$\boxed{1}$ $\vec{E} = x\,\hat{x} + z\,\hat{y} + (f(x,y) + z^2)\,\hat{z}$

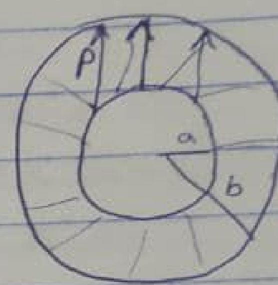
a. $\nabla \times \vec{E} = 0$

$$= \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{x} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{y} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{z}$$

$$= \left(\frac{\partial f}{\partial y} - 1\right)\hat{x} + \left(0 - \frac{\partial f}{\partial x}\right)\hat{y} + (0 - 0)\,\hat{z} = 0$$

$\Rightarrow \quad \frac{\partial f}{\partial y} = 1$ and $\frac{\partial f}{\partial x} = 0 \Rightarrow f(x,y) = y + c$, c is constant

$\boxed{3}$ $\vec{P}(\vec{r}) = \begin{cases} P_0\,\hat{z} & , a \le r \le b \\ 0 & , \text{otherwise} \end{cases}$



a. $\sigma_b = \vec{P}\cdot\hat{r} = -P_0\cos\theta$ at $r=a$

$\sigma_b = \vec{P}\cdot\hat{r} = P_0\cos\theta$ at $r=b$

$\rho_b = -\vec{\nabla}\cdot\vec{P} = 0$

b. $Q_{enc} = \sigma_a\, 4\pi a^2 = \cancel{P_0} - 4\pi a^2 P_0\cos\theta$

$\oint \vec{E}\cdot d\vec{a} = Q_{enc}/\epsilon_0$

$\Rightarrow \quad 4\pi r^2 E = \frac{-4\pi a^2 P_0\cos\theta}{\epsilon_0}$

c. $\Rightarrow \Rightarrow \vec{E} = -\frac{a^2}{\epsilon_0 r^2} P_0\cos\theta\,\hat{r}$

$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\oint_S \frac{\sigma_b}{r}\,da' \cancel{= \frac{1}{4\pi\epsilon_0}\int_{\phi=0}^{2\pi}\int^{\pi} \frac{P_0\cos\theta}{r}\, r^2\sin\theta\, d\theta\, d\phi}$

$\cancel{= \frac{2\pi P_0 r}{2\, 4\pi\epsilon_0}\int_{\theta=0}^{\pi}\cos\theta\sin\theta\, d\theta}$

or $\vec{V}(\vec{r}) = -\int_{\infty}^{r} \frac{-a^2}{\epsilon_0 r^2} P_0\cos\theta\, dr = -\frac{a^2}{\epsilon_0 r} P_0\cos\theta$

d. $\vec{D} = \epsilon_0\vec{E} + \vec{P}$

$$= \frac{-a^2}{r^2} P_0\cos\theta\,\hat{r} + P_0\,\hat{z}$$

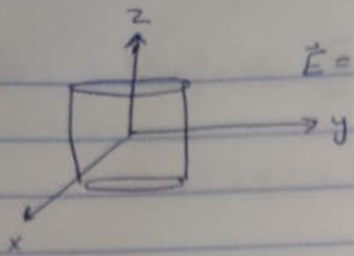
$\boxed{1}$ b. $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

$$(1 + 0 + 2z) = \frac{\rho}{\epsilon_0} \Rightarrow \rho = (1+2z)\epsilon_0$$

$$Q = \int_0^1\int_0^1\int_0^1 \rho \, d\tau = \int_0^1\int_0^1\int_0^1 (1+2z)\epsilon_0 \, dx\, dy\, dz$$

$$= \epsilon_0 \int_{z=0}^{1} (1+2z)\, dz = \epsilon_0 (z + z^2)\Big|_0^1 = 2\epsilon_0$$

$\boxed{4}$



$\vec{E} = E_0 \hat{x}$

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \varphi^2} = 0$$ — Laplace's eq in cylindrical coord. —

