Electromagnetic Theory I

Abdallah Sayyed-Ahmad

Department of Physics
Birzeit University

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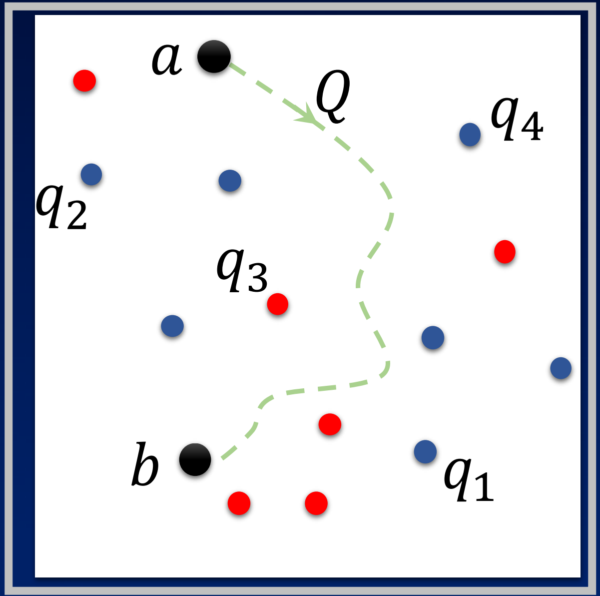
BIRZEIT UNIVERSITY

2.4.1 The work done to move a charge

- The work needed to move a charge from point a to point b is given by

$$W = \int_a^b \vec{F} \cdot d\vec{l} = -Q \int_a^b \vec{E} \cdot d\vec{l} = Q[V(b) - V(a)]$$

$\vec{F}$ opposite to $\vec{E}$



- **The potential difference between points a and b is equal to the work per unit charge required to carry a particle from a to b.**

$$[V(b) - V(a)] = W/Q$$

- To bring the charge Q from far away to its position at $\vec{r}$,

$V(a) = V(\infty)=0$ implies $W = QV(\vec{r})$ or Potential is potential energy per unit charge

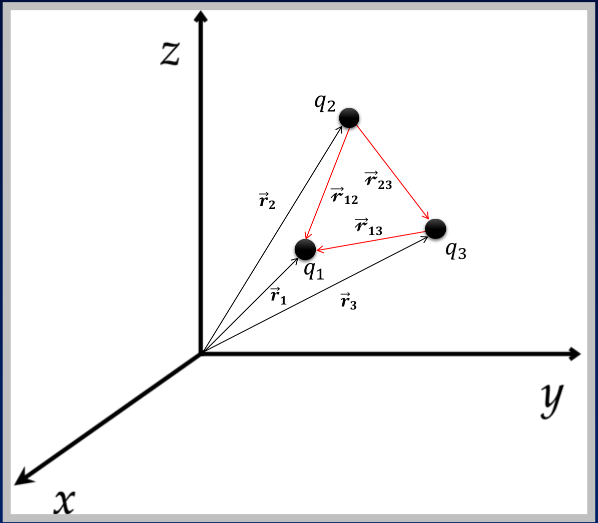
2.4.2 The energy of a charge distribution

- The work needed to assemble a collection of point charges

$$W_1 = q_1 V(\vec{r}_1) = 0; W_2 = q_2 V(\vec{r}_2) = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}}$$

$$W_3 = q_3 V(\vec{r}_3) = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\epsilon_0}\frac{q_2 q_3}{r_{23}}$$

$$W = W_1 + W_2 + W_3 = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\epsilon_0}\frac{q_2 q_3}{r_{23}}\right)$$



- In general for n charges

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j=1; j>i}^{n}\frac{q_i q_j}{r_{ij}} = \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j=1; j\neq i}^{n}\frac{q_i q_j}{r_{ij}} = \frac{1}{2}\sum_{i=1}^{n} q_i \sum_{j=1; j\neq i}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_j}{r_{ij}}$$

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r}_i)$$

2.4.3 The energy of a continuous charge distribution

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r}_i) \qquad W = \frac{1}{2}\int \rho V d\tau \qquad W = \frac{1}{2}\int \sigma V da \qquad W = \frac{1}{2}\int \lambda V dl$$

$$W = \frac{1}{8\pi\epsilon_0}\int\int \frac{\rho(\vec{r}_1)\rho(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|} d\tau_1\, d\tau_2$$

