BIRZEIT UNIVERSITY

Department of Mathematics

First Exam

Differential Equations (Math 331) **July 8, 2018**

Instructor: Dr. Ala Talahmeh **Time: 1:30 hours** **Summer 2018**

Name: ____________________ **Number:** ____________ **Section:** ____________

Question one (30 points). Circle the correct answer.

1. One of the following differential equations is **both** seperable and linear.

   (a) $y' - 4ty = e^t$

   (b) $\frac{1}{t}y' = t + y$

   (c) $y' = \sin(t - 3y)$

   (d) $t^2 y' = 2y$

$t^2\,dy = 2y\,dt$

$\frac{dy}{2y} = \frac{dt}{t^2}$ seperable.

OR $y' - \frac{2}{t^2}y = 0$ lin. in y

2. The **integrating factor** for the following first-order linear differential equation

$$\frac{1}{2}\frac{dy}{dx} - \frac{1}{x}y = x^2, \; x \neq 0 \text{ is}$$

   (a) $\mu(x) = \sqrt{x}$

   (b) $\mu(x) = x^{-2}$

   (c) $\mu(x) = x^2$

   (d) $\mu(x) = -2x$

$\frac{dy}{dx} - \frac{2}{x}y = 2x^2$

$\mu = e^{\int -\frac{2}{x}dx} = e^{-2\ln|x|} = x^{-2}$

3. **The behavior** of the solution for the differential equation $5y' + y = 0$

   (a) converges to -5

   (b) diverges

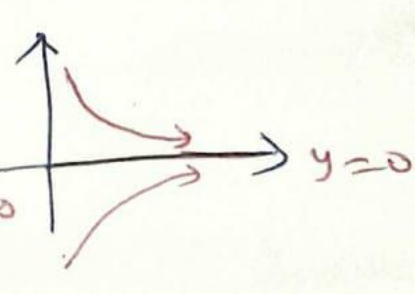
   (c) converges to 0

   (d) converges to $-\frac{1}{5}$

$5y' = -y$

$\Rightarrow y' = -\frac{y}{5}$

Equil. solution $-\frac{y}{5} = 0 \Rightarrow y = 0$



$y \to 0$ as $t \to \infty$

4. The differential equation

$$y'' + \left(\frac{x^2 \cdot \sin x}{e^2 \cdot \sqrt{\pi}}\right)(y')^3 + xy = 10 \text{ is}$$

(a) an ordinary, linear differential equation of order 2

(b) an ordinary, nonlinear differential equation of order 2

(c) an ordinary, linear differential equation of order 3

(d) a partial, linear differential equation of order 2

5. The general solution of the differential equation

$$\frac{dy}{dx} = 1 + x + y + xy \text{ is}$$

$$\frac{dy}{dx} = (1+x) + y(1+x)$$

$$\Rightarrow dy = (1+x)(1+y)\,dx$$

$$\Rightarrow \frac{dy}{y+1} = (1+x)\,dx$$

$$\Rightarrow \ln|y+1| = x + \frac{x^2}{2} + C$$

(a) $y = x + \frac{1}{2}x^2 + C$

(b) $y = \ln|1+x| \cdot y + C$

(c) $y = x + \frac{1}{2}x^2 + xy + \frac{1}{2}x^2 y + C$

(d) $\ln|1+y| = x + \frac{1}{2}x^2 + C$

6. The general solution of the differential equation

$$\underbrace{(3x^3y^2 + y^4 + 4xy^3)}_{N}dy + \underbrace{(3x^2y^3 + y^4)}_{M}dx = 0 \text{ is}$$

(a) $y = 5x^3y^3 + 5xy^4 + y^5$

(b) $5x^3y^3 + 5xy^4 + y^5 = C$

(c) $10x^3y^3 + 10xy^4 + y^5 = C$

(d) None of the above

$M_y = 9x^2y^2 + 4y^3$, $N_x = 9x^2y^2 + 4y^3$

$\Rightarrow$ eq is exact.

