Electromagnetic Theory I

Abdallah Sayyed-Ahmad

Department of Physics
Birzeit University

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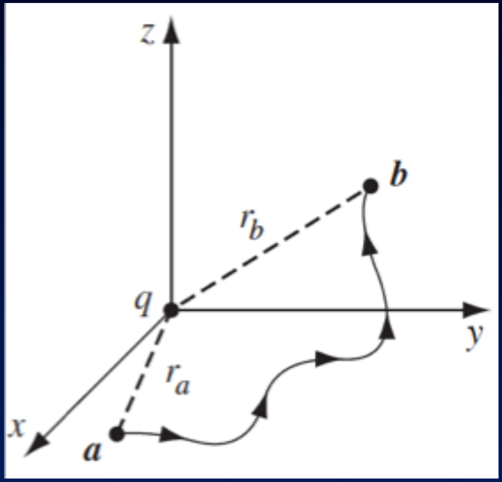
BIRZEIT UNIVERSITY

2.2.4 The curl of the Electric field

$$\vec{\mathrm{E}} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

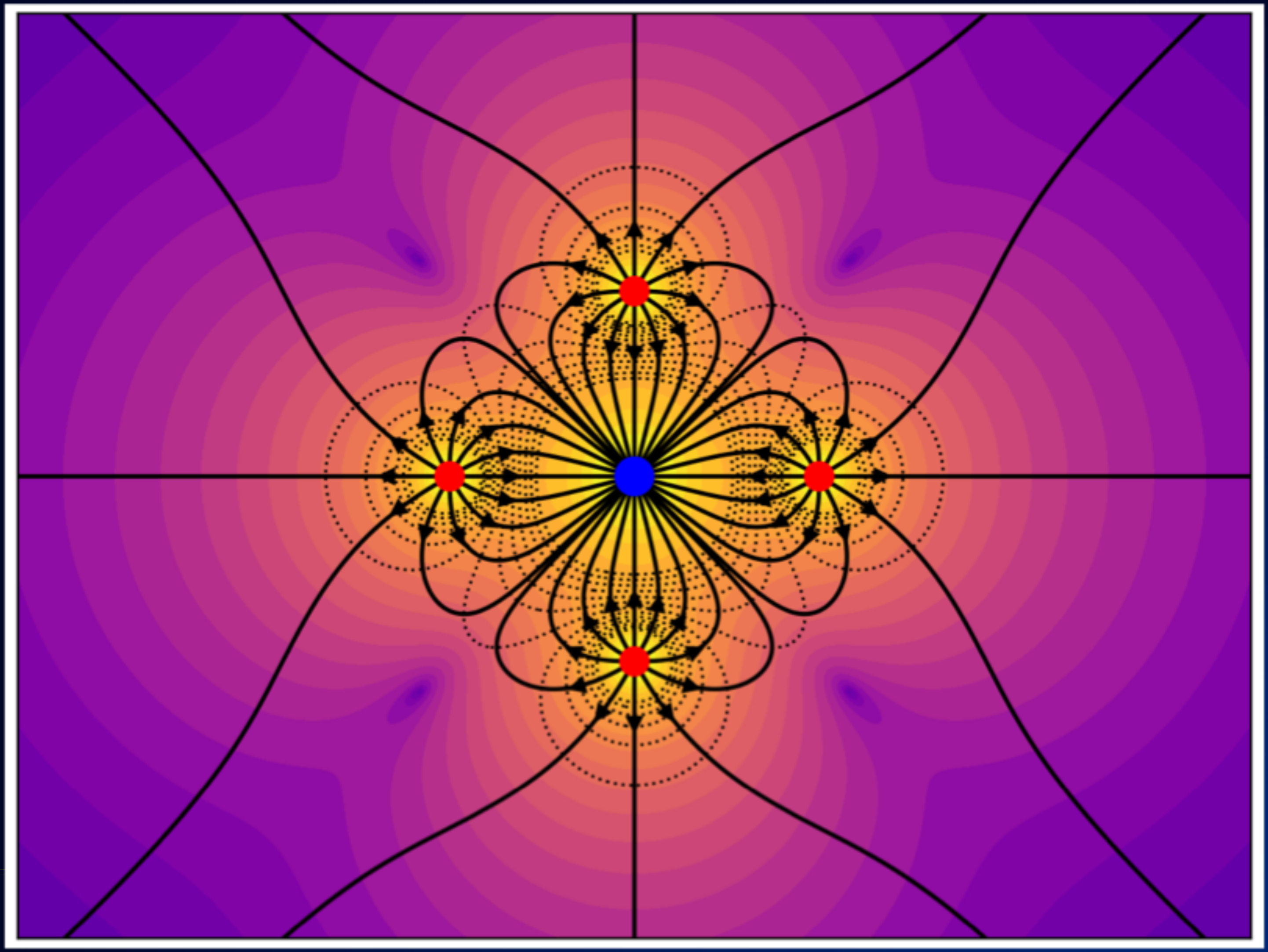
$$d\vec{l} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta\,\hat{\phi}$$



