Electromagnetic Theory I

Abdallah Sayyed-Ahmad

Department of Physics

Birzeit University

March 10, 2021

جامعة بيرزيت
BIRZEIT UNIVERSITY

Chapter 1: Vector Analysis

- ✓ Vector Algebra
- ✓ Differential Calculus
- ✓ Integral Calculus
- ✓ Curvilinear Coordinates
- ✓ The Dirac Delta Function
- ✓ The Theory of Vector Fields

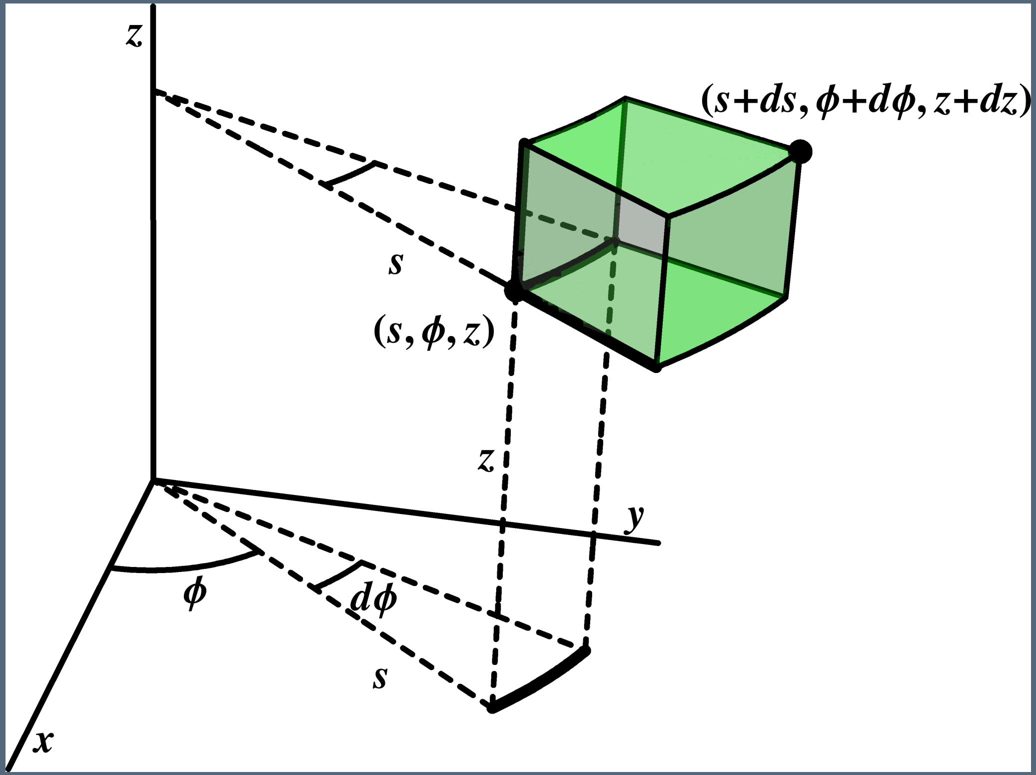
Cylindrical Coordinates

$x = s\cos\phi$
$y = s\sin\phi$
$z = z$

$s = \sqrt{x^2 + y^2}$
$\phi = \tan^{-1}\frac{y}{x}$

Unit Vectors

$\hat{\phi} = -\sin\phi\ \hat{x} + \cos\phi\ \hat{y}$
$\hat{s} = \cos\phi\ \hat{x} + \sin\phi\ \hat{y}$

