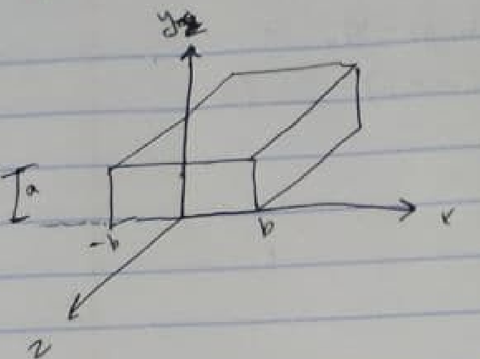
1



$$\nabla^2 V = 0 = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

- 2D-Problem: $V(x,y,z) = V(x,y)$
- BC's:

$V(-b,y) = V_0$, $V(b,y) = 2V_0$

$V(x,0) = 0$, $V(x,a) = 0$

Let $V(x,y) = X(x)\,Y(y)$

$$\nabla^2 V = 0 \Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\frac{X''}{X} = c_1 , \quad \frac{Y''}{Y} = c_2 , \quad c_1 = -c_2$$

$$\frac{X''}{X} = c_1 = k^2$$

$$\frac{Y''}{Y} = c_2 = -c_1 = -k^2$$

$$X(x) = A e^{kx} + B e^{-kx}$$

$$Y(y) = C\sin(ky) + D\cos(ky)$$

$V(x,0) = X(x)\,Y(0) = 0 \Rightarrow \cancel{X(x)}\ Y(0) = 0$ (Let $X(x) \neq 0$)

$$Y(0) = 0 \Rightarrow \boxed{0 = D}$$

$$V(x,a) = X(x)\,Y(a) = 0 \Rightarrow Y(a) = 0$$

$$Y(a) = 0 \Rightarrow 0 = C\sin(ka)$$

Let $C \neq 0 \Rightarrow ka = n\pi$, $n = 1, 2, \ldots$

$$\boxed{k = \frac{n\pi}{a}}$$

$$\cancel{V(x,y) = \sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty}\left[\alpha_n e^{\frac{n\pi}{a}x}\sin\left(\frac{n\pi}{a}y\right) + \beta_n e^{-\frac{n\pi}{a}x}\sin\left(\frac{n\pi}{a}y\right)\right]}$$

$$\cancel{V(b,y) = 2V_0 = \sum\left(\alpha_n \exp\left(\frac{n\pi}{a}b\right) + \beta_n \exp\left(-\frac{n\pi}{a}b\right)\right)\sin\left(\frac{n\pi}{a}y\right)}$$

$$V(x,y) = \sum_{n=1}^{\infty}\left(\alpha_n e^{\frac{n\pi x}{a}} + \beta_n e^{-\frac{n\pi}{a}x}\right)\sin\left(\frac{n\pi}{a}y\right)$$

$$V(b,y) = \sum\left(\alpha_n e^{\frac{n\pi b}{a}} + \beta_n e^{-\frac{n\pi}{a}b}\right)\sin\left(\frac{n\pi}{a}y\right) = 2V_0$$

$$\int_0^a 2V_0 \sin\left(\frac{m\pi}{a}y\right)dy = \sum_{n=1}^{\infty}\left(\alpha_n e^{\frac{n\pi}{a}b} + \beta_n e^{-\frac{n\pi}{a}b}\right)\underbrace{\int_0^a \sin\left(\frac{m\pi}{a}y\right)\sin\left(\frac{n\pi}{a}y\right)dy}$$

This integral $= 0$ if $m \neq n$ and $= \frac{a}{2}$ if $m = n$

$$\int_0^a 2V_0 \sin\left(\frac{m\pi}{a}y\right)dy = \left(\alpha_m e^{\frac{m\pi}{a}b} + \beta_m e^{-\frac{m\pi}{a}b}\right)\frac{a}{2}$$

$$2V_0\left(\frac{a - a\cos(m\pi)}{m\pi}\right) =$$

$$\Rightarrow \alpha_m e^{\frac{m\pi}{a}b} + \beta_m e^{-\frac{m\pi}{a}b} = \frac{4V_0}{m\pi}\underbrace{\left(1 - \cos(m\pi)\right)}_{\hookrightarrow\ m \text{ even} \Rightarrow \text{zero}} \quad (1)$$

Following the same procedures to $V(-b,y) = V_o$:

$\alpha_m e^{-\frac{m\pi b}{a}} + \beta_m e^{\frac{m\pi b}{a}} = \frac{2V_o}{m\pi}(1-\cos(m\pi))$ — (2)

Put (1) and (2) together :

(case m odd):

$\alpha_m e^{\frac{m\pi b}{a}} + \beta_m e^{-\frac{m\pi b}{a}} = \frac{4V_o}{m\pi}$

$\alpha_m e^{-\frac{m\pi b}{a}} + \beta_m e^{\frac{m\pi b}{a}} = \frac{2V_o}{m\pi}$

Let $\boxed{\frac{m\pi b}{a} = r}$, $\boxed{\frac{V_o}{m\pi} = q}$

$\Rightarrow \quad \alpha_m e^{r} + \beta_m e^{-r} = 2q$

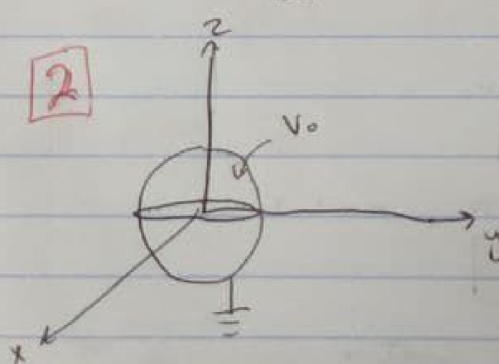
$\alpha_m e^{-r} + \beta_m e^{r} = q$

$\alpha_m = \frac{e^r(-1+2e^{2r})q}{-1+e^{4r}}$, $\beta_m = \frac{e^r(-2+e^{2r})q}{-1+e^{4r}}$

$\Rightarrow V(x,y) = \sum_{n=1}^{\infty}\left(\alpha_n e^{\frac{n\pi}{a}x} + \beta_n e^{-\frac{n\pi}{a}x}\right)\sin\left(\frac{n\pi}{a}y\right)$

where α_n, β_n as above.

2



$V_{inside}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$

$V_{outside}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$

□ $A_l = \frac{2l+1}{2R^l}\int_0^{\pi} V_o P_n(\cos\theta)\sin\theta\, d\theta$

Let $x = \cos x$, $dx = -\sin x\, dx$

$A_l = \frac{2l+1}{2R^l}\int_0^1 V_o P_n(x)\,dx$

$A_l = 0$ if l is even

if l is odd: $P_1(x) = x$, $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$ ...

$A_1 = \frac{3V_o}{2R}\int_0^1 x\,dx = \frac{3V_o}{4R}$

$A_3 = \frac{7V_o}{2R^3}\int_0^1 \left(\frac{5}{3}x^3 - \frac{3}{2}x\right)dx = \frac{-7}{8}\frac{V_o}{R^3}$

$\vdots$

□ $B_l = A_l R^{2l+1}$

same way ...