This can be rewritten in terms of the electric field

$$W = \frac{1}{2}\int \rho V d\tau = \frac{1}{2}\int (\epsilon_0\, \vec{\nabla}\cdot\vec{E}) V d\tau = \frac{\epsilon_0}{2}\int [\vec{\nabla}\cdot(V\vec{E}) - \vec{\nabla}V\cdot\vec{E}] d\tau$$

$$= \frac{\epsilon_0}{2}\int [\vec{\nabla}\cdot(V\vec{E}) + \vec{E}\cdot\vec{E}] d\tau = \frac{\epsilon_0}{2}\int E^2 d\tau + \frac{\epsilon_0}{2}\int V\vec{E}\cdot d\vec{a}$$

$\propto 1/r$

$$= \frac{\epsilon_0}{2}\int_{all\ space} E^2 d\tau$$

2.4.3 The energy of a continuous charge distribution

Example: Find the energy of a uniformly charged spherical shell of radius R and total charge q.

$$\vec{E}(\vec{r}) = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{r} & r \geq R \\ 0 & r < R \end{cases}$$

$$V(\vec{r}) = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{q}{r} & r \geq R \\ \frac{1}{4\pi\epsilon_0}\frac{q}{\boldsymbol{R}} & r < R \end{cases}$$

$$\boldsymbol{\rho}(\vec{\boldsymbol{r}}) = \frac{q}{4\pi R^2}\delta(r - R)$$

$$W = \frac{\epsilon_0}{2}\int_{all\ space} E^2 d\tau = \frac{q^2}{32\pi^2\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_R^{\infty}\frac{1}{r^4}\,r^{\mathbf{2}}\sin\theta\,drd\theta d\phi = -\frac{q^2}{8\pi\epsilon_0}\frac{1}{r}\Big|_R^{\infty} = \frac{q^2}{8\pi\epsilon_0 R}$$

or

$$W = \frac{1}{2}\int \rho V d\tau = W = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{q}{4\pi R^2}\delta(r - R)V(r)r^{\mathbf{2}}\sin\theta\,drd\theta d\phi = \frac{q^2}{8\pi\epsilon_0 R}$$

2.4.4 Comments on Electrostatic Energy

Can be + or -

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r}_i) \qquad W = \frac{1}{2}\int \rho V d\tau$$

BUT

$$W = \frac{\epsilon_0}{2}\int_{all\ space} E^2 d\tau > 0$$

What is happening???

Do we have an inconsistency in the equations!!

- It does not take into account the work needed to create the point charges in the first place; we started with point charges and simply found the work required to bring them together.

$$W = \frac{q^2}{32\pi^2\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{1}{r^4}r^2\sin\theta\cos\theta\, dr d\theta d\phi = \infty$$

- For an isolated charge, it indicates that the energy of a single point charge is infinite.
- The elementary particles change only their relative position (interaction energy).
- It is important how much energy it takes to make an elementary particle and the field it carries carries along.

The two formulation give similar results for a continuous distribution there is no distinction. This is because the amount of charge at a given point is extremely small, and its contribution to the potential is negligible.

2.4.4 Comments on Electrostatic Energy

Where is the energy stored?

$$W = \frac{1}{2}\int \rho V d\tau \qquad\qquad W = \frac{\epsilon_0}{2}\int_{all\ space} E^2 d\tau$$

In the charge distribution that is localized? OR in the field itself that spans whole space

- The difference is purely a matter of bookkeeping
- Total energy should be the same. The choice depends on the application
  - In radiation theory it is useful to regard the energy as being stored in the field, with a density $u\frac{\epsilon_0}{2}E^2$
  - In electrostatics it is commonly used it is stored in the charge, with a density $1/2\rho V$

2.4.4 Comments on Electrostatic Energy

The superposition principle?