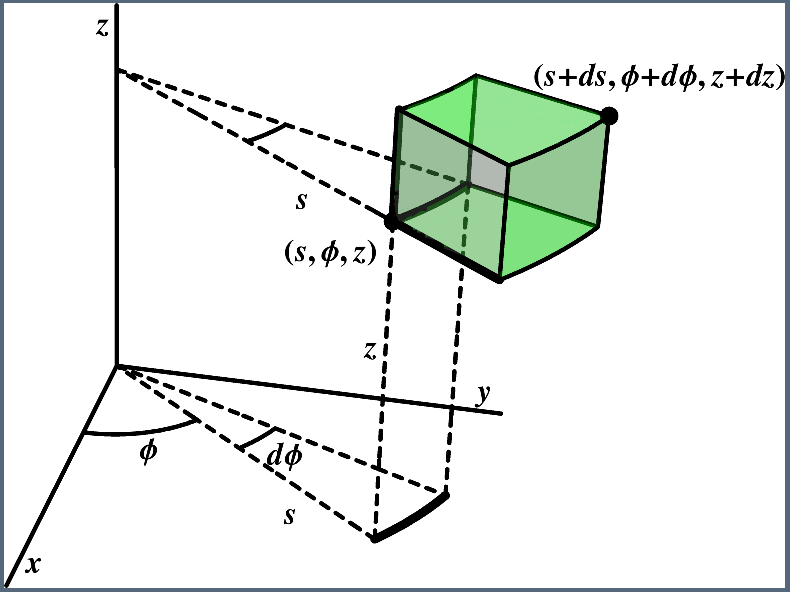


Note that $\hat{s}$ *and* $\hat{\phi}$ are not constant, e.g. $\frac{\partial \hat{s}}{\partial \phi} = \hat{\phi}$

Cylindrical Coordinates

Infinitesimal Displacement

$$d\vec{l} = dl_s\hat{s} + dl_\phi\hat{\phi} + dl_z\hat{z}$$
$$d\vec{l} = ds\hat{s} + sd\phi\hat{\phi} + dz\,\hat{z}$$



Infinitesimal Surface (it depends on orientation)

$$d\vec{a}_z = dl_s dl_\phi \hat{z} = sdsd\phi\,\hat{z} \quad (z \text{ is constant})$$

$$d\vec{a}_\phi = dl_s dl_z \hat{\phi} = dsdz\hat{\phi} \quad (\phi \text{ is constant})$$

$$d\vec{a}_s = dl_\phi dl_z \hat{s} = sdzd\phi\hat{s} \quad (s \text{ is constant})$$

Infinitesimal Volume

$$d\tau = sdsd\phi dz$$

Cylindrical Coordinates

Gradient

$$\vec{\nabla}\psi = \frac{\partial\psi}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial\psi}{\partial\phi}\hat{\phi} + \frac{\partial\psi}{\partial z}\hat{z}$$

Divergence

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{s}\frac{\partial}{\partial s}(sA_s) + \frac{1}{s}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

Curl

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \frac{\hat{s}}{s} & \hat{\phi} & \frac{\hat{z}}{s} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_s & sA_\phi & A_z \end{vmatrix}$$

Laplacian

$$\nabla^2\psi = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial\psi}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

