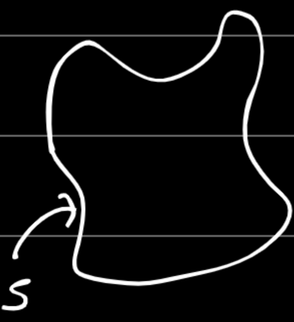
Gauss's Law:

Integral form: $\Phi_E = \oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$



Differential Form

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

@ spherical symmetry

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} \int_V \rho(\vec{r})\, d\tau$$

$\rho(\vec{r}) = \rho(r)$ ← spherical symmetry

$\rightarrow \vec{E}(\vec{r}) \rightarrow \vec{E}(r) \rightarrow \vec{E} = E(r)\,\hat{r}$

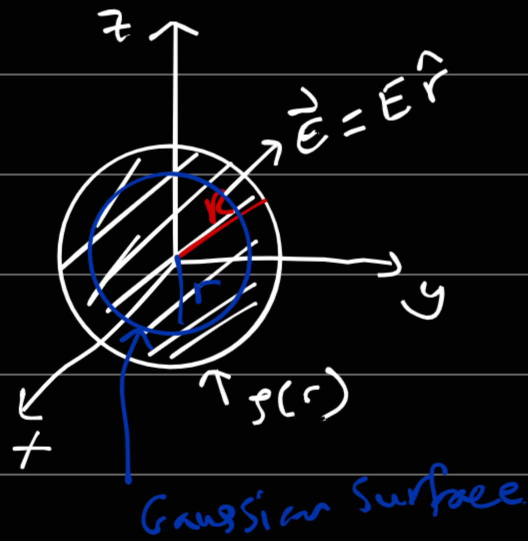
Note! $\rho(\vec{r}) = \begin{cases} \vec{a} \cdot \vec{r} & r < d \\ 0 & r \geq d \end{cases} \neq \rho(r)$

$\vec{a} = 2\hat{x} + 3\hat{y} + 4\hat{z}$

$\vec{r}_1 = 2\hat{x}$, $\vec{r}_2 = -2\hat{x}$

$$\rho(\vec{r}) = \frac{\sin(x)\cos(y)}{z}$$

Now,



Inside:

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

$$\int_0^{2\pi}\int_0^{\pi} E\,\hat{r}\cdot\left(r^2 \sin\theta \, d\theta \, d\varphi \, \hat{r}\right)$$

$$= \frac{1}{\varepsilon_0} \int_0^{r}\int_0^{2\pi}\int_0^{\pi} \rho(r)\, r^2 \sin\theta \, d\theta \, d\varphi$$

$$4\pi r^2 E = \frac{1}{\varepsilon_0} 4\pi \int_0^{r} \rho(r)\, r^2 dr$$

$$E = \frac{1}{\varepsilon_0 r^2} \int_0^{r} r^2 \rho(r)\, dr \qquad r < R$$

outside

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow 4\pi r^2 E = 4\pi \frac{1}{\varepsilon_0} \int_0^{R} r^2 \rho(r)\, dr$$

$$\Rightarrow E = \frac{1}{\varepsilon_0 r^2} \int_0^{R} r^2 \rho(r)\, dr$$

$$\vec{E}(\vec{r}) = \begin{cases} \dfrac{\hat{r}}{\varepsilon_0 r^2} \displaystyle\int_0^{r} r^2 \rho(r)\, dr & r \leq R \\[2ex] \dfrac{\hat{r}}{\varepsilon_0 r^2} \displaystyle\int_0^{R} r^2 \rho(r)\, dr & r > R \end{cases}$$

Example: $\rho(r) = \begin{cases} \frac{r}{R}\rho_0 & r \le R \\ 0 & r > 0 \end{cases}$

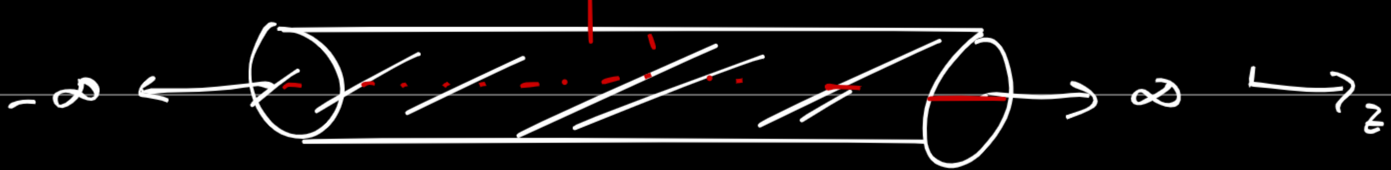
$$\vec{E}(\vec{r}) = \begin{cases} \frac{\hat{r}}{\varepsilon_0 r^2} \int_0^r \frac{r}{R}\rho_0 r^2 dr & r \le R \\ \frac{\hat{r}}{\varepsilon_0 r^2} \int_0^R \frac{r}{R}\rho_0 r^2 dr & r > R \end{cases}$$

