~~Key~~ Key Solutions

جامعة بيرزيت

BIRZEIT UNIVERSITY

Department of Mathematics

Homework II

Differential Equations (Math 331) **August 2, 2018**

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Name: ______________ **Number:** __________ **Section:** __________

Question One. Determine a lower bound for the radius of convergence of the series solutions about $x_0 = 3$ for the differential equation

$$2x(x-5)^2y'' + xy' + (x-5)y = 0.$$

Question Two. Consider the following IVP:

$$(2-x^2)y'' + 2(x-1)y' + 4y = 0, \quad y(0) = 6; \quad y'(0) = 2.$$

Find the first four nonzero terms of the series solution about the ordinary point $x_0 = 0$.

Question Three. Determine the singular points of the differential equation

$$(x^2-4)y'' + \frac{x}{x+2}y' + \frac{3}{x-2}y = 0,$$

and classify them as regular or irregular.

Question Four. Find two linearly independent power series solutions of the ODE

$$(x-2)y'' + (x-1)y' + y = 0.$$

Give the first three nonzero terms for each series solution.

Question Five.

(a) Find the Laplace transform of

$$f(t) = te^{-t}\sin^2(3t).$$

(b) Find the inverse Laplace transform of

$$F(s) = \frac{s^2+s-1}{2s^2+2s+1}$$

Question Six. Use the Laplace transform to solve the following BVP

$$y'' + 9y = \cos 2t, \quad y(0) = 1, \quad y\left(\frac{\pi}{2}\right) = -1.$$

Good Luck

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H.w #2

Q1) the Singular pts are $2x(x-5)^2=0$

$\Rightarrow x=0$ or $x=5$.

$\rho_1 = \text{dist}(3,5) = 2$, $\rho_2 = \text{dist}(3,0) = 3$.

$\Rightarrow$ minimal radius $= 2$.

Q2) $(2-x^2)y'' + 2(x-1)y' + 4y = 0$, $y(0)=6$, $y'(0)=2$

let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$

$\boxed{a_0 = 6}$, $\boxed{a_1 = 2}$, $a_2 = \frac{y''(0)}{2!}$, where

$(2-x^2)y'' = -2(x-1)y' - 4y$ — (*)

put $x=0 \Rightarrow$ $2y''(0) = 2y'(0) - 4y(0) = 2(2) - 4(6) = -20$

$\Rightarrow y''(0) = -10 \Rightarrow \boxed{a_2 = \frac{-10}{2!} = -5}$

To find a_3, we differentiate (*) w.r.to x,

$-2xy'' + (2-x^2)y''' = -2y' - 2(x-1)y'' - 4y'$

put $x=0 \Rightarrow$ $2y'''(0) = -2y'(0) + 2y''(0) - 4y'(0)$

$\Rightarrow y'''(0) = \frac{-2(2) + 2(-10) - 4(2)}{2}$

$= \frac{-4-20-8}{2} = -16$

$\boxed{a_3 = \frac{y'''(0)}{3!} = \frac{-16}{6} = -\frac{8}{3}}$.

$\therefore y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$= 6 + 2x - 5x^2 - \frac{8}{3}x^3 + \cdots$

Q3) $y'' + \frac{x}{(x+2)(x^2-4)} y' + \frac{3}{(x-2)(x^2-4)} y = 0$

or $$\boxed{y'' + \frac{x}{(x+2)^2(x-2)} y' + \frac{3}{(x-2)^2(x+2)} y = 0}$$

$\Rightarrow p(x) = \frac{x}{(x+2)^2(x-2)}$, $q(x) = \frac{3}{(x-2)^2(x+2)}$

Singular pts $\boxed{x=-2}$, $\boxed{x=2}$.

For $\boxed{x=-2}$, $\lim_{x\to -2} (x+2)\,p(x) = \lim_{x\to -2} \frac{x}{(x+2)(x-2)}$

= infinite limit.

$\Rightarrow x=-2$ is irregular singular point.