$$\int_{\vec{r}_a}^{\vec{r}_b} \vec{\mathrm{E}}\cdot d\vec{l} = \int_{\vec{r}_a}^{\vec{r}_b} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr = -\frac{1}{4\pi\epsilon_0}\frac{q}{r}\Bigg|_{r_a}^{r_b}$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) \rightarrow \oint \vec{\mathrm{E}}\cdot d\vec{l} = 0 \rightarrow \vec{\nabla}\times\vec{E} = 0$$

2.3 Electric Potential



2.3.1 Introduction to Potential

- The goal is to simplify the problem from a vector equation to a scalar one.
- We have previously shown

$$\vec{\nabla}\times\vec{E} = 0 \Leftrightarrow \oint \vec{E}\cdot d\vec{l} = 0$$

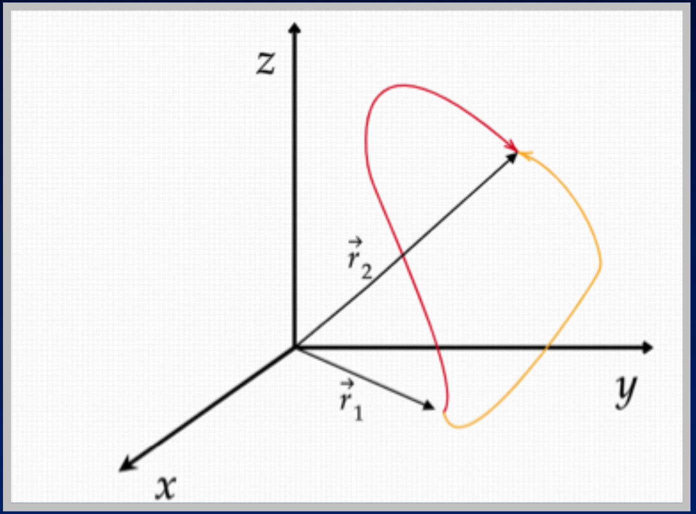
- This allows us to define a scalar function called Electric Potential

$$V(\vec{r}) = -\int_{\mathcal{O}}^{\vec{r}} \vec{E}\cdot d\vec{l}$$

Where $\mathcal{O}$ is some standard reference point.

2.3.1 Introduction to Potential

- The potential difference between two points $\vec{r}_2$ and $\vec{r}_1$ is given by



$$V(\vec{r}_2) - V(\vec{r}_1) = -\int_{\mathcal{O}}^{\vec{r}_2} \vec{\mathrm{E}} \cdot d\vec{l} + \int_{\mathcal{O}}^{\vec{r}_1} \vec{\mathrm{E}} \cdot d\vec{l}$$

$$= -\int_{\vec{r}_1}^{\vec{r}_2} \vec{\mathrm{E}} \cdot d\vec{l}$$

2.3.1 Introduction to Potential

- But we also know that the potential difference between two points $\vec{r}_2$ and $\vec{r}_1$ is given by the fundamental theorem of gradients

$$V(\vec{r}_2) - V(\vec{r}_1) = \int_{\vec{r}_1}^{\vec{r}_2} \vec{\nabla} V \cdot d\vec{l}$$

- Therefore, the electric field can be written as a gradient of a scalar function

$$\vec{E} = -\vec{\nabla} V$$

2.3.1 Introduction to Potential

Example: Can the electric field $\vec{E} = x\,\hat{x} + xy\,\hat{y} + z\,\hat{z}$ be realized.