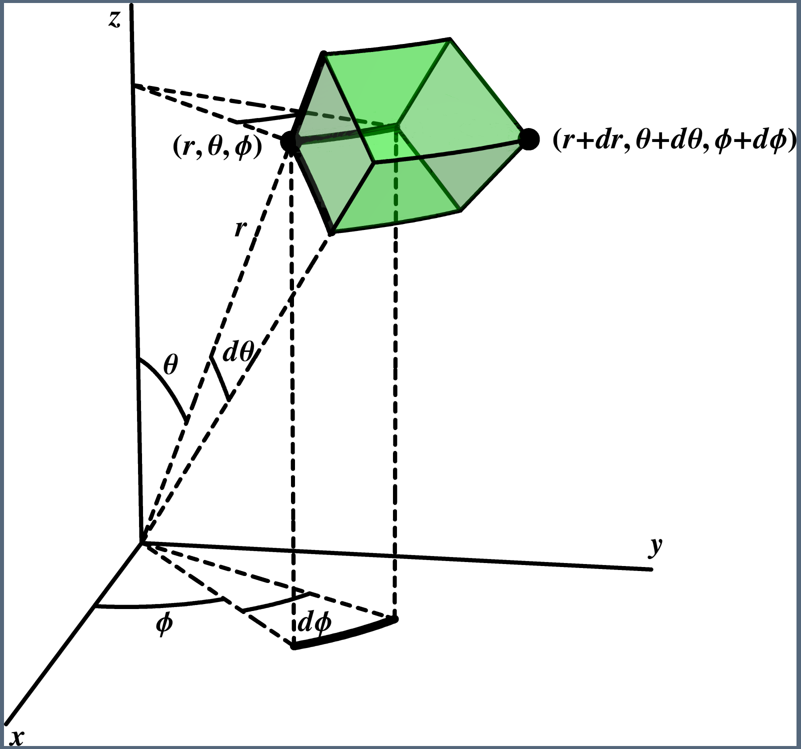
Spherical Coordinates

$$x = r\cos\phi\sin\theta$$
$$y = r\sin\phi\sin\theta$$
$$z = r\cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}}$$
$$\phi = \tan^{-1}\frac{y}{x}$$

Unit Vectors

$$\hat{\phi} = -\sin\phi\ \hat{x} + \cos\phi\ \hat{y}$$
$$\hat{r} = \frac{\vec{r}}{r} = \cos\phi\sin\theta\ \hat{x} + \sin\phi\sin\theta\ \hat{y} + \cos\theta\ \hat{z}$$
$$\hat{\theta} = \hat{\phi}\times\hat{r} = \cos\theta\cos\phi\ \hat{x} + \cos\theta\sin\phi\ \hat{y} - \sin\theta\ \hat{z}$$



Note that $\hat{r}, \hat{\theta}$ *and* $\hat{\phi}$ are not constant, e.g. $\frac{\partial\hat{r}}{\partial\theta} = \hat{\theta}$

Spherical Coordinates

Infinitesimal Displacement

$$d\vec{l} = dl_r\hat{r} + dl_\theta\hat{\theta} + dl_\phi\hat{\phi}$$

$$d\vec{l} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta\, d\phi\hat{\phi}$$

Infinitesimal Surface (it depends on orientation)

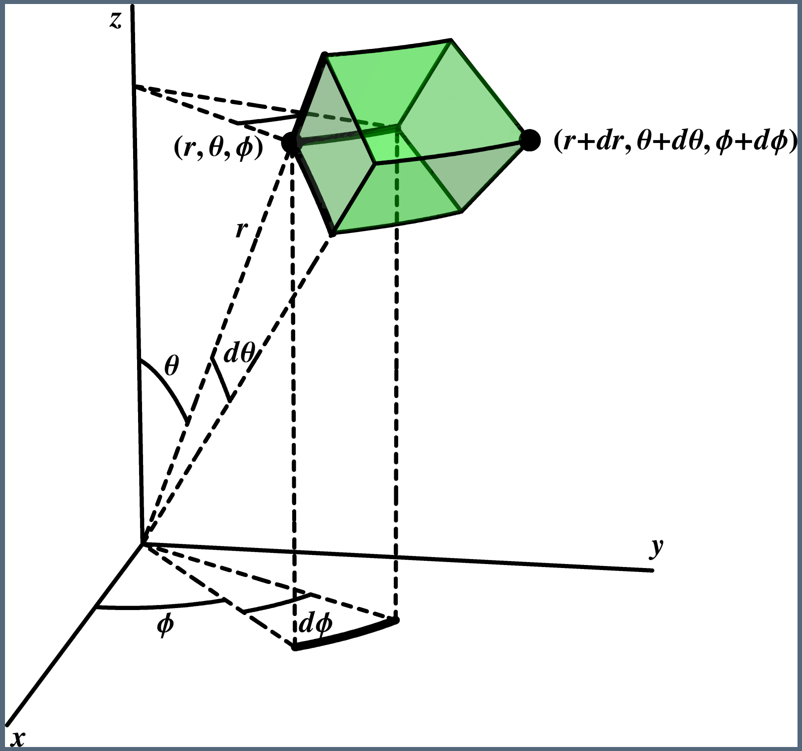
$$d\vec{a}_r = dl_\theta dl_\phi\hat{r} = r^2\sin\theta\, d\theta d\phi\hat{r} \quad (r\ is\ constant)$$

$$d\vec{a}_\theta = dl_r dl_\phi\hat{\theta} = r\sin\theta\, drd\phi\hat{\theta} \quad (\theta\ is\ constant)$$

$$d\vec{a}_\phi = dl_r dl_\theta\hat{\phi} = rdrd\theta\hat{\phi} \quad (\phi\ is\ constant)$$