$$W_{tot} = \frac{\epsilon_0}{2}\int E^2 d\tau = \frac{\epsilon_0}{2}\int \left|\vec{E}_1 + \vec{E}_2\right|^2 d\tau = \frac{\epsilon_0}{2}\int \left(E_1^2 + E_2^2 + 2\vec{E}_1 \cdot \vec{E}_2\right) d\tau = W_1 + W_2 + W_{12}$$

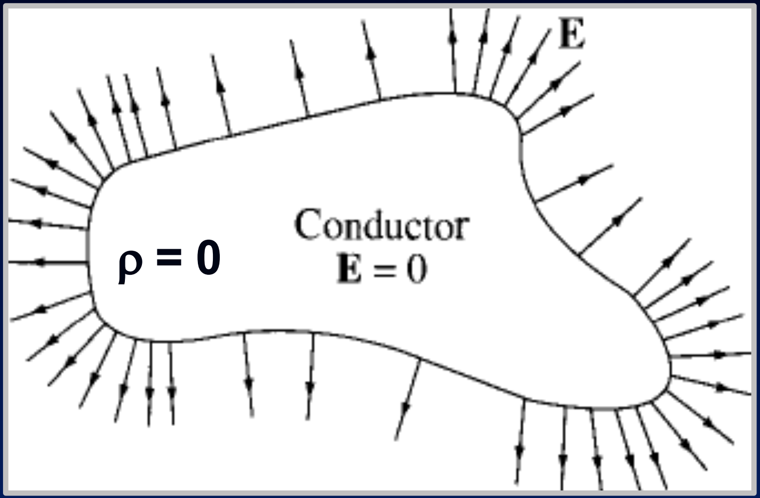
$$W_{12} = \epsilon_0 \int \vec{E}_1 \cdot \vec{E}_2 d\tau$$

2.5 Conductors: Basic Properties

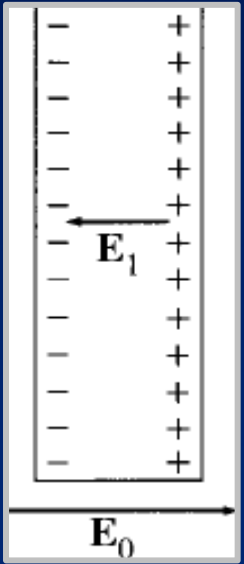
- The electric field and net charge density is zero inside a conductor

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}; \vec{E} = 0 \rightarrow \rho = 0$$

- Any net charge resides on the surface
- A conductor is an equipotential surface
- The electric field is perpendicular on the surface



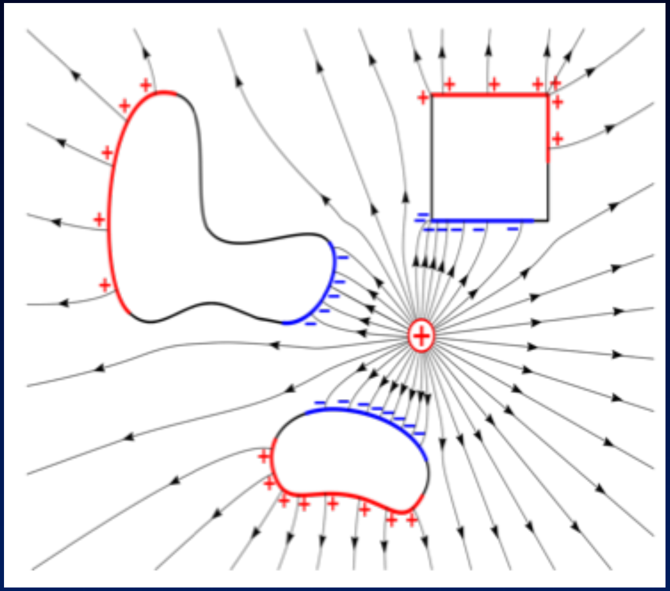
- When a conductor into an external electric field $\vec{E}_0$, *induced charges* produce a field $\vec{E}_1$.
- *$\vec{E}_1$ tends to cancel $\vec{E}_0$.*

