$r^2 \sin\theta \, d\theta \, dr \, d\phi$

$\boxed{1}$ $\int_\Omega \left(\frac{\vec{r}}{r^3}\right) \cdot \vec{\nabla}\left(\frac{r^4 - R^4}{r^5 + 7r^3 + 2}\right) d\tau = I$

We know that : $\nabla \cdot (f\vec{A}) = f(\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla f)$

Let $\vec{A} = \frac{\vec{r}}{r^3}$, $f = \frac{r^4 - R^4}{r^5 + 7r^3 + 2}$

$I = \int \vec{A} \cdot \nabla f \, d\tau = \int \nabla \cdot (f\vec{A}) d\tau - \int f (\nabla \cdot \vec{A}) d\tau$

$= \int \nabla \cdot \left(\frac{r^4 - R^4}{r^5 + 7r^3 + 2} \frac{\hat{r}}{r^4}\right) d\tau - \int \frac{r^4 - R^4}{r^5 + 7r^3 + 2} \left(\nabla \cdot \frac{\hat{r}}{r^4}\right)$

$\nabla \cdot \frac{\hat{r}}{r^4} = \nabla \cdot \left(\frac{\hat{r}}{r^2} \frac{1}{r^2}\right) = \left(\nabla \cdot \frac{\hat{r}}{r^2}\right) \frac{1}{r^2} + \frac{\hat{r}}{r^2} \cdot \nabla\left(\frac{1}{r^2}\right)$

$= 4\pi \delta^3(\vec{r}) \frac{1}{r^2} - \frac{2}{r^5}$

$\Rightarrow I = \int \frac{r^4 (r^5 + 7r^3 + 2)(4r^3 \cancel{\quad}) - (r^4 - R^4)(5r^4 + 21r^2)}{(r^5 + 7r^3 + 2)^2 \, r^8} d\tau$

$2\pi R^4$

$- \int \frac{r^4 - R^4}{r^5 + 7r^3 + 2} \frac{4\pi}{r^2} \delta^3(\vec{r}) + \int 2 \frac{r^4 - R^4}{(r^5 + 7r^3 + 2) r^5}$

~~We~~ We have $\int_{\phi=0}^{2\pi} d\phi = 2\pi$

$\int_{\theta=0}^{\pi} \sin\theta \, d\theta = 4\pi$

The second term is $\frac{R^4}{2} 4\pi = 2\pi R^4$

OR

$I = \int_\Omega \left(\frac{\hat{r}}{r^2}\right) \cdot \nabla\left(\frac{r^4 - R^4}{r^5 + 7r^3 + 2}\right) dr \, d\phi \, d\theta$

Let $A = \frac{\hat{r}}{r^2}$, $f = $

$4\pi \delta^3(\vec{r})$

$I = \int \nabla \cdot \left(f \frac{\hat{r}}{r^2}\right) d\tau - \int f \left(\nabla \cdot \frac{\hat{r}}{r^2}\right) d\tau$

$\sin\theta \, dr \, d\theta \, d\phi$

$-f(4\pi)\Big|_{r=0} = R^4 (2\pi)$

$= \int \nabla \cdot \left(\frac{r^4 - R^4}{r^2 (r^5 + 7r^3 + 2)}\right) + 2\pi R^4$

$= \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi \int_0^R \frac{(r^7 + 7r^5 + 2r^2)(4r^3) - (r^4 - R^4)(7r^6 + 35r^4 + 4r)}{(r^7 + 7r^5 + 2r^2)^2} dr + 2\pi R^4$

$= (4\pi)(2\pi)(2\pi R^4) \int_0^R \cdots$

2 a. $V(x,0)=0$

$V(x,d)=0$

$V(0,y)=V(y)=\sin\left(\frac{3\pi}{d}y\right)$

$V=0$ as $x\to\infty$

$$\nabla^2 V=\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0$$

Let $V(x,y)=X(x)\,Y(y)$

$$\Rightarrow \underbrace{\frac{X''(x)}{X(x)}}_{m^2}+\underbrace{\frac{Y''(y)}{Y(y)}}_{-m^2}=0$$