let $\boxed{\psi_x = 3x^2y^3 + y^4}$ — ①

$\boxed{\psi_y = 3x^3y^2 + y^4 + 4xy^3}$ — ②

From ①

$\boxed{\psi(x,y) = x^3y^3 + y^4x + g(y)}$

$\Rightarrow \psi_y = 3x^3y^2 + 4y^3x + g'(y)$

$\overset{②}{=} 3x^3y^2 + y^4 + 4xy^3$

$\Rightarrow g'(y) = y^4 \Rightarrow \boxed{g(y) = \frac{y^5}{5}}$

$\Rightarrow$ Solution is

$x^3y^3 + y^4x + \frac{y^5}{5} = C$

7. Consider the following IVP

$$(\ln t) \cdot y' - y = \frac{3}{(t-2)(t-5)}, \quad y\left(\frac{3}{2}\right) = 4.$$

The largest interval in which the solution is certain to exist is

(a) $1 < t < 5$

(b) $-\infty < t < 2$

(c) $0 < t < 2$

(d) $1 < t < 2$

$$y' \underbrace{- \frac{1}{\ln t}}_{p} y = \overbrace{\frac{3}{(\ln t)(t-2)(t-5)}}^{q}$$

p is Cont. on $(0, \infty)$, q is Cont. on $t \neq 1$, $t \neq 2$, $t \neq 5$, $t > 0$

8. The **The values of** b for which the IVP

$$\frac{dy}{dx} = \frac{\sqrt{y-6x}}{x^2+1}, \quad y(5) = b$$

has a unique solution are

(a) $b > 30$

(b) $b \geq 30$

(c) $b > \frac{5}{6}$

(d) $b > \frac{6}{5}$

$$f(x,y) = \frac{\sqrt{y-6x}}{x^2+1}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y-6x}\,(x^2+1)}$$

f & f_y Cont. in

$R = \{(x,y) \mid y > 6x\}$

$(5,b) \in R \Rightarrow b > 6(5)$ or $b > 30$

9. Let P be the fish population in a certain large lake. Assume that the rate of increase in the population due to births is 40 % per year and the death rate is 10% per year. Fishermen harvest the fish at the rate of 10,000 fish per year. The differential equation that describes this model is

(a) $\frac{dP}{dt} = 0.3P - 10,000$

(b) $\frac{dP}{dt} = 4P - 10,000$

(c) $\frac{dP}{dt} = 0.3 - \frac{P}{10,000}$

(d) $\frac{dP}{dt} = \frac{3}{10P} - 10,000P$

$$\frac{dP}{dt} = \frac{40}{100}P - \frac{10}{100}P - 10,000$$

or $\frac{dP}{dt} = 0.3P - 10,000.$

10. One of the following is a **solution** for the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x.$$

(a) $y = 2e^x + 1$

(b) $y = (x+1)e^{-x}$

(c) $y = (2x+1)e^{-x}$

(d) $y = (x+1)e^x$

$y' = (x+1)e^x + 1 \cdot e^x$

$\Rightarrow y' = (x+2)e^x$

$\Rightarrow y'' = (1)e^x + (x+2)e^x$

$y'' = (x+3)e^x$

$L.H.S \therefore y'' - y' = (x+3)e^x - (x+2)e^x$

$= e^x(x+3-x-2) = e^x = R.H.S$

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(a) $\ln|1+y| = x + \frac{1}{2}x^2 + C$
(b) $y = \ln|1+x| \cdot y + C$
(c) $y = x + \frac{1}{2}x^2 + xy + \frac{1}{2}x^2y + C$
(d) $y = x + \frac{1}{2}x^2 + C$

2. One of the following is a **solution** for the differential equation

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$$(3x^3y^2 + y^4 + 4xy^3)dy + (3x^2y^3 + y^4)dx = 0 \quad \text{is}$$

(a) $y = 5x^3y^3 + 5xy^4 + y^5$

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(d) None of the above

10. The differential equation

$$y'' + \left(\frac{x^2\cdot\sin x}{e^2\cdot\sqrt{\pi}}\right)(y')^3 + xy = 10 \quad \text{is}$$

(a) an ordinary, nonlinear differential equation of order 2

(b) an ordinary, linear differential equation of order 2

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Question two (5 points). Solve the following differential equation

$$(x\sin y+\cos y)y' + (x+y)\sin y = 0.$$

1 point

$$\Rightarrow \boxed{(\underbrace{x\sin y + \cos y}_{N})\,dy + (\underbrace{(x+y)\sin y}_{M})\,dx = 0} \quad (*)$$

$M_y = (x+y)\cos y + 1 - \sin y$

$N_x = \sin y.$

$\Rightarrow M_y \neq N_x$ is not exact.