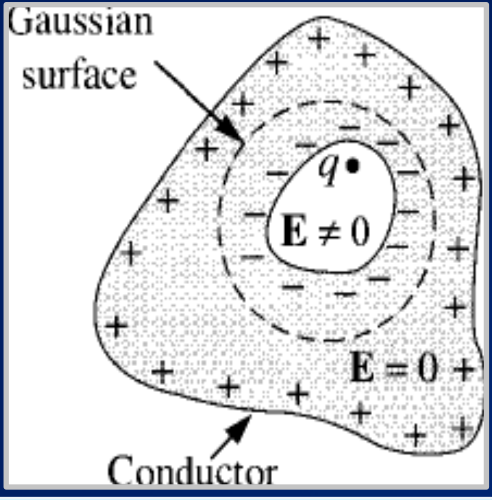


2.5.2 Induced Charges

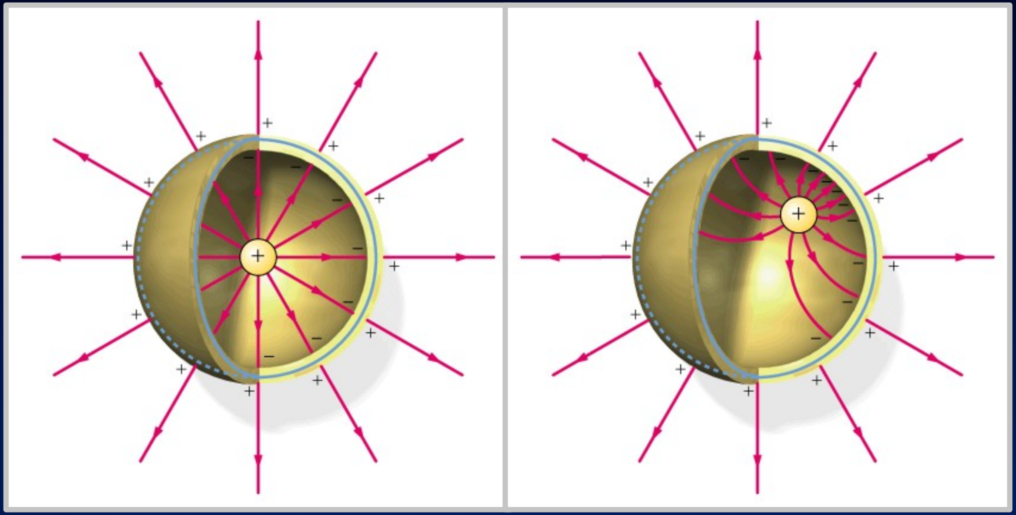
- If you hold a charge $+q$ near an uncharged conductor, the two will attract one another. The negative induced charge is closer to q.
- There is a net force of attraction

- If there is some cavity in the conductor that contain a charge,
- The electric field in the cavity will not be zero.
- No external fields penetrate the conductor; they are canceled at the outer surface by the induced charge.



