Electromagnetic Theory I

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Chapter 3: Potentials

- Laplace Equation
- The Method of Images
- Separation of Variables
- Multipole Expansion

3.1 Laplace Equation

The goal is to find the electric field of a given stationary charge distribution using

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\, d\tau'$$

We can simplify this by exploiting symmetry and using Gauss 's law, but for most practical cases finding V is recommended

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')}{\mathscr{r}}\, d\tau'$$

However this integral is often too tough to solve analytically. Therefore, the problem can be cast as a partial differential equation called Poisson's Equation (with appropriate boundary conditions)

$$\nabla^2 V = -\rho/\epsilon_0$$

We are sometimes interested in finding the potential in a region where there is no charge density. In this case, Poisson's equation reduces to Laplace's equation:

$$\nabla^2 V = 0$$

3.1.2 Laplace Equation

In generalized coordinates

$$\nabla^2 V = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial x_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial V}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_2 h_1}{h_3}\frac{\partial V}{\partial x_3}\right)\right) = 0$$

In Spherical coordinates

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

In Cylindrical coordinates

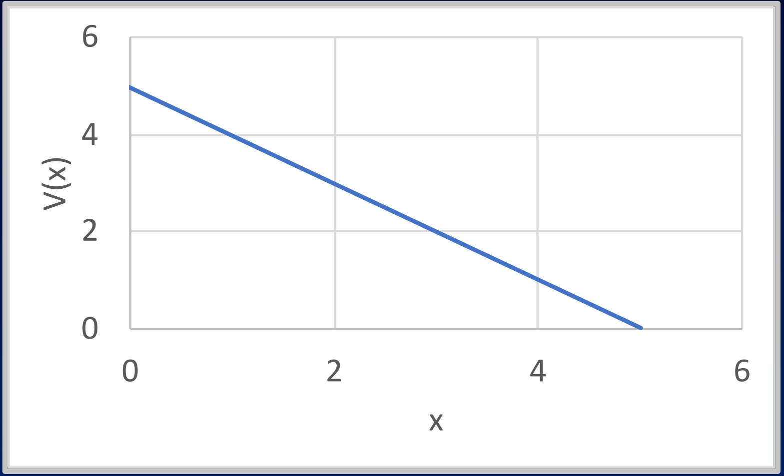
$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

In Cartesian coordinates

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

3.1.2 Laplace Equation in One Dimension

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} = 0$$



The general solution

$$V(x) = mx + b$$

m and b can be determined from the boundary conditions

Notes on the result:

$$V(x) = \frac{1}{2}[V(x+a) + V(x-a)]$$

Laplace's equation tolerates no local maxima or minima; Extreme values of V must occur at the end points

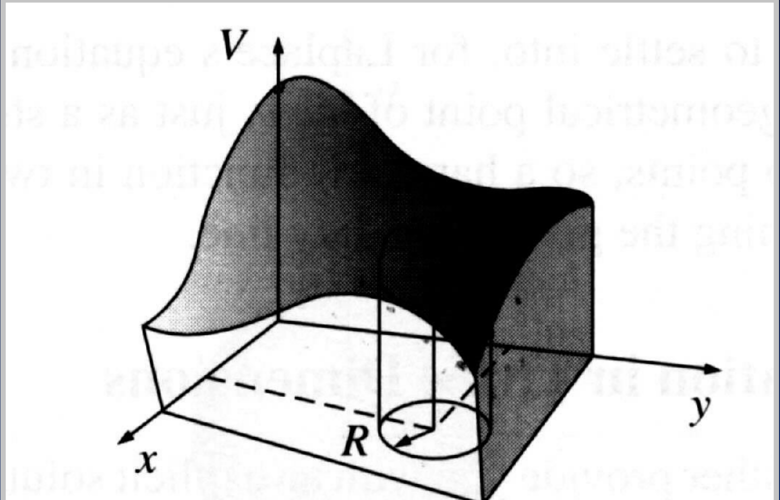
3.1.3 Laplace Equation in Two Dimensions

