2}} + \cancel{1}\right) a_n x^{n+\frac{1}{2}} + \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}} = 0 .$$

$$x^{\frac{1}{2}}\left[\sum_{n=0}^{\infty}(2n^2-n)\, a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1}\right] = 0 .$$

$$\sum_{n=0}^{\infty}(2n^2-n)\, a_n x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0 ,$$

$$0 \cdot a_0 x^0 + \sum_{n=1}^{\infty}\left[n(2n-1)a_n + a_{n-1}\right] x^n = 0 ,$$

$$\Rightarrow \quad a_n = -\frac{a_{n-1}}{n(2n-1)}, \quad n = 1, 2, \ldots$$

$$a_1 = -\frac{a_0}{1 \cdot 1} = -a_0 .$$

$$a_2 = -\frac{a_1}{2 \cdot 3} = \frac{a_0}{(1 \cdot 2)(1 \cdot 3)} = \frac{a_0}{6} .$$

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$$a_3 = -\frac{a_2}{3 \cdot 5} = -\frac{a_0}{(1 \cdot 2 \cdot 3)(1 \cdot 3 \cdot 5)} = -\frac{a_0}{90}$$

$\vdots$

$$y = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$

$$= x^{\frac{1}{2}}\left[a_0 + a_1 x + a_2 x^2 + \cdots\right.$$

$$= x^{\frac{1}{2}}\left[a_0 - a_0 x + \frac{a_0}{6}x^2 - \frac{a_0}{90}x^3 + \cdots\right.$$

$$= a_0 \; x^{\frac{1}{2}}\left[1 - x + \frac{1}{6}x^2 - \frac{1}{90}x^3 + \cdots\right]$$

y_2

$\Rightarrow$ H.W.

Ex. (H.W) Find the first three nonzero terms of the series solution of the eq.

$$4x\,y'' + 2y' + y = 0$$ about $x = 0$.

which corresponds to the larger indicial roots of the D.E.

Ans. $\boxed{r_1 = 0}$, $\boxed{r_2 = \frac{1}{2}}$

For $\boxed{r_1 = 0}$ $\Rightarrow$ the recurrence relation is

$$a_n = \frac{-1}{2n(2n-1)}a_{n-1}, \quad n = 1, 2, \dots$$

For $\boxed{r_2 = \frac{1}{2}}$ $\Rightarrow$ the recurrence relation is

$$a_n = \frac{-1}{2n(2n+1)}a_{n-1}, \quad n = 1, 2, \dots$$