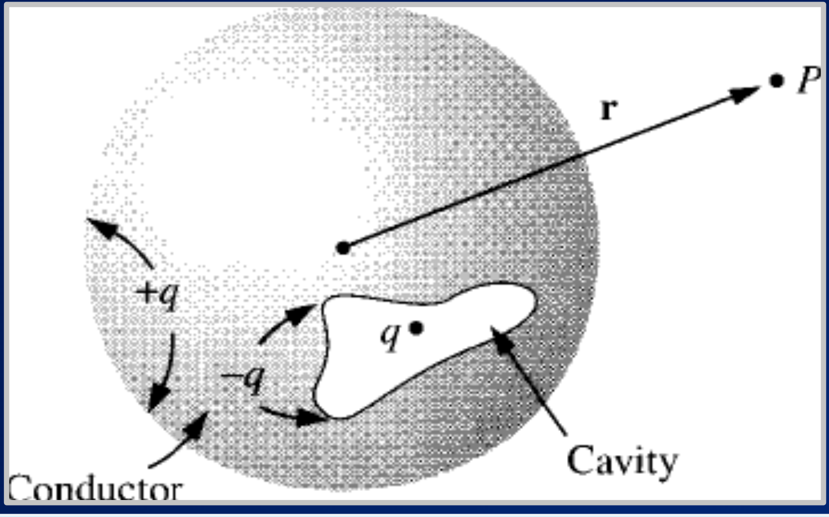


2.5.2 Induced Charges



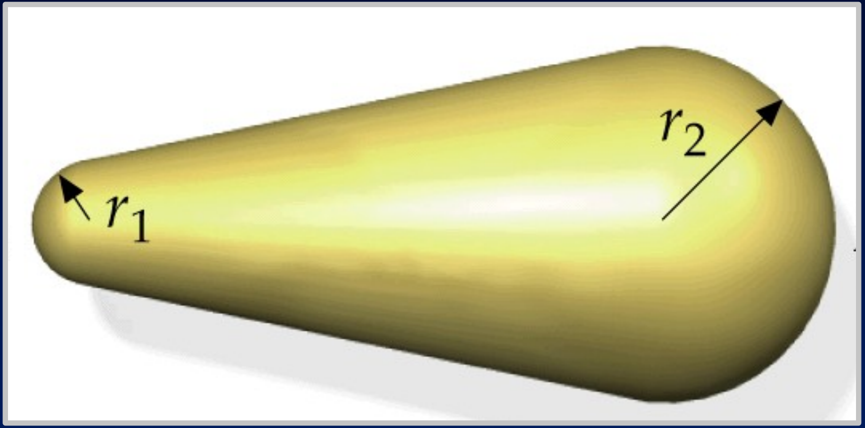
Conductors conceal all the information regarding the nature of their interior

Example: Find the electric field of a metallic sphere of radius R and has a cavity that contains a point charge q



2.5 Conductors

- Exercise: Compare the field strength at the two ends of a lightning-rod



2.5.3 Surface Charge and Force on a conductor

For a conductor, $E^{\perp}_{below} = E^{\parallel}_{below} = \mathbf{0}$

$$\longrightarrow \quad E^{\perp}_{above} = \frac{\sigma}{\epsilon 0};\, E^{\parallel}_{above} = 0$$

The electric field just out side the conductor

$$\vec{E}_{above} = \frac{\sigma}{\epsilon 0}\hat{n}$$

The surface charge density on a conductor can be determined

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$$

2.5.4 Capacitors

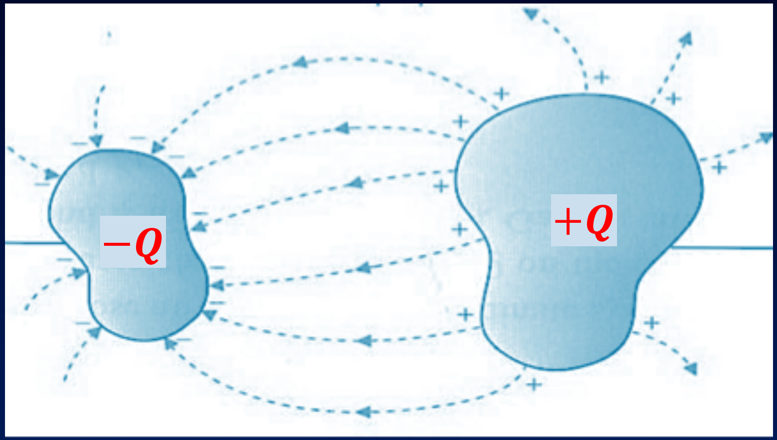
$$\Delta V = V(+) - V(-) = -\int_{-}^{+} \vec{E} \cdot d\vec{l}$$

$$Recall, \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau'$$

Doubling $\boldsymbol{Q}$ *does* double ρ everywhere
Therefore, $\boldsymbol{E}$ *is proportional to* $\boldsymbol{Q}$

