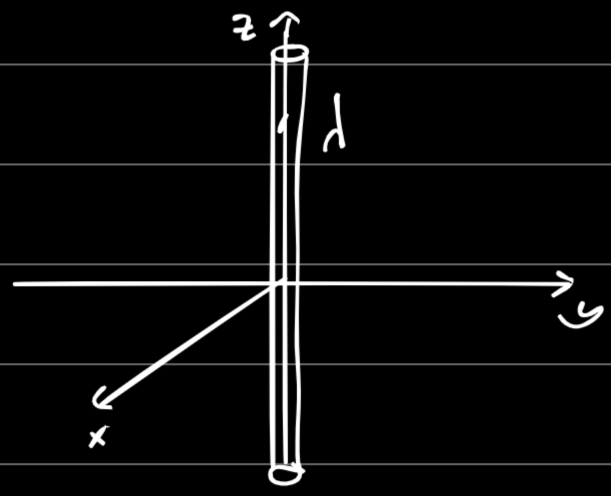
Example: write the charge density of a uniformly charge rod that lies along the z-axis

$\rho(\vec{r}) = \underline{\lambda}\,\delta(x)\,\delta(y) \leftarrow$

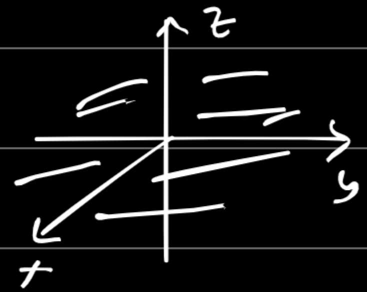
$\rho(\vec{s}) = C\,\delta(s) \leftarrow$

$= \frac{\lambda}{s}\,\delta(s)$



Example: Uniform surface charge density on the xy-plane (σ)

$\boxed{\rho(\vec{r})} = \underline{\sigma}\,\delta(z)$ Cartesian



$\rho(\vec{r}) = \frac{C}{r^2 \sin\theta}\,\delta\left(\theta - \frac{\pi}{2}\right)$ spherical coordinates

Example. Find $I = \int (r^2+1)\,\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) d\tau$ over a sphere of radius R

Method 1:

$\vec{\nabla}\cdot\left([r^2+1]\,\frac{\hat{r}}{r^2}\right) = \vec{\nabla}(r^2+1)\cdot\frac{\hat{r}}{r^2} + (r^2+1)\,\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)$

$\Rightarrow I = \int \vec{\nabla}\cdot\left[(r^2+1)\,\frac{\hat{r}}{r^2}\right] d\tau - \int \vec{\nabla}(r^2+1)\cdot\frac{\hat{r}}{r^2}\,d\tau$

$I = \oint \frac{(r^2+1)}{r^2}\,\hat{r}\cdot d\vec{a} - \int 2r\,\hat{r}\cdot\frac{\hat{r}}{r^2}\,d\tau$

$$I = \int_0^{\pi}\int_0^{2\pi} \frac{R^2+1}{\cancel{R^2}} \cancel{R^2} \sin\theta \, d\theta \, d\varphi - \int_0^{R}\int_0^{\pi}\int_0^{2\pi} \frac{2r}{\cancel{r^2}} \cancel{r^2} \sin\theta \, d\varphi \, d\theta \, dr$$

$$= 4\pi \cdot (R^2+1) - R^2 \, 4\pi$$

$$= 4\pi$$

method 2:

$$I = \int (r^2+1) \underbrace{\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right)}_{4\pi\delta(\vec{r})} d\tau$$

$$I = \int (r^2+1) \, 4\pi \, \delta(\vec{r}) \, d\tau$$

$$= 4\pi (r^2+1)\Big|_{r=0} = 4\pi$$

Helmholtz Theorem:

$$\vec{\nabla}\cdot\vec{F} = \vec{C}$$

$$\vec{\nabla}\times\vec{F} = \vec{D}, \quad \vec{\nabla}\cdot\vec{D} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0$$

and $\vec{F}$ is specfied on the boundaries,

then $\vec{F}$ is uniquely determined everywhere

In particular, if $\vec{D}$ and $\vec{C}$ vanishes as $r \to \infty$ faster than $\frac{1}{r^2}$, then $\vec{F}$ is given by