A static electric field must satisfy $\vec{\nabla}\times\vec{E} = 0$

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & xy & z \end{vmatrix}$$

$$= \left(\frac{\partial z}{\partial y} - \frac{\partial xy}{\partial z}\right)\hat{x} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)\hat{y} + \left(\frac{\partial xy}{\partial x} - \frac{\partial x}{\partial y}\right)\hat{z} = y\hat{z} \neq 0$$

2.3.2 Comments on the potential

- Name origin
  - Potential and Potential Energy are completely different things and they should have different names, but it is used for historical reasons. Although there is a connection as we shall see later.
- **Advantage of using the potential**
  - As we have said earlier finding the potential first usually make the problem easier. To find the electric field you just take the gradient of the potential $\left(\vec{E} = -\vec{\nabla}V\right)$
  - This is amazing!! We can get $\vec{E}$ (three components) from a scalar V (one component)
  - This is the case because the three components of $\vec{E}$ are not really independent since $\vec{\nabla}\times\vec{\mathrm{E}} = 0$ (i.e. $\frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial z} = \frac{\partial E_z}{\partial y}$ and $\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}$)

2.3.2 Comments on the potential

- **The choice of the reference point $\mathcal{O}$**
  - The choice of reference point $\mathcal{O}$ is kind of arbitrary.
  - Changing reference points only adds a constant to the potential.

$$V'(\vec{r}) = -\int_{\mathcal{O}'}^{\vec{r}} \vec{\mathrm{E}} \cdot d\vec{l} = -\int_{\mathcal{O}}^{\vec{r}} \vec{\mathrm{E}} \cdot d\vec{l} - \int_{\mathcal{O}}^{\mathcal{O}} \vec{\mathrm{E}} \cdot d\vec{l} = V(\vec{r}) + constant$$

  - Changing reference will not affect the potential difference

$$V'(\vec{r}_2) - V'(\vec{r}_2) = V(\vec{r}_2) + c - V(\vec{r}_1) - c = V(\vec{r}_2) - V(\vec{r}_1)$$

  - Changing reference will not affect the electric field

$$\vec{E}' = -\vec{\nabla} V' = -\vec{\nabla}(V + c) = -\vec{\nabla} V + 0 = \vec{E}$$

  - We usually choose the refence point at ∞ and set it to zero (i.e. $V(\mathcal{O} = \infty) = 0$).

2.3.2 Comments on the potential

- **The potential obeys the super position principle**
  - The choice of reference point $\mathcal{O}$ is kind of arbitrary.
  - Changing reference points only adds a constant to the potential.

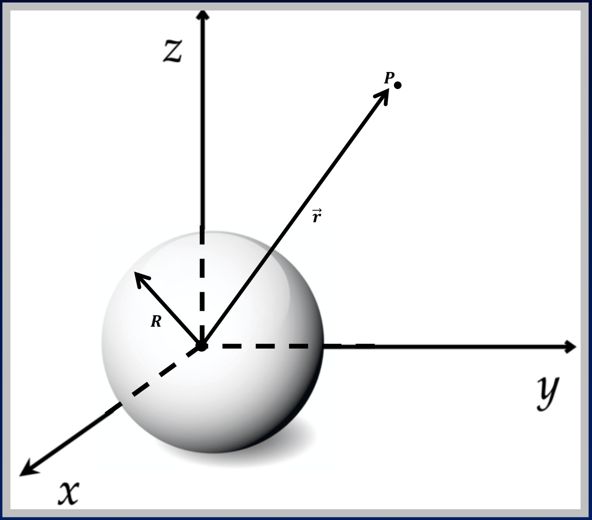
$$V(\vec{r}) = -\int_{\mathcal{O}}^{\vec{r}} \vec{\mathrm{E}} \cdot d\vec{l} = -\int_{\mathcal{O}}^{\vec{r}} (\vec{\mathrm{E}}_1 + \vec{\mathrm{E}}_2 + \cdots) \cdot d\vec{l} = V_1(\vec{r}) + V_2(\vec{r}) + \cdots$$

- **Units of the potential**

$$Volt = J/C = Nm/C$$

2.3.2 Comments on the potential

Example: Find the potential inside and outside a spherical shell of radius R with a uniform surface charge of total q.



We can easily show using Gauss's Law that

$$\vec{E} = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^2}\,\hat{r} & r \geq R \\ 0 & r < R \end{cases}$$

*We choose the reference point at infinity.