$V = S(s)\,\phi(\varphi)$

$$\underbrace{\frac{s}{S}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right)}_{\lambda} + \underbrace{\frac{1}{\phi}\frac{\partial^2 \phi}{\partial \varphi^2}}_{-\lambda} = 0$$

let $\lambda = m^2$: $s^2 S'' + s S' - m^2 S = 0$ $\qquad \phi'' + m^2 \phi = 0$

$S = A s^m + B s^{-m}$ $\qquad \phi(\varphi) = C\cos m\varphi + D\sin m\varphi$

$\Rightarrow$ ~~$V(s,\varphi)=$~~ let $\lambda = 0$: $\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) = 0$ $\qquad \phi'' = 0$

$S(s) = a_0 \ln(s) + b_0$

$$\Rightarrow V(s,\varphi) = a_0\ln(s) + b_0 + \sum_{m=1}^{\infty}\left(A_m s^m + \frac{B_m}{s^m}\right)\left(C_m\cos m\varphi + D_m \sin m\varphi\right)$$

Inside: $a_0 = 0$, $B_n = 0$, let $b_0 = 0$, and let $D_m = 0$

Outside: $a_0 = 0$, $A_n = 0$, let $b_0 = 0$ (otherwise, it is blowing up!)

$$\Rightarrow V_{in}(s,\varphi) = \sum_{m=1}^{\infty} A_m s^m (\cancel{C_m}\cos m\varphi)$$

$$V_{out}(s,\varphi) = \underbrace{V_{external}}_{V_{ext} = E_0 s\cos\varphi} + \sum_{m=1}^{\infty} B_m s^{-m}(\cancel{C_m}\cos m\varphi)$$

We have: $\begin{cases} V_{in} = V_{out} \rightarrow \text{at } s = R & \text{— (1)} \\ \dfrac{\partial V_{out}}{\partial s} - \dfrac{\partial V_{in}}{\partial s} = \dfrac{-\sigma}{\epsilon_0} \text{ at } s = R & \text{— (2)} \\ V(s\to\infty) = E_0\, s\cos\varphi & \text{— (3)} \end{cases}$

Using condition (1):

$$\sum_{m=1}^{\infty} A_m R^m \cos m\varphi = E_0 R\cos\varphi + \sum_{m=1}^{\infty} B_m R^{-m}\cos m\varphi$$

$$\Rightarrow A_1 R = E_0 R + B_1\frac{1}{R}, \quad B_m = R^{2m} A_m \quad \text{— (4)}$$

$\boxed{4}$ a.) So $\quad V_{in}(s,\varphi) = \sum_{m=1}^{\infty} A_m s^m \cos\varphi$

$$V_{out}(s,\varphi) = \sum_{m=1}^{\infty} \left(A_m R^{2m} s^{-m} \cos m\varphi\right) + E_0 s \cos\varphi$$

b.) $\cancel{\sigma = -\epsilon_0 \frac{\partial V}{\partial s}\Big|_{s=R} +}$

$\cancel{-\epsilon_0 \sum m A_m}$

from (2): $\sigma = -\epsilon_0 \left[\frac{\partial V_{out}}{\partial s} - \frac{\partial V_{in}}{\partial s} \right]\Big|_{s=R}$

$$= -\epsilon_0 \left[\sum_{m=1}^{\infty} \left(-m A_m R^{2m} R^{-m-1} \cos m\varphi\right) - \sum_{m=1}^{\infty} \left(m A_m R^{m-1} \cos\varphi\right) \right]$$

$$= +\epsilon_0 \left[\sum_{m=1}^{\infty} m A_m \cos m\varphi \left(2R^{m-1}\right) \right]$$

We need another condition to determine the A_m's :)