$$\vec{F} = -\frac{1}{4\pi}\vec{\nabla}\left[\int \frac{\vec{\nabla}'\cdot\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\tau'\right] + \frac{1}{4\pi}\vec{\nabla}\times\left[\int \frac{\vec{\nabla}'\times\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\tau'\right]$$

$$\vec{F} = -\vec{\nabla}V(\vec{r}) + \vec{\nabla}\times\vec{A}(\vec{r})$$

Recall, $\vec{E} = -\vec{\nabla}V$ ← scalar potential

$\vec{B} = \vec{\nabla}\times\vec{A}$ ← vector potential

Theorem 1: if $\vec{F}$ is irrotational (i.e. $\vec{\nabla}\times\vec{F}=0$), then

① $\vec{F}$ can be written as a gradient of a scalar function

$$\vec{F} = -\vec{\nabla}V$$

② $\int \vec{F}\cdot d\vec{l}$ is independent of path

③ $\oint \vec{F}\cdot d\vec{l} = 0$

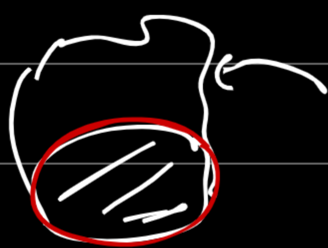
④ $\vec{\nabla}\times\vec{F} = 0$

e.g. $\oint \vec{F}\cdot d\vec{l} \underset{\substack{\uparrow \\ \text{Stokes's} \\ \text{Theorem}}}{=} \int(\vec{\nabla}\times\vec{F})\cdot d\vec{a} = \int \vec{0}\cdot d\vec{a} = 0$

Theorem 2: if $\vec{F}$ is divergenless (solenoidal), then

① $\vec{\nabla}\cdot\vec{F} = 0$

② $\int \vec{F}\cdot d\vec{a}$ is independent of the surface for given boundary line



③ $\oint \vec{F} \cdot d\vec{a} = 0$

$\updownarrow$

④ $\vec{F}$ can be written as a curl of a vector field i.e. $\vec{F} = \vec{\nabla} \times \vec{A}$

e.g. $\oint \vec{F} \cdot d\vec{a} \underset{\text{Divergence Theorem}}{=} \int \vec{\nabla} \cdot \vec{F} \, d\tau = \int \underbrace{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A})}_{0} \, d\tau$

$= 0$

Example: uniform charge density on a sphere of radius R

$$\sigma = \frac{Q}{4\pi R^2}$$

$$\rho(\vec{r}) = k_1\, \delta(x^2+y^2+z^2-R^2)$$

$$\rho(\vec{r}) = \underset{\sigma}{k_2}\, \delta\left(\sqrt{x^2+y^2+z^2}-R\right)$$

$$\int_{\text{all space}} \rho(\vec{r})\, d\tau = Q$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} k_2\, \delta\left(\sqrt{x^2+y^2+z^2}-R\right) dx\,dy\,dz = Q$$

$r = \sqrt{x^2+y^2+z^2}$

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$$\longrightarrow \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} k_2\, \delta(r-R)\, r^2 \sin\theta\, \underline{d\varphi\, d\theta}\, dr$$

$$= 4\pi k_2 \int_0^{\infty} r^2 \delta(r-R)\, dr$$

$$= 4\pi k_2 R^2 = Q \Rightarrow k_2 = \sigma$$

$$\int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\varphi = 4\pi$$

Example: Show that

$$\int_V \vec{\nabla}\psi \, d\tau = \oint_S \psi \, d\vec{a}$$

constant but arbitrary vector

$$\underline{\vec{c} \cdot \oint \psi \, d\vec{a}} = \oint \vec{c} \cdot (\psi \, d\vec{a}) = \oint \psi \vec{c} \cdot d\vec{a}$$

$$= \int_V \vec{\nabla} \cdot (\psi \vec{c}) \, d\tau$$

$$= \int_V \left[\vec{\nabla}\psi \cdot \vec{c} + \psi \underbrace{\vec{\nabla} \cdot \vec{c}}_{0} \right] d\tau$$

$$= \int_V \vec{c} \cdot \vec{\nabla}\psi \, d\tau = \underline{\vec{c} \cdot \int \vec{\nabla}\psi \, d\tau}$$