$$\Delta V \propto Q$$

$$\mathrm{C} = \mathrm{Q}/\Delta V$$

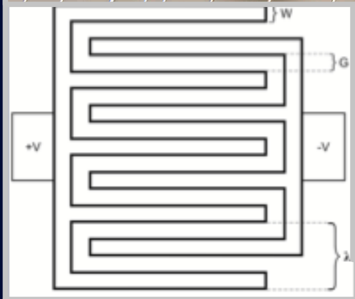


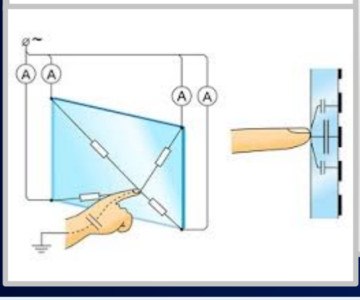
Work needed to charge a capacitor

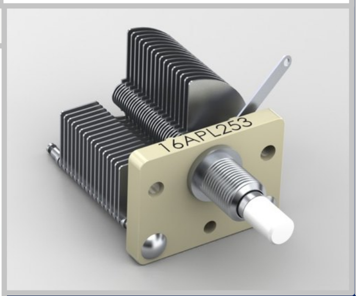
$$dW = V dq = \frac{q}{C} dq$$

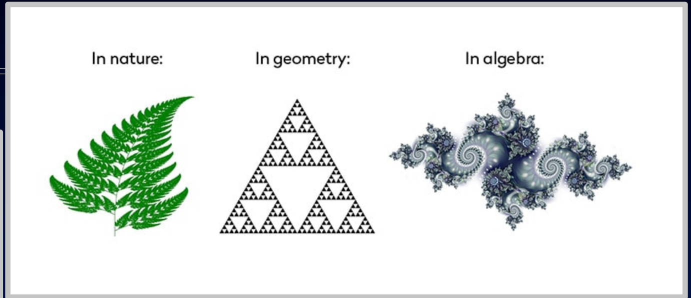
$$W = \int_0^Q \frac{q}{C} dq\, Vd = \frac{1}{2}\frac{Q^2}{C}$$

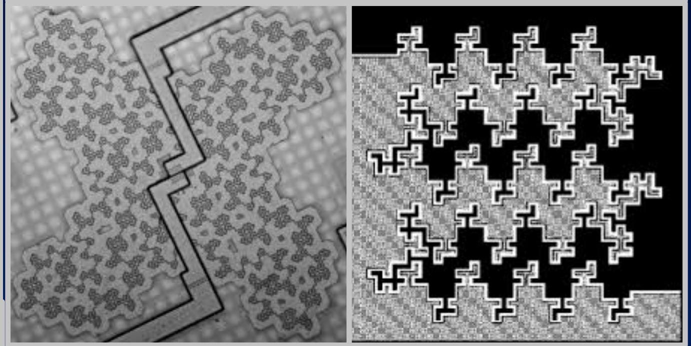
2.5.4 Capacitors

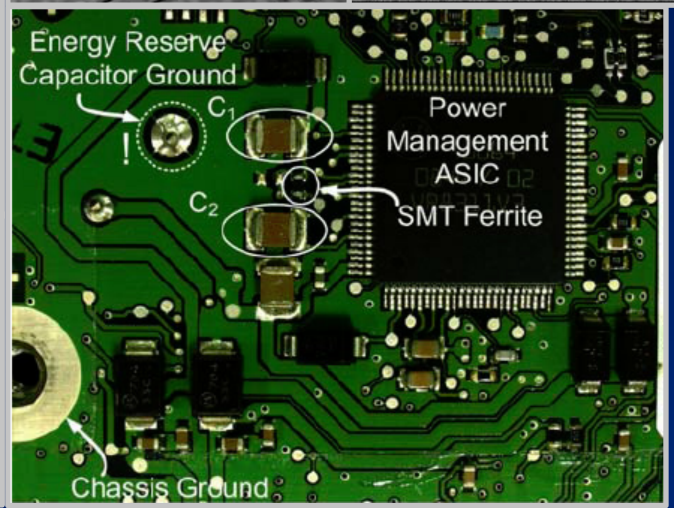












Example: Find the capacitance of a parallel plate capacitor of cross section area A and separation distance d