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

The general solution is satisfied by harmonic functions u,v:

$f(z) = u(x, y) + iv(x, y)$ is analytic

$$\leftrightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; and \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$



Potential, Rubber membrane, Soap film

Notes on the solution:

$$V(x, y) = \frac{1}{2\pi R} \oint_{circle} V dl$$

Laplace's equation tolerates no local maxima or minima; Extreme values of V must occur at the boundary points

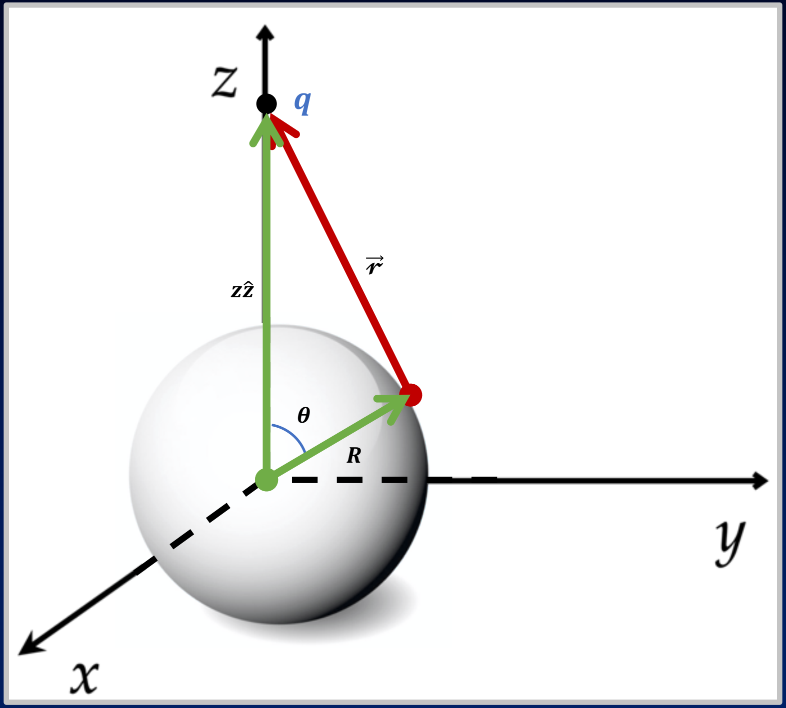
3.1.4 Laplace Equation in Three Dimensions

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Notes on the solution:

$$V(\vec{r}) = \frac{1}{4\pi R^2} \oint_{sphere} V da$$

The value of V at point $\vec{r}$ is the average of V over a spherical surface of radius R centered at $\vec{r}$:



V has no local maxima or minima; all extrema occur at the boundaries

Earnshaw's Theorem:

A charged particle cannot be held in a stable equilibrium
by electrostatic forces alone.

3.1.4 Laplace Equation in Three Dimensions

Proof:

Let us first start by calculating the average potential over a spherical surface of radius R due to a *single* point charge q located outside the sphere at distance z.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