Now, for r $\geq R$,

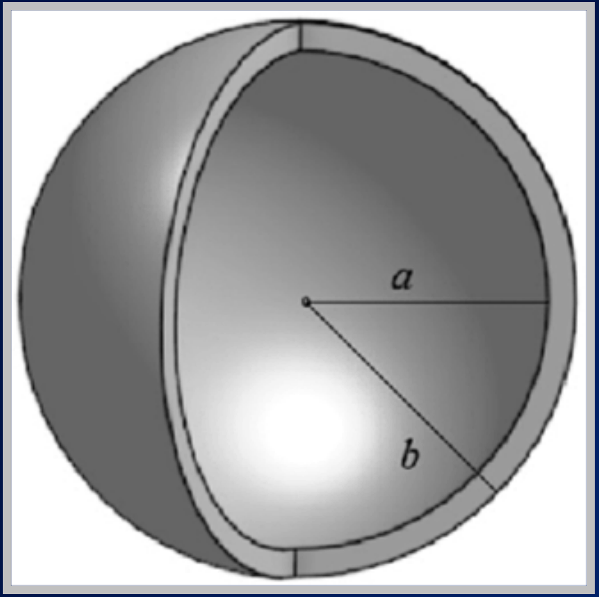
$$V(\vec{r}) = -\int_{\mathcal{O}}^{\vec{r}} \vec{\mathrm{E}}\cdot d\vec{l} = -\int_{\infty}^{r} \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\,dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\bigg|_{\infty}^{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

Also, for r $< R$,

$$V(\vec{r}) = -\int_{\mathcal{O}}^{\vec{r}} \vec{\mathrm{E}}\cdot d\vec{l} = -\int_{\infty}^{R} \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\,dr' - \int_{R}^{r} 0\,dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\bigg|_{\infty}^{R} = \frac{1}{4\pi\epsilon_0}\frac{q}{R}$$

2.3.2 Comments on the potential

Exercise: Find the potential everywhere for a spherical shell of inner radius a and outer radius b with a uniform volume charge of total q.



2.3.4 The potential of a localized charge distribution

We will reduce the two equations needed for finding the electric field into one equation for the potential

$$\vec{\nabla}\times\vec{\mathrm{E}} = 0 \Leftrightarrow \vec{E} = -\vec{\nabla}V$$

Hence, Gauss's Law in the differential form gives

$$\vec{\nabla}\cdot\vec{\mathrm{E}} = -\vec{\nabla}\cdot\vec{\nabla}V = \rho/\epsilon_0$$

Or it gives what is called Poisson Equation

$$\nabla^2 V = -\rho/\epsilon_0$$

In regions of space were there is no charge density, the above equation reduces to Laplace equation

$$\nabla^2 V = 0$$

2.3.4 The Potential of Localized Charge Distribution

Based on the assumption that the reference point is at infinity:

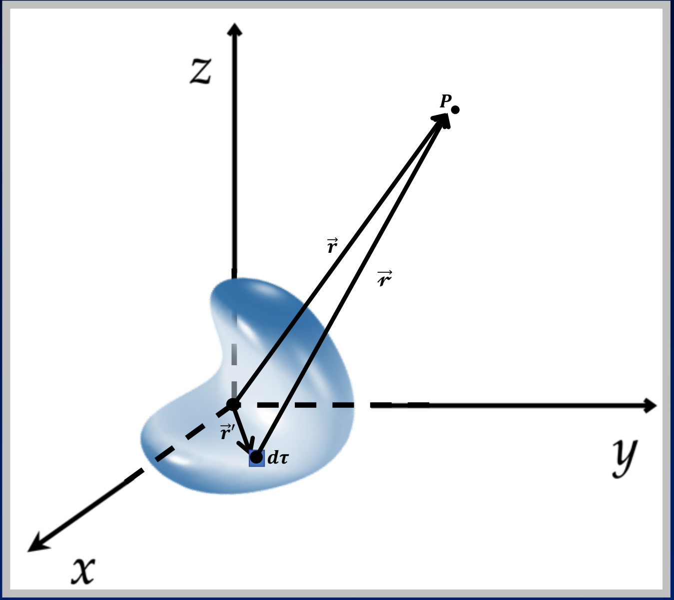
The potential of a point charge q:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r}'|} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}$$

The potential of a collection of point charges:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\vec{r}-\vec{r}_i{}'|} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{\mathscr{r}_i}$$

The potential of a continuous charge distribution:



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{dq}{\mathscr{r}}$$

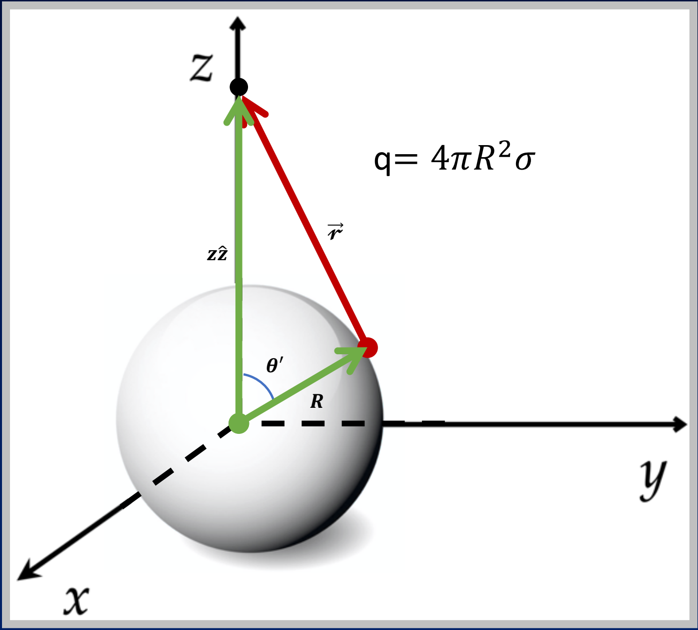
For a volume, surface, or line charge:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}')}{\mathscr{r}}d\tau', V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma(\vec{r}')}{\mathscr{r}}da', V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\lambda(\vec{r}')}{\mathscr{r}}dl'$$

2.3.4 The Potential of Localized Charge Distribution

Example: Find the potential of a uniformly charged spherical shell of radius R.

$$da' = R^2 sin\theta' d\theta' d\phi'; r' = R; \mathscr{r} = \sqrt{R^2 + z^2 - 2Rzcos\theta'}$$



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\vec{r}')}{\mathscr{r}} da'$$

$$= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi} \frac{\sigma}{\sqrt{R^2 + z^2 - 2Rzcos\theta'}} R^2 sin\theta' d\theta' d\phi'$$

$$V(\vec{r}) = \frac{\sigma R^2}{2\epsilon_0}\int_0^{\pi} \frac{sin\theta' d\theta'}{\sqrt{R^2 + z^2 - 2Rzcos\theta'}} = \frac{\sigma R^2}{2\epsilon_0}\int_0^{\pi} \frac{sin\theta' d\theta'}{\sqrt{R^2 + z^2 - 2Rzcos\theta'}}$$

2.3.4 The Potential of Localized Charge Distribution

$$V(\vec{r}) = \frac{\sigma R^2}{2\epsilon_0} \int_{-2Rz}^{2Rz} \frac{du}{2Rz\sqrt{R^2 + z^2 + u}} = \frac{\sigma R}{2\epsilon_0 z} \sqrt{R^2 + z^2 + u}\,\Big|_{-2Rz}^{2Rz}$$

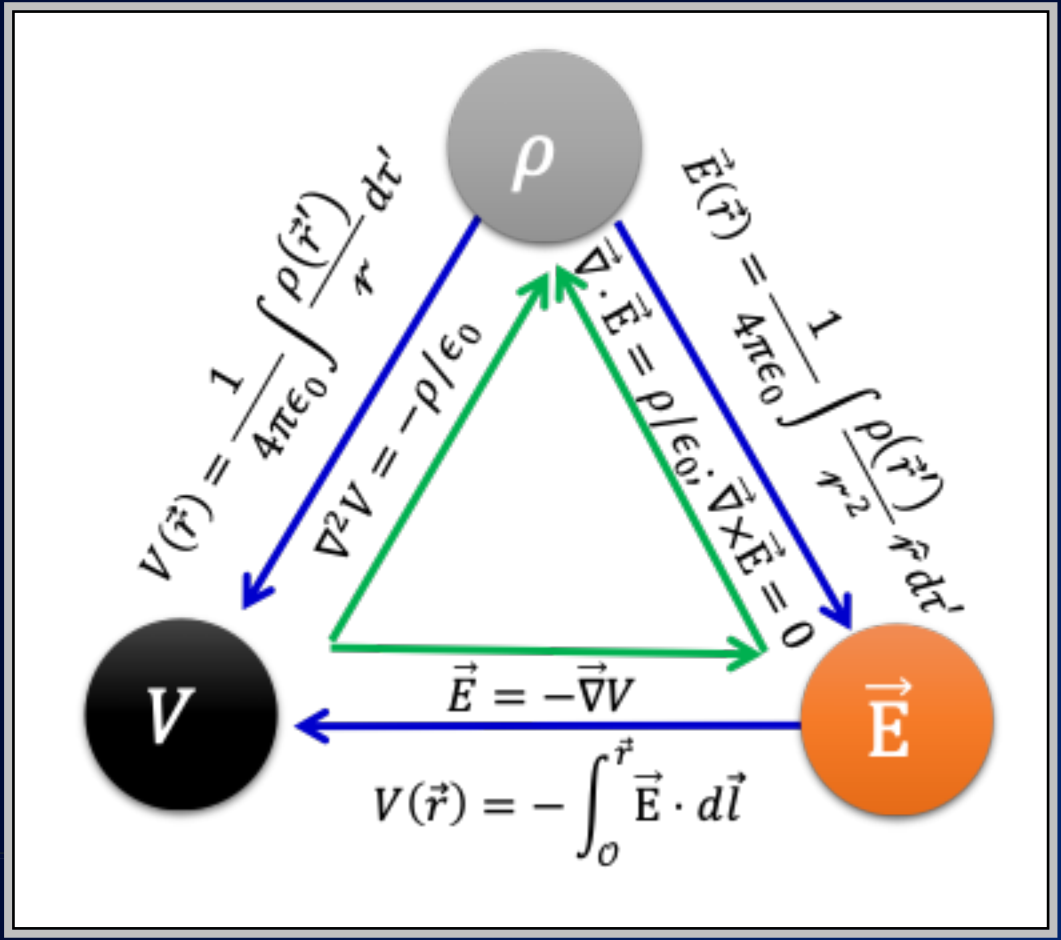
$$V(\vec{r}) = \frac{\sigma R}{2\epsilon_0 z}\left(\sqrt{R^2 + z^2 + 2Rz} - \sqrt{R^2 + z^2 - 2Rz}\right)$$