$\therefore$ since $\vec{c}$ is an arbitrary vector, we conclude $\oint \psi \, d\vec{a} = \int_V \vec{\nabla}\psi \, d\tau$

Area of a surface $A = \int_S da$

Also, we can define "vector Area"

$$\vec{A} = \int_S d\vec{a} = \int_S \hat{n}\, da$$

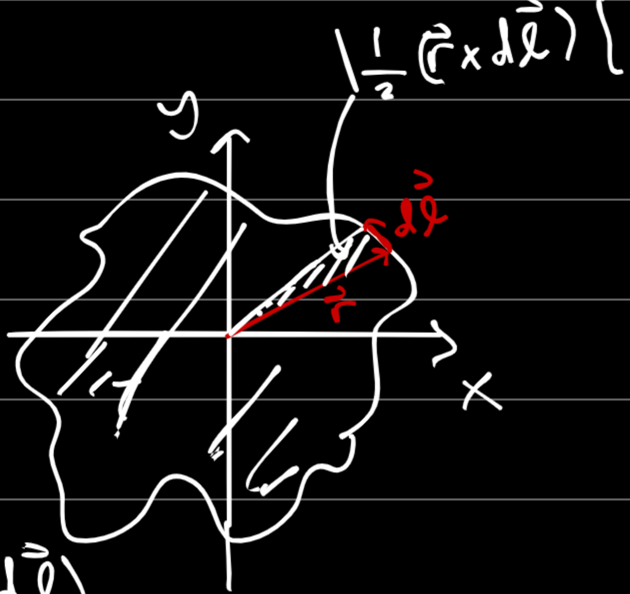
Note: ① $\vec{A} = \oint_S d\vec{a} = 0$

$\vec{c}$ is a constant vector

$$\vec{c} \cdot \oint_S d\vec{a} = \oint_S \vec{c} \cdot d\vec{a} = \int_V \vec{\nabla} \cdot \vec{c}\, d\tau = \underline{\underline{0}}$$

Divergence theorem

② $\underline{\vec{A}} = \underline{\frac{1}{2} \oint \vec{r} \times d\vec{l}}$



$$\underline{\vec{c}} \cdot \underline{\oint \vec{r} \times d\vec{l}} = \oint \underline{\vec{c} \cdot (\vec{r} \times d\vec{l})}$$

$$= \oint \underline{(\vec{c} \times \vec{r}) \cdot d\vec{l}}$$

$$= \int_S \vec{\nabla} \times (\vec{c} \times \vec{r}) \cdot d\vec{a} = \int_S 2\vec{c} \cdot d\vec{a}$$

$$= 2\vec{c} \cdot \int_S d\vec{a}$$

$$\vec{\nabla}\times(\vec{c}\times\vec{r}) = (\vec{r}\cdot\vec{\nabla})\vec{c} - (\vec{c}\cdot\vec{\nabla})\vec{r} + \vec{c}\,\underline{\vec{\nabla}\cdot\vec{r}} - \vec{r}\,\vec{\nabla}\cdot\vec{c}$$

$$= 0 - \vec{c} + 3\vec{c} - 0 = 2\vec{c}$$

$$(\vec{c}\cdot\vec{\nabla})\vec{r} = \left[c_x\frac{\partial}{\partial x} + c_y\frac{\partial}{\partial y} + c_z\frac{\partial}{\partial z}\right]\left[x\hat{x} + y\hat{y} + z\hat{z}\right]$$

$$= c_x\hat{x} + c_y\hat{y} + c_z\hat{z} = \vec{c}$$

$$\therefore \quad 2\vec{A} = \oint \vec{r}\times d\vec{l} \Rightarrow \vec{A} = \frac{1}{2}\oint \vec{r}\times d\vec{l}$$

$$* \quad (\vec{r}\cdot\vec{\nabla})\vec{c} = \left[(x\hat{x} + y\hat{y} + z\hat{z})\cdot\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\right.$$

$$\left[c_x\hat{x} + c_y\hat{y} + c_z\hat{z}\right]$$

$$= \left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right]\left[c_x\hat{x} + c_y\hat{y} + c_z\hat{z}\right]$$

$$= 0$$