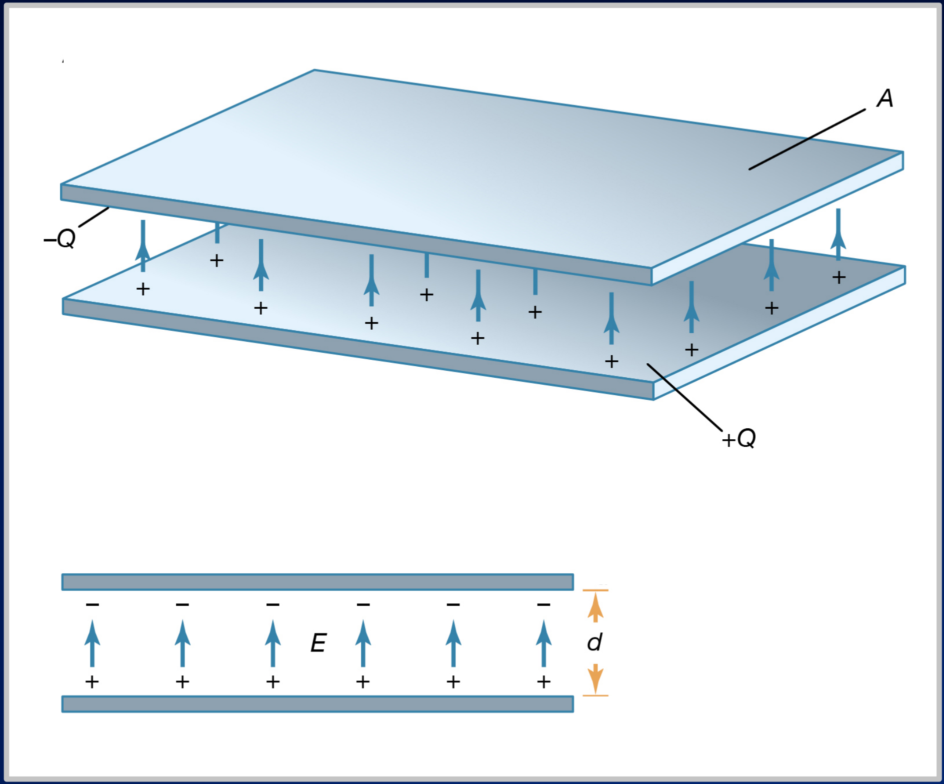
$$\sigma = \frac{Q}{A}$$

$$\vec{E} = \begin{cases} \frac{\sigma}{\epsilon_0}\hat{z} & 0 < z < d \\ 0 & otherwise \end{cases}$$

$$\Delta V = V(+) - V(-) = -\int_{-}^{+} \vec{E} \cdot d\vec{l}$$

$$\Delta V = -\int_{d}^{0} \frac{\sigma}{\epsilon_0} dz = \frac{\sigma d}{\epsilon_0} = \frac{Qd}{A\epsilon_0}$$

$$\mathrm{C} = \frac{\mathrm{Q}}{\Delta V} = \frac{Q}{\left(\frac{Qd}{A\epsilon_0}\right)} = \epsilon_0 \frac{A}{d}$$



Example: Find the capacitance of a two concentric metallic spheres of inner radius a and outer radius b

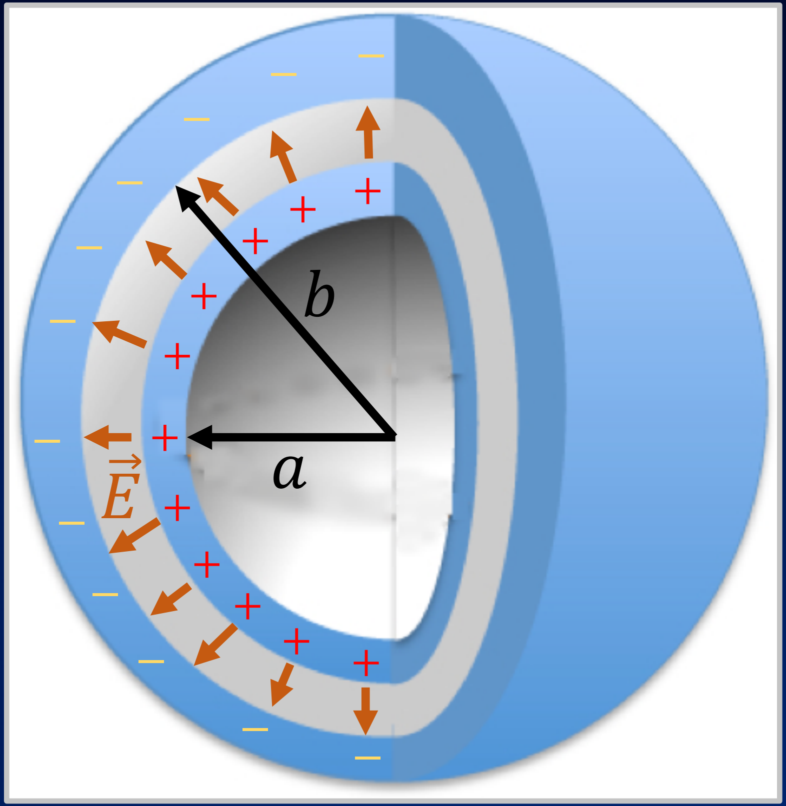
$$\vec{E} = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r^2}\hat{r} & a < r < b \\ 0 & otherwise \end{cases}$$