$Y''+m^2Y=0$

$Y(y)=A\sin(my)+B\cos(my)$

$V(x,0)=0\Rightarrow Y(0)=0\Rightarrow B=0$

$V(x,d)=0\Rightarrow Y(d)=0\Rightarrow$ md $=n\pi$, $n=1,2,\ldots$

$m=\frac{n\pi}{d}$

$\Rightarrow Y(y)=A_n\sin\left(\frac{n\pi}{d}y\right)$

$X''-m^2X=0$

$X(x)=Ce^{mx}+De^{-mx}$

$V=0$ as $x\to\infty\Rightarrow C=0$

$\Rightarrow X(x)=D_n e^{-\frac{n\pi}{d}x}$

$$\boxed{V(x,y)=\sum_{n=1}^{\infty}C_n e^{-\frac{n\pi}{d}x}\sin\left(\frac{n\pi}{d}y\right)}$$

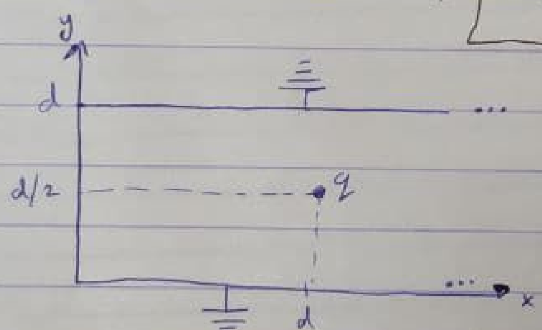
$V(0,y)=\sin\left(\frac{3\pi}{d}y\right)=\sum_{n=1}^{\infty}C_n\sin\left(\frac{n\pi}{d}y\right)$

$C_3=1$

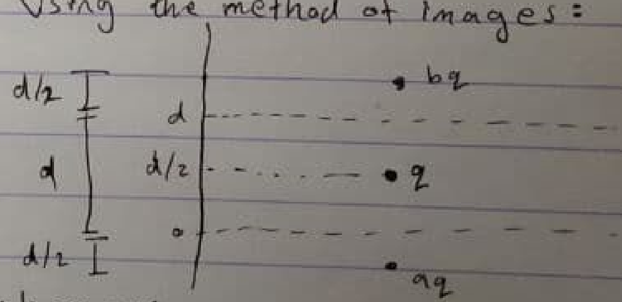
$C_n=0,\ \forall n\neq 3$

$$\Rightarrow\boxed{V(x,y)=e^{-\frac{3\pi}{d}x}\sin\left(\frac{3\pi}{d}y\right)}$$

b.



Using the method of images:



Let us consider these system of charges:

$q(d,d/2)$, $aq(d,-d/2)$, $bq(d,3d/2)$ where a,b real numbers

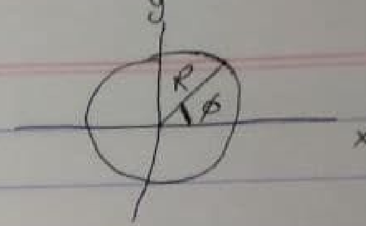
Conditions: $V(x,0)=0\ \to\ \frac{2aq}{d}+\frac{2bq}{3d}+\frac{2q}{d}=0$

$V(x,d)=0\ \to\ \frac{2a}{3}\frac{q}{d}+2b\frac{q}{d}+\frac{2q}{d}=0$

$$\left.\begin{aligned}a+\frac{b}{3}&=-1\\ \frac{a}{3}+b&=-1\end{aligned}\right\}\to a=b=-\frac{3}{4}$$

The Force on $+q$:

$$F=\frac{1}{4\pi\epsilon_0}\left[\frac{q\left(-\frac{3}{4}q\right)}{d^2}+\frac{q\left(-\frac{3}{4}q\right)}{d^2}\right]=\frac{1}{4\pi\epsilon_0}\,\frac{-3}{2}\,\frac{q^2}{d^2}$$



$\boxed{3}$ a. $\rho = \lambda_0 \cos^3\phi \, \delta(s-R)\, \delta(z)$

b. $\cancel{Q = \int_{-\infty}^{\infty}\int_{0}^{R}\int_{0}^{2\pi} \lambda_0 \cos^3\phi \; s \, \delta(s-R)\, \delta(z)\, d\phi\, ds\, dz}$