- $\dfrac{M_y - N_x}{N} = \dfrac{(x+y)\cos y + \cancel{\sin y} \cancel{-\sin y}}{x\sin y + \cos y} \neq g(x)$. " x alone.

2 pts

- $\dfrac{M_y - N_x}{-M} = \dfrac{(x+y)\cos y + \cancel{\sin y} - \cancel{\sin y}}{(x+y)\sin y} = \cot y.$

$\Rightarrow \mu(y) = e^{-\int \cot y\,dy} = e^{-\ln|\sin y|} = \csc y,\ 0<y<\pi.$

Multiply $(*)$ by $\mu(y) = \csc y$,

$\boxed{(x+\cot y)\,dy + (x+y)\,dx = 0}$. exact (verify!!).

2 points

$\Rightarrow$

$\psi_x = x+y$

$\psi = \int (x+y)\,dx + g(y)$

$\boxed{\psi = \frac{x^2}{2} + yx + g(y)}$

$\psi_y = x + \cot y$

$\Rightarrow \psi_y = \cancel{x} + g'(y) = \cancel{x} + \cot y \Rightarrow g(y) = \int \cot y\,dy$

$\boxed{g(y) = \ln|\sin y|}$

$\therefore\ \frac{x^2}{2} + xy + \ln|\sin y| = c$ is an implicit solution of $(*)$.

Question three (5 points). Find the **explicit** solution of the following IVP

$$\frac{dy}{dx} = y(xy^3 - 1), \quad y(0) = 1.$$

3 points

$\Rightarrow \boxed{\frac{dy}{dx} + y = xy^4}$ (1) Bernoulli with $n = 4$.

let $v = y^{1-n}$ or $y = v^{\frac{1}{1-n}} = v^{-\frac{1}{3}}$

$\Rightarrow \boxed{y = v^{-\frac{1}{3}}}$ (2) $\Rightarrow \boxed{\frac{dy}{dx} = -\frac{1}{3} v^{-\frac{4}{3}} \frac{dv}{dx}}$ (3)

(2) & (3) into (*),

$$-\frac{1}{3} v^{-\frac{4}{3}} \frac{dv}{dx} + v^{-\frac{1}{3}} = x v^{-\frac{4}{3}}$$

$\Rightarrow \boxed{\frac{dv}{dx} - 3v = -3x}$ (4) is lin. in v.

2 points

$$\mu = e^{\int -3\,dx} = e^{-3x}$$

$$\therefore v(x) = e^{3x}\left[\int -3x\, e^{-3x}\,dx + c\right]$$

$$y^{-3} = e^{3x}\left[x e^{-3x} + \frac{1}{3} e^{-3x} + c\right]$$

or $\boxed{y^{-3} = x + \frac{1}{3} + c\, e^{3x}}$

$y(0) = 1 \Rightarrow 1 = 0 + \frac{1}{3} + c \Rightarrow c = \frac{2}{3}$

$$\Rightarrow y = \frac{1}{\sqrt[3]{x + \frac{1}{3} + \frac{2}{3} e^{3x}}}$$

$-3x \quad e^{-3x}$

$-3 \quad \frac{e^{-3x}}{-3}$

$0 \quad \frac{e^{-3x}}{9}$

Question Four (5 points). Use the **method of successive approximations** to solve

$$\boxed{\frac{dy}{dx} = 3(y+1), \quad y(0) = 0.} \quad (*)$$

$(*)$ Can be written as: $\quad \phi_n(t) = \int_0^t f(s, \phi_{n-1}(s))\, ds.$

where $f(t,y) = 3(y+1).$

choose $\boxed{\phi_0(t) = 0}$, next $\phi_1(t) = \int_0^t f(s,0)\, ds$

$$\Rightarrow \phi_1(t) = \int_0^t 3\, ds = 3t \Rightarrow \boxed{\phi_1(t) = 3t}$$

$$\phi_2(t) = \int_0^t f(s, \phi_1(s))\, ds = \int_0^t f(s, 3s)\, ds$$

$$= \int_0^t 3(3s+1)\, ds$$

$$= 9\frac{s^2}{2} + 3s \Big|_0^t = \boxed{\frac{9}{2}t^2 + 3t = \phi_2(t)}$$

$\vdots$

$$\phi_n(t) = (3t) + \frac{(3t)^2}{2!} + \frac{(3t)^3}{3!} + \cdots + \frac{(3t)^n}{n!}$$

$$\phi_n(t) = \sum_{n=0}^{\infty} \frac{(3t)^n}{n!} - 1 = e^{3t} - 1$$

Converge by ratio test (verify)

$$\Rightarrow \lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} (e^{3t} - 1) = e^{3t} - 1$$

$\therefore \phi(t) = e^{3t} - 1$ is the solution of $(*)$.

