جامعة بيرزيت
BIRZEIT UNIVERSITY

Department of Mathematics

Homework 1

Ordinary Differential Equations (Math 331) **July 1, 2018**

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Question one: Solve the following ODE.

1. $xdy = \left(xe^{\frac{y}{x}} + y + x\right)dx.$

2. $\frac{dy}{dx} = \frac{xy+3x-y-3}{xy-2x+4y-8}, \quad y \neq 2,\ y \neq -3.$

انتبه

3. $x^2 \sin\left(\frac{y^2}{x^2}\right) - 2y^2 \cos\left(\frac{y^2}{x^2}\right)dx + 2xy\cos\left(\frac{y^2}{x^2}\right)dy = 0.$

4. $x^2(x-2)\frac{dy}{dx} + x(x-2)y = 2, \quad y(1) = 1.$

5. $(6x+1)y^2y' + 2y^3 + 3x^2 = 0.$

6. $y' = \tan^2(x+y)$. (**Hint:** Let $u = x+y$)

7. $(t^2+1)y' - 4t(y+\sqrt{y}) = 0$. (Try by two methods!)

8. $y' = 2 + \sqrt{y-2x+3}$. (**Hint:** Let $u = y-2x+3$)

9. $(x\sin y + \cos y)y' + (x+y)\sin y = 0.$

10. $\frac{1}{z}\frac{dz}{dx} + \frac{1}{x}\ln z = 3x$. (**Hint:** Let $u = \ln z$)

11. $(x^2+1)y' + 3x^3y = 6xe^{-\frac{3}{2}x^2}, \quad y(0) = 1.$

12. $x(x^2+4)\ln y\, y' = 12y^{-2}.$

Question Two: An object with temperature $108°F$ was placed outside where the temperature is $-20°F$. At 00 : 00 the temperature of the object is $60°F$ and at 00 : 02 the temperature of the object is $30°F$. At what time was the object placed outside?

Good Luck

①

H.W # 1

Question one 1) $x\,dy = (x e^{\frac{y}{x}} + y + x)\,dx$

$$\Rightarrow \frac{dy}{dx} = \frac{x e^{\frac{y}{x}} + y + x}{x} = e^{\frac{y}{x}} + \frac{y}{x} + 1 = F\left(\frac{y}{x}\right)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = e^{y/x} + \left(\frac{y}{x}\right) + 1} \text{ — ①} \quad \therefore \text{ homog}$$

let $\boxed{y = vx}$ — ② $\Rightarrow \boxed{\frac{dy}{dx} = x\frac{dv}{dx} + v}$ — ③

put ② & ③ into ①, we get

$$x\frac{dv}{dx} + \cancel{v} = e^{v} + \cancel{v} + 1$$

$$\Rightarrow \int \frac{dv}{e^{v}+1} = \int \frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{e^{-v}}{1+e^{-v}}\,dv = \ln|x| + c$$

put $u = 1 + e^{-v}$ ... continue.

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2) $$\frac{dy}{dx} = \frac{x(y+3) - (y+3)}{x(y-2) + 4(y-2)} = \frac{(y+3)(x-1)}{(y-2)(x+4)}$$

$$\Rightarrow \boxed{\int \frac{y-2}{y+3}\,dy = \int \frac{x-1}{x+4}\,dx} \rightarrow (*)$$

For $\int \frac{y-2}{y+3}\,dy = \int \frac{y+3-5}{y+3}\,dy = \int \left[1 - \frac{5}{y+3}\right] dy$

$= y - 5\ln|y+3| + c.$

For $\int \frac{x-1}{x+4}\,dx = \int \frac{x+4-5}{x+4}\,dx$

$= \int \left[1 - \frac{5}{x+4}\right] dx$

$= x - 5\ln|x+4| + c.$

Return to (*), we get

$y - 5\ln|y+3| = x - 5\ln|x+4| + c$ is the solution.