Infinitesimal Volume

$$d\tau = r^2\sin\theta\, drd\theta d\phi$$



Spherical Coordinates

Gradient

$$\vec{\nabla}\psi = \frac{\partial \psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \phi}\hat{\phi}$$

Divergence

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r)\hat{r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\ A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

Curl

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \frac{\hat{r}}{r^2\sin\theta} & \frac{\hat{\theta}}{r\sin\theta} & \frac{\hat{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta\, A_\phi \end{vmatrix}$$

Laplacian

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\,\frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2}$$

Dirac Delta Function

(i) The divergence of $\frac{\hat{\boldsymbol{r}}}{r^2}$

$$\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(\frac{1}{r^2}\right)\right)=0\ for\ r\neq 0$$

$$\int_V \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)d\tau=\oint_S\left(\frac{\hat{r}}{r^2}\right)\cdot d\vec{a}=\int\int\frac{1}{r^2}r^2\sin\theta\, d\theta d\phi=4\pi$$

Does this mean that the divergence theorem is false?
What's going on here?

Dirac Delta Function

(ii) The one dimensional Dirac delta function

$$\delta(x-a)=\begin{cases}0 & x\neq a\\ \infty & x=a\end{cases}$$

$$\int_{-\infty}^{\infty}\delta(x-a)f(x)dx=f(a)\rightarrow\int_{-\infty}^{\infty}\delta(x-a)dx=1$$

(iii) Representations of Dirac delta function

$$\delta(x)=\lim_{\gamma\rightarrow 0}\frac{1}{\pi}\frac{\gamma}{x^2+\gamma^2}$$

$$\delta(x)=\lim_{\gamma\rightarrow 0}\frac{1}{\sqrt{\pi}}e^{-\frac{x^2}{\gamma^2}}$$

$$\delta(x-a)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{iq(x-a)}dq$$

Dirac Delta Function

(iv) Properties of Dirac delta function

$$(a)\ \delta(kx) = \frac{1}{|k|}\delta(x)$$

$$(b)\ \int_{-\infty}^{\infty} f(x)\,\delta^{(n)}(x-a)dx = (-1)^n f^{(n)}(a) \rightarrow \int_{-\infty}^{\infty} f(x)\,\delta'(x-a)dx = -f'(a)$$

$$(c)\ \delta\big(f(x)\big) = \sum_{i=1}^{N} \frac{\delta(x-x_i)}{|f'(x_i)|}$$

$$(d)\ \delta - function\ \ has\ units\ of\ 1/length$$

Dirac Delta Function

(v) The three dimensional Dirac delta function

Cartesian

$$\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$$

$$\delta^3(\vec{r} - \vec{r}') = \delta(x - x')\delta(y - y')\delta(z - z')$$

$$\int_V \delta(\vec{r} - \vec{r}')\, d\tau = \begin{cases} 0 & \vec{r}' \text{ is inside } V \\ 1 & \vec{r}' \text{is outside } V \end{cases}$$

Cylindrical

$$\delta^3(\vec{s} - \vec{s}') = \frac{1}{s}\delta(s - s')\delta(\phi - \phi')\delta(z - z')$$

Spherical

$$\delta^3(\vec{r} - \vec{r}') = \frac{1}{r^2 \sin\theta}\delta(r - r')\delta(\theta - \theta')\delta(\phi - \phi')$$

$$\delta^3(\vec{r} - \vec{r}') = \frac{1}{r^2}\delta(r - r')\delta(\cos\theta - \cos\theta')\delta(\phi - \phi')$$