$$\Delta V = V(+) - V(-) = -\int_{-}^{+} \vec{E}\cdot d\vec{l}$$

$$\Delta V = -\int_b^a \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}dr = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}\bigg|_b^a$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

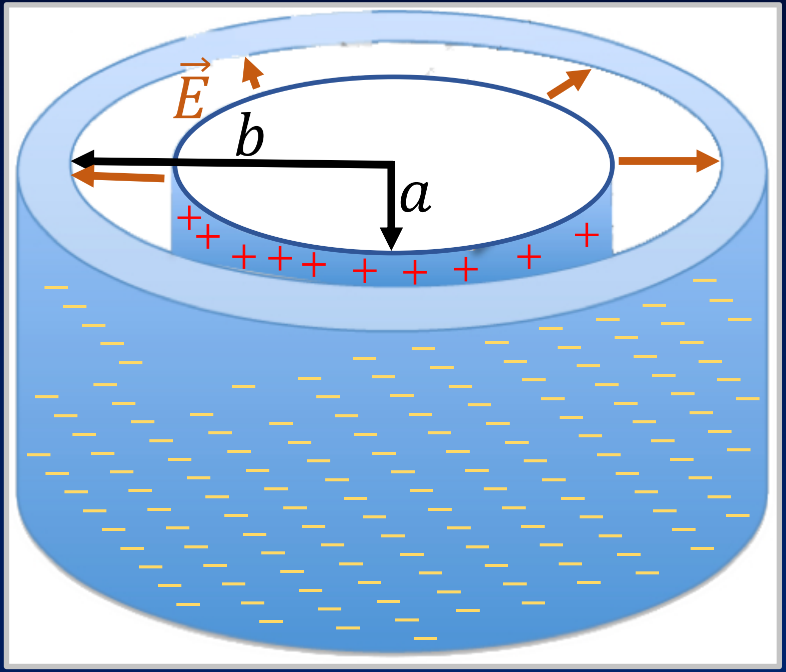
$$\mathrm{C} = \frac{\mathrm{Q}}{\Delta V} = \frac{Q}{\left(\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)\right)} = \epsilon_0\frac{4\pi ab}{(b-a)}$$



Example: Find the capacitance of a two concentric metallic spheres of inner radius a and outer radius b

$$\vec{E} = \begin{cases} \frac{1}{2\pi L\epsilon_0}\frac{Q}{s}\hat{s} & a < s < b \\ 0 & otherwise \end{cases}$$

$$\Delta V = V(+) - V(-) = -\int_{-}^{+} \vec{E}\cdot d\vec{l}$$



$$\Delta V = -\int_b^a \frac{1}{2\pi L\epsilon_0}\frac{Q}{s}ds = \frac{Q}{2\pi L\epsilon_0} ln(s)\Big|_a^b$$

$$= \frac{Q}{2\pi L\epsilon_0} ln\left(\frac{b}{a}\right)$$

$$\mathrm{C} = \frac{\mathrm{Q}}{\Delta V} = \frac{Q}{\left(\frac{Q}{2\pi L\epsilon_0} ln\left(\frac{b}{a}\right)\right)} = \epsilon_0 \frac{2\pi L}{ln\left(\frac{b}{a}\right)}$$