(3) $\boxed{\left[x^2 \sin\left(\frac{y^2}{x^2}\right) - 2y^2 \cos\left(\frac{y^2}{x^2}\right)\right] dx + 2xy \cos\left(\frac{y^2}{x^2}\right) dy = 0}$ (*)

homog. let $\boxed{y = vx}$ — (1) $\Rightarrow$ $\boxed{\frac{dy}{dx} = v + x\frac{dv}{dx}}$ — (2)

(1) & (2) into (*),

$\left[x^2 \sin(v^2) - 2v^2x^2 \cos(v^2)\right] dx + 2x^2 v \cos(v^2)\left[v\,dx + x\,dv\right] = 0$

$\Rightarrow x^2 \sin(v^2)\,dx \cancel{- 2v^2x^2 \cos(v^2)\,dx} \cancel{+ 2x^2v^2 \cos(v^2)\,dx}$

$+ 2x^3 v \cos(v^2)\,dv = 0.$

$\Rightarrow x^2 \sin(v^2)\,dx = -2x^3 v \cos(v^2)\,dv.$

or $+\frac{1}{x}\,dx = -2\frac{v\cos(v^2)}{\sin(v^2)}\,dv$

or $\int \frac{1}{x}\,dx = -2\int v \cot(v^2)\,dv$

let $u = v^2$ ... continue.

4 $\frac{dy}{dx} + \frac{1}{x}y = \frac{2}{x^2(x-2)}$, $y(1)=1$ linear in y.

$$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x, \quad x>0.$$

$$\Rightarrow y = \frac{1}{x}\left[\int \frac{2}{x^2(x-2)} \cdot x \, dx + c\right]$$

$$= \frac{1}{x}\left[\int \frac{2}{x(x-2)} dx + c\right]$$

now, $\boxed{\frac{2}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}}$

$$\Rightarrow 2 = A(x-2) + Bx.$$

$\boxed{x=2} \Rightarrow 2 = 2B \Rightarrow B=1$

$x=0 \Rightarrow 2 = -2A \Rightarrow A=-1$

$$\Rightarrow y = \frac{1}{x}\left[\int \left(\frac{-1}{x} + \frac{1}{x-2}\right) dx + c\right]$$

$$= \frac{1}{x}\left[-\ln|x| + \ln|x-2| + c\right]$$

$$\Rightarrow y = \frac{1}{x}\ln\left|\frac{x-2}{x}\right| + \frac{c}{x}$$

$y(1) = \ln 1 + c = 1 \Rightarrow c=1$

$$\therefore y = \frac{1}{x}\ln\left|\frac{x-2}{x}\right| + \frac{1}{x}.$$

5 $(6x+1)y^2y' + 2y^3 + 3x^2 = 0$

$$\Rightarrow \boxed{y' + \frac{2}{(6x+1)}y = \frac{-3x^2}{6x+1}y^{-2}} \quad -(1)$$

is Bernoulli with $n = -2$

$$\text{let } \boxed{y = v^{\frac{1}{1-n}} = v^{\frac{1}{3}}} \quad -(2)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{3}v^{-\frac{2}{3}}\frac{dv}{dx}} \quad -(3)$$

(2) & (3) into (1),

$$\frac{1}{3}v^{-\frac{2}{3}}\frac{dv}{dx} + \frac{2}{6x+1}v^{\frac{1}{3}} = \frac{-3x^2}{6x+1}v^{-\frac{2}{3}}.$$

Divide by $\frac{1}{3}v^{-\frac{2}{3}}$ to get

$$\boxed{\frac{dv}{dx} + \frac{6}{6x+1}v = -\frac{9x^2}{6x+1}} \quad \text{is lin in } v$$

Now, you can continue (please do it!).