For $\boxed{x=2}$, $\lim_{x\to 2} (x-2)\,p(x) = \lim_{x\to 2} \cancel{(x-2)} \frac{x}{(x+2)^2\cancel{(x-2)}}$

$= \frac{2}{16} = \frac{1}{8}$. exists

and

$\lim_{x\to 2} (x-2)^2 q(x) = \lim_{x\to 2} \cancel{(x-2)^2} \frac{3}{(x+2)\cancel{(x-2)^2}} = \frac{3}{4}$

exists

$\Rightarrow x=2$ is regular singular point.

Q4) $(x-2)y'' + (x-1)y' + y = 0$; Take $x_0 = 0$ ordinary pt.

Let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Subst.

$$(x-2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + (x-1)\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} - 2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

$$\Rightarrow \sum_{n=1}^{\infty} (n+1)(n) a_{n+1} x^n - 2\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

$$\Rightarrow -2(2)(1)a_2 x^0 - a_1 x^0 + a_0 x^0 + \sum_{n=1}^{\infty}\left[n(n+1)a_{n+1} - 2(n+2)(n+1)a_{n+2} + n a_n - (n+1)a_{n+1} + a_n\right]x^n = 0$$

$$\Rightarrow \boxed{-4a_2 - a_1 + a_0 = 0} \quad \text{and}$$

$$\left[n(n+1) - (n+1)\right]a_{n+1} - 2(n+2)(n+1)a_{n+2} + (n+1)a_n = 0, \quad n \geq 1.$$

$$\Rightarrow \boxed{a_2 = \frac{a_0 - a_1}{4}},$$

$$\cancel{(n+1)}(n-1)\, a_{n+1} - 2(n+2)\cancel{(n+1)}\, a_{n+2} + \cancel{(n+1)}\, a_n = 0$$

$$\Rightarrow \boxed{a_{n+2} = \frac{(n-1)\, a_{n+1} + a_n}{2(n+2)}, \quad n \geq 1.}$$

is the recurrence relation.

$$n=1 \Rightarrow \boxed{a_3 = \frac{a_1}{2(3)} = \frac{1}{6} a_1}$$

$$\boxed{n=2} \Rightarrow a_4 = \frac{a_3 + a_2}{2(4)} = \frac{\frac{1}{6}a_1 + \frac{a_0 - a_1}{4}}{8}$$

$$= \left[\left(\frac{1}{6} - \frac{1}{4}\right) a_1 + \frac{1}{4} a_0\right] \cdot \frac{1}{8}$$

$$\boxed{a_4 = \frac{-1}{96} a_1 + \frac{1}{32} a_0}.$$

$$\therefore\ y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_0 + a_1 x + \left(\frac{1}{4}a_0 - \frac{1}{4}a_1\right)x^2 + \left(-\frac{1}{96}a_1 + \frac{1}{32}a_0\right)x^4 + \frac{1}{6}a_1 x^3 + \cdots$$

$$= a_0\left(1 + \frac{1}{4}x^2 + \frac{1}{32}x^4 + \cdots\right)$$

$$+ a_1\left(x - \frac{1}{4}x^2 - \frac{1}{96}x^4 + \frac{1}{6}x^3 + \cdots\right)$$

$$\Rightarrow y_1 = 1 + \frac{1}{4}x^2 + \frac{1}{32}x^4 + \cdots$$

$$y_2 = x - \frac{1}{4}x^2 - \frac{1}{96}x^4 + \frac{1}{6}x^3 + \cdots$$

(5)

Q5) a) $\mathcal{L}\{ t e^{-t} \sin^2(3t) \}$

$= \mathcal{L}\{ t e^{-t} (\frac{1}{2} - \frac{1}{2}\cos 6t) \}$

$= \frac{1}{2}\mathcal{L}\{ t e^{-t} \} - \frac{1}{2}\mathcal{L}\{ t e^{-t}\cos 6t \}$

$= \frac{1}{2}\mathcal{L}\{t\}_{s \to s+1} - \frac{1}{2}(-1)^1 \frac{d}{ds}\mathcal{L}\{ e^{-t}\cos 6t \}$

$= \frac{1}{2}\frac{1}{s^2}\Big|_{s \to s+1} + \frac{1}{2}\frac{d}{ds}\left(\frac{s+1}{(s+1)^2+36} \right)$