$$\vec{E}(\vec{r}) = \begin{cases} \frac{r^2 \rho_0}{4\varepsilon_0 R}\hat{r} & r \le R \\ \frac{R^3 \rho_0}{\varepsilon_0 r^2}\hat{r} & r > R \end{cases}$$

③ Cylindrical Symmetry

$\rho(\vec{s}) = \rho(s)$ ← It does not depend on φ and z

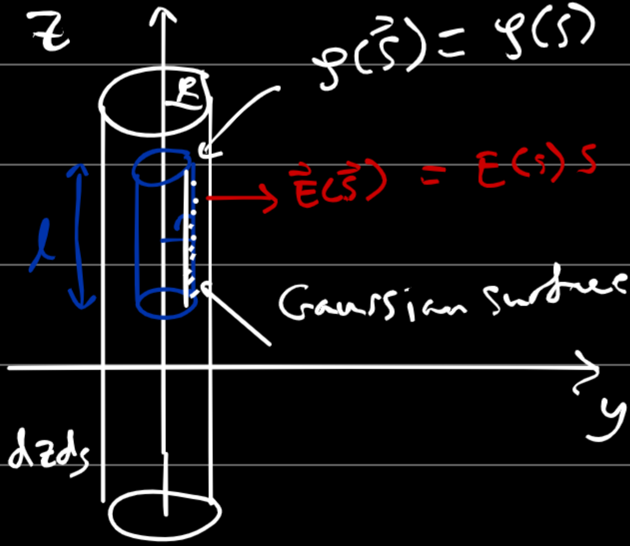
$\vec{E}(\vec{s}) = \underline{E(s)\hat{s}}$



e.g. $\rho(\vec{s}) = \begin{cases} -k s^2 & s \le R \\ 0 & \text{otherwise} \end{cases} = \rho(s)$ ✓

$\rho(\vec{s}) = \begin{cases} \vec{a}\cdot\vec{s} & s \le R \\ 0 & s > R \end{cases} \neq \rho(s)$ ✗

inside (i.e. $r \leq R$)



$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

$$\int_0^{2\pi} E\hat{s} \cdot (l s \, d\phi \, \hat{s}) = \frac{1}{\varepsilon_0} \int_0^s \int_0^l \int_0^{2\pi} \rho(s)\, s \, d\phi \, dz \, ds$$

$$\cancel{2\pi} \cancel{l} s E = \frac{\cancel{2\pi} \cancel{l}}{\varepsilon_0} \int_0^s s \rho(s) \, ds$$

$$\Rightarrow \vec{E} = \frac{1}{s \varepsilon_0} \int_0^s s \rho(s) \, ds \; \hat{s}$$

outside : $r \geq R$

$$\vec{E} = \frac{1}{s \varepsilon_0} \int_0^R s \rho(s) \, ds \; \hat{s}$$

$$\vec{E} = E(s)\hat{s} = \begin{cases} \dfrac{\hat{s}}{s \varepsilon_0} \displaystyle\int_0^s s \rho(s) \, ds & s \leq R \\[2ex] \dfrac{\hat{s}}{s \varepsilon_0} \displaystyle\int_0^R s \rho(s) \, ds & s \geq R \end{cases}$$

Example: consider a cylinder of radius R that have a volume charge density given by

$$\rho(\vec{s}) = -k s^2$$

$$\vec{E}(\vec{s}) = E(s)\hat{s} = \begin{cases} \dfrac{\hat{s}}{s\varepsilon_0}\displaystyle\int_0^s s(-ks^2)\,ds & s \le R \\ \dfrac{\hat{s}}{s\varepsilon_0}\displaystyle\int_0^R s(-ks^2)\,ds & s \ge R \end{cases}$$

$$= \begin{cases} \dfrac{-ks^3}{4\varepsilon_0}\hat{s} & s \le R \\ \dfrac{-kR^4}{4\varepsilon_0 s}\hat{s} & s \ge R \end{cases}$$