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6 $y' = \tan^2(x+y)$ — (*)

$$\text{let } u = x+y \Rightarrow \boxed{y = u - x} \quad -(1)$$

$$\boxed{\frac{dy}{dx} = \frac{du}{dx} - 1} \quad -(2)$$

(1) & (2) into (*),

$$\frac{du}{dx} - 1 = \tan^2 u \Rightarrow \frac{du}{dx} = 1 + \tan^2 u$$

$$\Rightarrow \frac{du}{\sec^2 u} = dx$$

5

$$\Rightarrow \int \cos^2 u \, du = \int dx$$

$$\Rightarrow \int \frac{1+\cos(2u)}{2} \, du = x + c$$

$$\Rightarrow \frac{1}{2}\left(u + \frac{\sin(2u)}{2}\right) = x + c$$

$$\Rightarrow \frac{1}{2}\left(x+y + \frac{\sin(2x+2y)}{2}\right) = x + c$$

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7 $(t^2+1)\,y' - 4t\,(y+\sqrt{y}) = 0$

Method 1 (Bernoulli done in the ~~note~~ class (see it))

Method 2 Seperable

$$(t^2+1)\frac{dy}{dt} = 4t\,(y+\sqrt{y})$$

$$\Rightarrow \int \frac{dy}{y+\sqrt{y}} = \int \frac{4t}{t^2+1}$$

$$\Rightarrow \int \frac{dy}{y+\sqrt{y}} = 2\ln(t^2+1) + c$$

Try!! (let $w = \sqrt{y}$)

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8 $y' = 2 + \sqrt{y-2x+3}$ — (*)

let $\boxed{u = y-2x+3}$ — (1) $\Rightarrow y = u + 2x - 3$

$$\boxed{\frac{dy}{dx} = \frac{du}{dx} + 2} \text{ — (2)}$$

(1) & (2) in (*) $\quad \frac{du}{dx} \cancel{+2} = \cancel{2} + \sqrt{u} \Rightarrow \int u^{-\frac{1}{2}} du = \int dx$

6

$\Rightarrow \quad 2u^{\frac{1}{2}} = x + c$

$\Rightarrow \quad 2\sqrt{y - 2x + 3} = x + c.$

Try for this $\quad y' = \sqrt{y - 2x + 3} \quad$??!

(9) $\underbrace{(x \sin y + \cos y)}_{N} dy + \underbrace{(x+y) \sin y}_{M} dx = 0$ — ($*$)

$\frac{\partial M}{\partial y} = (x+y) \cos y + \sin y.$

$\frac{\partial N}{\partial x} = \sin y \quad \Rightarrow \quad M_y \neq N_x$ is not exact.

Now, $\quad \frac{M_y - N_x}{M} = \frac{(x+y)\cos y + \cancel{\sin y} - \cancel{\sin y}}{(x+y) \sin y} = \cot y.$

$\Rightarrow \quad \mu(y) = e^{-\int \cot y} = \csc y.$

multiply both sides of ($*$) by $\mu = \csc y.$

$\boxed{(x + \cot y)\, dy + (x+y)\, dx = 0} \; \text{—} \; (**)$

($**$) is exact now you can continue.

(10) $\frac{1}{z} \frac{dz}{dx} + \frac{1}{x} \ln z = 3x$ — ($*$)

let $\boxed{u = \ln z}$ — (1) $\Rightarrow \boxed{\frac{du}{dx} = \frac{1}{z} \frac{dz}{dx}}$ — (2)

(1) & (2) into ($*$), $\boxed{\frac{du}{dx} + \frac{1}{x} u = 3x}$ is lin in u

continue.

11 $(x^2+1)y' + 3x^3 y = 6x\, e^{-\frac{3}{2}x^2}$, $y(0)=1$

$$\frac{dy}{dx} + \frac{3x^3}{x^2+1} y = \frac{6x}{x^2+1} e^{-\frac{3}{2}x^2}$$ lin. in y.