We can calculate the total charge using λ :

$$Q = \int \lambda \, dl = \int_0^{2\pi} \lambda_0 \cos^3\phi \, d\phi \qquad (\text{on the ring})$$

$$\downarrow$$

$$= \lambda_0 \int_0^{2\pi} [1-\sin^2\phi] \cos\phi \, d\theta \qquad \begin{matrix} dz = 0 \\ ds = 0 \end{matrix}$$

$$= \lambda_0 \left[\sin\phi - \frac{\sin^3\phi}{3}\right]\Bigg|_0^{2\pi} = 0$$

$$\Rightarrow Q = \text{zero}$$

c. $\vec{p} = \int \vec{r}' \rho(\vec{r}')\, d\tau'$ $\quad \to d\tau = s\, ds\, dz\, d\phi$, $\quad \to \vec{r}' = s\hat{s} + z\hat{z}$

$$= \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{R} (s\hat{s} + z\hat{z})\, \lambda_0 \cos^3\phi \, \delta(s-R)\, \delta(z) \; s\, ds\, d\phi\, dz$$

$$= \iiint \hat{s} s^2 \lambda_0 \cos^3\phi \, \delta(s-R)\, \delta(z)\, ds\, d\phi\, dz$$
$$+ \iiint z\, \lambda_0 \cos^3\phi \, \delta(s-R)\, \delta(z)\, ds\, d\phi\, dz$$

The second term is zero, since $\int_{-\infty}^{\infty} z\, \delta(z)\, dz = 0$

$$\cancel{\Rightarrow \vec{p} = \int_{-\infty}^{\infty} \delta(z)\, dz \int_{s=0}^{R} \lambda_0 s^2 \delta(s-R)\, ds \int_0^{2\pi} \lambda_0 \cos^3\phi\, d\phi}$$

we have : $\hat{s} = \cos\phi\, \hat{i} + \sin\phi\, \hat{j}$

$$\int_{-\infty}^{\infty} \delta(z)\, dz = 1$$

$$\int_{s=0}^{R} \lambda_0 s^2 \delta(s-R)\, ds = \lambda_0 R^2$$

By parts

$$\Rightarrow \vec{p} = \lambda_0 R^2 \left[\int_0^{2\pi} \cos^4\phi \, d\phi \; \hat{i} + \int_0^{2\pi} \sin\phi \cos\phi \, d\phi \; \hat{j}\, d\phi \right]$$

$$= \lambda_0 R^2 \left[\left(\cos^3\phi \sin\phi \Big|_0^{2\pi} + \int_0^{2\pi} 3\cos^2\phi \sin^2\phi \, d\phi \right)\hat{i} + \int_0^0 x\, dx \; \hat{j} \right]$$

$$= \lambda_0 R^2 \left(\frac{3\pi}{4} \hat{j} \right)$$

$\boxed{4}$ $\vec{E} = (\vec{c} \cdot \vec{r})\,\vec{c}$

$\frac{\rho}{\epsilon_0} = \vec{\nabla} \cdot \vec{E}$, let $\vec{c} = c_1 \hat{x} + c_2 \hat{y} + c_3 \hat{z}$

$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$

$(\vec{c} \cdot \vec{r}) = c_1 x + c_2 y + c_3 z$

$\vec{E} = (\vec{c} \cdot \vec{r})\,\vec{c} = (c_1 x + c_2 y + c_3 z)(c_1 \hat{x} + c_2 \hat{y} + c_3 \hat{z})$

$\Rightarrow \frac{\rho}{\epsilon_0} = \vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$

$\frac{\partial E_x}{\partial x} = c_1^2$, $\frac{\partial E_y}{\partial y} = c_2^2$, $\frac{\partial E_z}{\partial z} = c_3^2$