$\cancel{V_{in}(s,\varphi) = \sum_{m=1}^{\infty} A_m s^m \cos\varphi}$

$\boxed{5}$

$$\vec{A}(\vec{r}) = \begin{cases} \underbrace{\tfrac{1}{2}\mu_0 \rho_0 \omega}\left(\frac{rR^2}{3} - \frac{r^3}{5}\right)\sin\theta\,\hat{\phi} & , r \le R \\ \underbrace{\tfrac{1}{15}\mu_0 \rho_0 \omega}\,\frac{R^4}{r^2}\sin\theta\,\hat{\phi} & , r > R \end{cases}$$

a. $\vec{B} = \nabla\times\vec{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi \sin\theta)\right)\hat{r}$

$$-\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right)\hat{\theta}$$

$B_{in} = \frac{1}{r\sin\theta}\left[2K\left(r\frac{R^2}{3} - \frac{r^3}{5}\right)\cos\theta\sin\theta\right]\hat{r}$; where $\boxed{K \equiv \frac{1}{2}\mu_0\rho_0\omega}$

$$-\frac{1}{r}\left[Kr\left(\frac{R^2}{3} - \frac{3r^2}{5}\right)\sin\theta + K\left(\frac{rR^2}{3} - \frac{r^3}{5}\right)\sin\theta\right]\hat{\theta}$$

$B_{out} = \frac{1}{r\sin\theta}\left[2\lambda\frac{R^4}{r^2}\sin\theta\cos\theta\right]\hat{r}$; where $\boxed{\lambda \equiv \frac{1}{15}\mu_0\rho_0\omega}$

$$-\frac{1}{r}\left[\lambda\frac{-2R^4}{r^2}\sin\theta + \lambda\frac{R^4}{r^2}\sin\theta\right]\hat{\theta}$$

Simplify:

$$B_{in} = 2K\left(\frac{R^2}{3} - \frac{r^2}{5}\right)\cos\theta\,\hat{r} - K\sin\theta\left(\frac{2R^2}{3} - \frac{4}{5}r^2\right)\hat{\theta}$$

$$B_{out} = 2\lambda\frac{R^4}{r^3}\cos\theta\,\hat{r} + \lambda\frac{R^4}{r^3}\sin\theta\,\hat{\theta}$$

b. $m = I\int da$

$I = \int\vec{J}\cdot d\vec{a}$

$\vec{J} = \frac{-1}{\mu_0}\nabla^2\vec{A} = \frac{-1}{\mu_0}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_\phi}{\partial\theta}\right)\right]\hat{\phi}$

or $\vec{B}_{dip}(\vec{r}) = \vec{\nabla}\times\vec{A}_{dip} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$

$= 2\lambda\frac{R^4}{r^3}\cos\theta\,\hat{r} + \lambda\frac{R^4}{r^3}\sin\theta\,\hat{\theta} \equiv \frac{\lambda R^4}{r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$

$\Rightarrow \frac{\mu_0 m}{4\pi \cancel{r^3}} = \frac{\lambda R^4}{\cancel{r^3}} \Rightarrow \boxed{m = \frac{4\pi}{\mu_0}\lambda R^4}$

where λ as in above

$\boxed{2}$ a. $I=0$

$\vec{\nabla}\cdot B = 0$ ✓

$\vec{\nabla}\times\vec{B} = \frac{-\partial B_x}{\partial y}\hat{z} = -\hat{z} = \mu_0 \vec{J}$

$\vec{B} = y\hat{x}$?

$\Rightarrow \vec{J} = \frac{-1}{\mu_0}\hat{z}$

But we have $I=0$:

$I = \int \vec{J}\cdot d\vec{a} = 0 \Rightarrow \vec{J} = 0$

so, it is not possible to have $\vec{B} = y\hat{x}$

b. $\vec{J} = \vec{J}_0\, e^{-r^2}$

Continuity equation : $\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0$

Steady current $\rightarrow \frac{\partial \rho}{\partial t} = 0$

$\Rightarrow \vec{\nabla}\cdot\vec{J} = 0$ (must)

the r component of $\vec{J}_0$

$\vec{\nabla}\cdot\vec{J} = \frac{1}{r^2}\frac{\partial(r^2 \vec{J}_{0,r}\, e^{-r^2})}{\partial r} \neq 0$

So, it is not possible