$$V(\vec{r}) = \frac{\sigma R}{2\epsilon_0 z}\left(\sqrt{(R + z)^2} - \sqrt{(R - z)^2}\right)$$

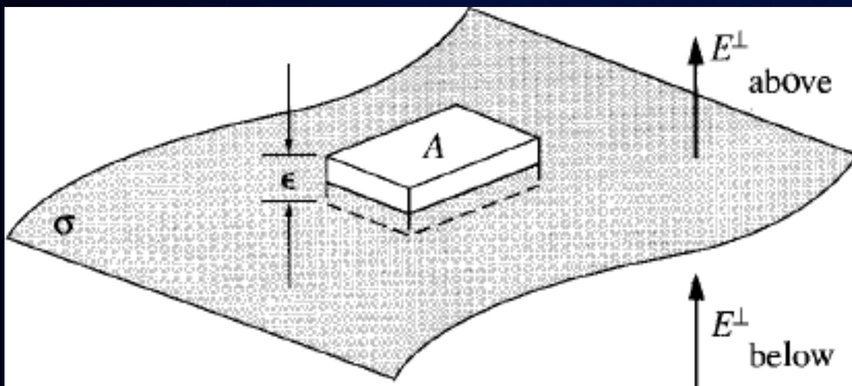
$$V(\vec{r}) = \begin{cases} \dfrac{\sigma R^2}{\epsilon_0 z} & z \geq R \\ \dfrac{\sigma R}{\epsilon_0} & z \leq R \end{cases}$$

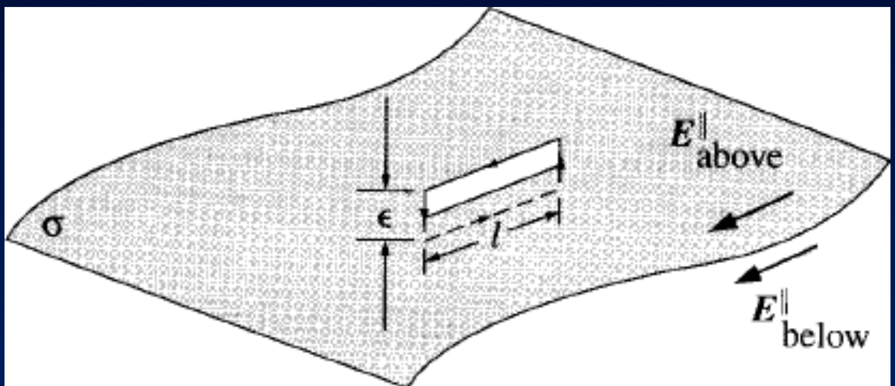
2.3.5 Summary of Electrostatics

All derived from Coulomb Law and Superposition principle



2.3.5 of Electrostatics Boundary Conditions





$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon 0}$$

The *normal component of* $\vec{E}$ is discontinuous

$$E^{\parallel}_{above} - E^{\parallel}_{below} = 0$$

The *tangential component of* $\vec{E}$ is continuous

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon 0}\hat{n}$$