$\Rightarrow \frac{\rho}{\epsilon_0} = c_1^2 + c_2^2 + c_3^2 = |\vec{c}|^2$

$\rho = \epsilon_0 |\vec{c}|^2$

$\boxed{5}$ c. $\underset{E_{out}}{\vec{E}}\Big|_{r=R+} - \underset{E_{in}}{\vec{E}}\Big|_{r=R-} = \frac{\sigma(\theta)}{\epsilon_0}\hat{r}$

$$\Rightarrow \frac{\sigma(\theta)}{\epsilon_0} = -\frac{136}{105}\frac{V_0}{R} - \frac{400}{105}\frac{V_0}{R}P_2(\cos\theta) + \frac{72}{35}\frac{V_0}{R}P_4(\cos\theta)$$

$\boxed{5}$ $V(\theta) = V_0 \sin^4(\theta)$

a. ▫ First, inside the sphere :

$$V_{in}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$V_{in}(R,\theta) = V_0(1-\cos^2\theta)^2 = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta)$$

$$\sin^4\theta = (1-\cos^2\theta)^2 = 1 - 2\cos^2\theta + \cos^4\theta$$

$$= \frac{8}{35}\left[\frac{35}{8}\cos^4\theta + \frac{35}{8} - \frac{2\times 35}{8}\cos^2\theta\right]$$

$$= \frac{8}{35}\left[\underbrace{\frac{35}{8}\cos^4\theta - \frac{30}{8}\cos^2\theta + \frac{3}{8}}_{} - 5\cos^2\theta + \overset{\frac{17}{3}}{4}\right]$$

$$= \frac{8}{35}\left[P_4(\cos\theta) - \frac{10}{3}P_2(\cos\theta) - \left(\frac{5}{3}+4\right)P_0(\cos\theta)\right]$$

$$\left(\text{since } \cos^2\theta = \frac{2}{3}P_2(\cos\theta) + \frac{1}{3}P_0(\cos\theta)\right)$$

Applying the condition:

$$V_0\sin^4\theta = V_0\left[\frac{8}{35}P_4(\cos\theta) - \frac{80}{105}P_2(\cos\theta) - \frac{136}{105}P_0(\cos\theta)\right]$$

$$= \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta)$$

$$A_0 = -\frac{136}{105}V_0, \quad A_2 = \frac{-80}{105R^2}V_0, \quad A_4 = \frac{8}{35R^4}V_0, \quad A_l = 0 \quad \forall l \neq 0, 2, 4$$

$$\Rightarrow V_{in}(r,\theta) = -\frac{136}{105}V_0 + -\frac{80}{105R^2}V_0 r^2 P_2(\cos\theta) + \frac{8}{35R^4}V_0 r^4 P_4(\cos\theta)$$

▫ Outside :

$$V_{out}(R,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}P_l(\cos\theta) = V_0\sin^4\theta$$

$$\frac{8}{35}V_0 = \frac{B_4}{R^5}, \quad \frac{-80}{105}V_0 = \frac{B_2}{R^3}, \quad \frac{-136}{105}V_0 = \frac{B_0}{R}$$

$$\Rightarrow V_{out}(r,\theta) = \frac{-136}{105}\frac{R}{r}V_0 - \frac{80}{105}\frac{R^3}{r^3}V_0 P_2(\cos\theta) + \frac{8}{35}\frac{R^5}{r^5}V_0 P_4(\cos\theta)$$

b. $E_{in} = -\frac{\partial V_{in}}{\partial r}\hat{r} = -\vec{\nabla}V = \frac{+80}{105R^2}V_0(2r)P_2(\cos\theta)\hat{r} - \frac{8}{35R^4}V_0(4r^3)P_4(\cos\theta)\hat{r}$

$$= \left[\frac{160}{105R^2}V_0 r P_2(\cos\theta) - \frac{32}{35R^4}V_0 r^3 P_4(\cos\theta)\right]\hat{r}$$

$$E_{out} = \frac{-\partial V_{out}}{\partial r}\hat{r} = \left[\frac{136}{105}\frac{R}{r^2}V_0 - \frac{240}{105}\frac{R^3}{r^4}V_0 P_2(\cos\theta) + \frac{40}{35}\frac{R^5}{r^6}V_0 P_4(\cos\theta)\right]\hat{r}$$

$$E = \begin{cases} E_{in}, & r < R \\ E_{out}, & r > R \end{cases}$$