$= \frac{1}{2}\frac{1}{(s+1)^2} + \frac{1}{2}\left[\frac{[(s+1)^2+36]\cdot 1 - (s+1)\cdot 2(s+1)}{((s+1)^2+36)^2} \right]$

$= \frac{1}{2}\frac{1}{(s+1)^2} + \frac{1}{2}\frac{-(s+1)^2+36}{((s+1)^2+36)^2}$

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b) $\mathcal{L}^{-1}\left\{ \frac{s^2+s-1}{2s^2+2s+1} \right\}$

$$\begin{array}{r|l} & \frac{1}{2} \\ \hline 2s^2+2s+1 & s^2+s-1 \\ & -s^2 \mp s \mp \frac{1}{2} \\ \hline & -\frac{3}{2} \end{array}$$

$= \mathcal{L}^{-1}\left\{ \frac{1}{2} - \frac{\frac{3}{2}}{2s^2+2s+1} \right\}$

$= \frac{1}{2}\mathcal{L}^{-1}\{1\} - \frac{3}{2}\mathcal{L}^{-1}\left\{ \frac{1}{2(s^2+s+\frac{1}{4}-\frac{1}{4})+1} \right\}$

$$= \frac{1}{2}\delta(t) - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{2(s+\frac{1}{2})^2+\frac{1}{2}}\right\}$$

$$= \frac{1}{2}\delta(t) - \frac{3}{4}\frac{2}{1}\mathcal{L}^{-1}\left\{\frac{1\cdot\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{1}{4}}\right\}$$

$$= \frac{1}{2}\delta(t) - \frac{3}{2}e^{-\frac{1}{2}t}\sin(\tfrac{1}{2}t)$$

Q6) $y'' + 9y = \cos 2t$, $y(0)=1$, $y(\frac{\pi}{2}) = -1$

Sol. Take L.T:

$$s^2Y - sy(0) - y'(0) + 9Y = \frac{s}{s^2+4}$$

$$(s^2+9)Y - s - k = \frac{s}{s^2+4}$$ (we assume that $y'(0) = k$). we will find it later

$$\Rightarrow Y = \left(s + k + \frac{s}{s^2+4}\right)\cdot\frac{1}{s^2+9}$$

$$\boxed{Y = \frac{s}{s^2+9} + \frac{k}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}}$$

Now, $$\frac{s}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

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$\Rightarrow \quad s = (As+B)(s^2+9) + (Cs+D)(s^2+4).$

$$\left.\begin{aligned} \Rightarrow s^3 &: A + C = 0 \\ s^2 &: B + D = 0 \\ s &: 9A + 4C = 1 \\ \text{Constants: } s^0 &: 9B + 4D = 0 \end{aligned}\right\} \Rightarrow \boxed{A=\tfrac{1}{5}}, \boxed{B=0}, \boxed{C=-\tfrac{1}{5}}, \boxed{D=0}$$

check!!

$$\therefore \quad Y = \frac{s}{s^2+9} + \frac{k}{3}\cdot\frac{3}{s^2+9} + \frac{1}{5}\frac{s}{s^2+4} - \frac{1}{5}\frac{s}{s^2+9}$$

$$\Rightarrow y = \mathcal{L}^{-1}\{Y\}.$$

$$= 1\cos(3t) + \frac{k}{3}\sin(3t) + \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t$$

$$\boxed{y = \frac{4}{5}\cos(3t) + \frac{k}{3}\sin(3t) + \frac{1}{5}\cos(2t)}$$

It remains to find k, we apply $y(\frac{\pi}{2}) = -1$

$$\Rightarrow -1 = \frac{4}{5}\cos\left(\frac{3\pi}{2}\right) + \frac{k}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{5}\cos(\pi)$$

$$-1 = 0 + \frac{k}{3}(-1) + \frac{1}{5}(-1)$$

$$\Rightarrow -\frac{4}{5} = -\frac{k}{3} \Rightarrow \boxed{k = \frac{12}{5}}.$$

$$\therefore \quad y = \frac{4}{5}\cos(3t) + \frac{4}{5}\sin(3t) + \frac{1}{5}\cos(2t).$$

is the solution.

Good Luck