Question Five (5 points). Solve the following **homogeneous** differential equation

$$y(\ln x - \ln y)dx = (x\ln x - x\ln y - y)dy.$$

No need to show it is homog.

$$\Rightarrow y \ln\left(\frac{x}{y}\right) dx = \left[x \ln\left(\frac{x}{y}\right) - y\right] dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y \ln\left(\frac{x}{y}\right)}{x \ln\left(\frac{x}{y}\right) - y} = \frac{y \ln\left(\frac{x}{y}\right)}{x\left(\ln\left(\frac{x}{y}\right) - \frac{y}{x}\right)}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{\frac{y}{x} \ln\left(\frac{x}{y}\right)}{\ln\left(\frac{x}{y}\right) - \frac{y}{x}}} \quad \text{homog.} \quad -(1)$$

2 points

let $\boxed{y = vx}$ $-(2)$ $\Rightarrow \boxed{\frac{dy}{dx} = v \cdot 1 + x\frac{dv}{dx}}$ $-(3)$

(2) & (3) into (1)

$$v + x\frac{dv}{dx} = \frac{v\ln\left(\frac{1}{v}\right)}{\ln\left(\frac{1}{v}\right) - v}$$

1 points

$$\Rightarrow \cancel{v\ln\left(\frac{1}{v}\right)} - v^2 + \left(x\ln\left(\frac{1}{v}\right) - xv\right)\frac{dv}{dx} = \cancel{v\ln\left(\frac{1}{v}\right)}$$

$$\Rightarrow x(-\ln v - v)\,dv = v^2\,dx$$

or $$\int -\frac{\ln v - v}{v^2}\,dv = \int \frac{1}{x}\,dx$$

2 points

$$\Rightarrow -\int v^{-2}\ln v - \int v^{-1}\,dv = \int x^{-1}\,dx$$

$u = \ln v$ $\quad dh = v^{-2}dv$

$du = \frac{1}{v}dv$ $\quad h = \frac{v^{-1}}{-1}$

$$\Rightarrow \frac{\ln v}{v} - \int v^{-2}\,dv - \ln|v| = \ln|x| + c$$

$$\Rightarrow \frac{x}{y}\ln\left(\frac{y}{x}\right) + \frac{x}{y} - \ln\left|\frac{y}{x}\right| = \ln|x| + c$$

is the solution.

Question six (5 points). A thermometer is taken from a room where the temperature is $20°C$ to the outside where the temperature is $25°C$. After one minute, the thermometer reading is $23°C$. When exactly will the reading of the thermometer be $24.68°C$?

$$\frac{dT}{dt} = k(T - T_m), \quad T_m = 25, \quad T(0) = 20$$

$$T(1) = 23$$

$$T(t) = 24.68 \Rightarrow t = ??$$

(1.5 points)

$$\Rightarrow \int \frac{dT}{T-25} = \int k\,dt$$

$$\ln|T-25| = kt + c \Rightarrow T - 25 = Ae^{kt}$$

$$\Rightarrow T(t) = 25 + Ae^{kt}$$

$$T(0) = 25 + A = 20 \Rightarrow \boxed{A = -5}$$

$$T(1) = 25 - 5e^{k} = 23 \Rightarrow 5e^{k} = 2$$

$$k = \ln\left(\tfrac{2}{5}\right)$$

$$\therefore \quad T(t) = 25 - 5e^{\ln\left(\frac{2}{5}\right)t}$$

(2.5 points)

$$24.68 = 25 - 5\left(\tfrac{2}{5}\right)^t$$

$$\Rightarrow 5\left(\tfrac{2}{5}\right)^t = +0.32 \Rightarrow \left(\tfrac{2}{5}\right)^t = \frac{32}{500} = \frac{64}{1000}$$

$$\Rightarrow (0.4)^t = (0.4)^3$$

$$\Rightarrow \boxed{t = 3}$$

(1 point)

Good Luck