$$\mu(x) = e^{\int \frac{3x^3}{x^2+1} dx}$$

$$\begin{array}{r} 3x \\ x^2+1 \overline{)\, 3x^3 \qquad} \\ -\ 3x^3 + 3x \\ \hline -3x \end{array}$$

$$= e^{\int \left(3x - \frac{3x}{x^2+1}\right) dx}$$

$$= e^{\frac{3x^2}{2} - \frac{3}{2}\ln|x^2+1|}$$

$$= e^{\frac{3x^2}{2}} \cdot e^{\ln(x^2+1)^{-\frac{3}{2}}} = (x^2+1)^{-\frac{3}{2}} e^{\frac{3x^2}{2}}.$$

$$\therefore y = (x^2+1)^{\frac{3}{2}} e^{-\frac{3x^2}{2}} \left[\int \frac{6x}{x^2+1} \cancel{e^{-\frac{3}{2}x^2}} \cdot (x^2+1)^{-\frac{3}{2}} \cancel{e^{\frac{3x^2}{2}}} dx + c \right]$$

$$= (x^2+1) e^{-\frac{3x^2}{2}} \left[\int 6x (x^2+1)^{-\frac{5}{2}} dx + c \right]$$

let $u = x^2+1$ .... continue.

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12 $x(x^2+4) \ln y \frac{dy}{dx} = 12 y^{-2}$ (seperable).

$$\Rightarrow \int y^2 \ln y\, dy = \int \frac{12}{x(x^2+4)} dx$$

For $\int y^2 \ln y\, dy$, $u = \ln y$ $dv = y^2 dy$

$du = \frac{1}{y} dy$ $v = \frac{y^3}{3}$

$$\Rightarrow \int y^2 \ln y \, dy = \frac{y^3}{3}\ln y - \int \frac{y^2}{3}\,dy.$$

$$= \boxed{\frac{y^3}{3}\ln y - \frac{y^3}{9} + c} \quad — \ 1$$

For $\int \frac{12}{x(x^2+4)}\,dx$, $\frac{12}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+c}{x^2+4}$

$$\Rightarrow 12 = A(x^2+4) + Bx^2 + cx$$

$$\Rightarrow \boxed{A + B = 0}, \quad \boxed{C = 0}, \quad 4A = 12$$

$$A = 3$$

$$\Rightarrow B = -3$$

$$\therefore \int \frac{12}{x(x^2+4)}\,dx = \int \left[\frac{3}{x} - \frac{3x}{x^2+4}\right] dx$$

$$= \boxed{3\ln|x| - \frac{3}{2}\ln(x^2+4) + c} \quad 2$$

1 & 2 $\Rightarrow$ the solution of the d.e is

$$\frac{y^3}{3}\ln y - \frac{y^3}{9} = 3\ln|x| - \frac{3}{2}\ln(x^2+4) + c.$$

Q2) $\frac{dT}{dt} = k(T - T_m) \Longrightarrow \int \frac{dT}{T - T_m} = \int k\,dt$

$$\Rightarrow \ln|T - T_m| = kt + c$$

$$\Rightarrow \boxed{T = T_m + Ae^{kt}}$$

$T(0) = -20 + A = 60 \Rightarrow A = 80°$

$$\Rightarrow T = -20 + 80e^{kt}$$

$T_m = -20°F$

$T(0) = 60°F$

$T(2) = 30°F$

$T(t) = 108 \Rightarrow t = ??$

$$T(2) = -20 + 80e^{2k} = 30$$

$$\Rightarrow 80e^{2k} = 50 \Rightarrow 2k = \ln\left(\frac{5}{8}\right)$$

$$\Rightarrow \boxed{k = \frac{1}{2}\ln\left(\frac{5}{8}\right)}$$

$$\therefore T(t) = -20 + 80e^{\frac{1}{2}\ln\left(\frac{5}{8}\right)t}$$

$$108 = -20 + 80\left(\frac{5}{8}\right)^{\frac{t}{2}}$$

$$128 = 80\left(\frac{5}{8}\right)^{\frac{t}{2}}$$

$$\frac{128}{80} = \frac{64}{40} = \frac{32}{20} = \frac{16}{10} = \frac{8}{5}$$

$$\Rightarrow \frac{128}{80} = \left(\frac{5}{8}\right)^{\frac{t}{2}}$$

$$\Rightarrow \left(\frac{8}{5}\right) = \left(\frac{5}{8}\right)^{t/2} \Rightarrow \frac{t}{2} = -1 \Rightarrow t = -2$$

$\Rightarrow$ The object was placed at 23:58 p.m.