$$V(0) = \frac{1}{4\pi\epsilon_0}\frac{q}{z}$$

$$r = \sqrt{z^2 + R^2 - 2zR\cos\theta}$$

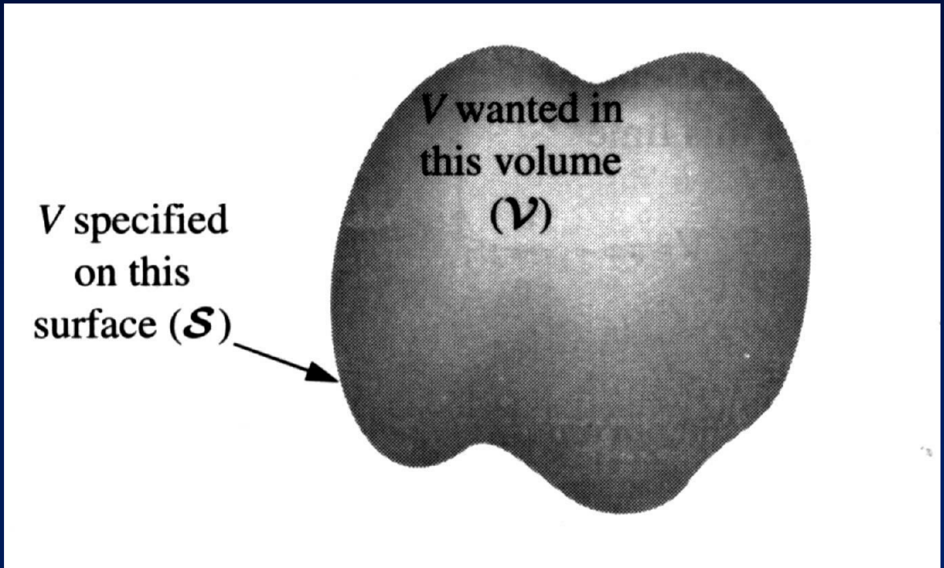
$$V_{avg} = \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\frac{R^2\sin\theta\ d\theta d\phi}{\sqrt{z^2 + R^2 - 2zR\cos\theta}}$$

$$= \frac{1}{2zR}\frac{q}{4\pi\epsilon_0}\sqrt{z^2 + R^2 - 2zR\cos\theta}\Big|_0^{\pi} = \frac{1}{4\pi\epsilon_0}\frac{q}{z}$$

3.1.5 Boundary Conditions and Uniqueness Theorem

First Uniqueness Theorem:

The solution to Laplace's equation in some volume V is uniquely determined if V is specified on the boundary surface S.



The potential in a volume V is uniquely determined if

a) the charge density in the region, and
b) the values of the potential on all boundaries are specified.

3.1.5 Boundary Conditions and Uniqueness Theorem

Proof:

Suppose there were two solutions to Laplace's equation V_1 and V_2 : Their difference $V_3 = V_2 - V_1$ will obey Laplace's equation.

$$if\ \nabla^2 V_1 = 0\ and\ \nabla^2 V_2 = 0, then\ \nabla^2 V_3 = \nabla^2 V_2 - \nabla^2 V_1 = 0$$

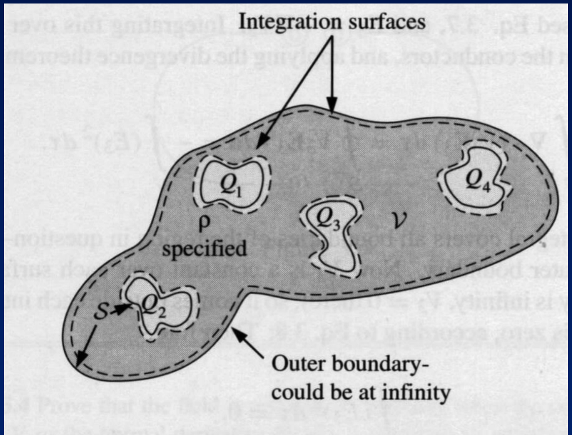
V_3 takes the value zero on all boundaries (since V_1 and V_2 are equal there).
But Laplace's equation allows no local maxima or minima all extrema occur on the boundaries.
So the maximum and minimum of V_3 are both zero. Therefore V_3 must be zero everywhere,

$$if\ \nabla^2 V_1 = -\frac{\rho}{\epsilon_0}\ and\ \nabla^2 V_2 = -\frac{\rho}{\epsilon_0}, then\ \nabla^2 V_3 = \nabla^2 V_2 - \nabla^2 V_1 = 0$$

3.1.6 Boundary Conditions and Uniqueness Theorem

Second Uniqueness Theorem:

- There are other circumstances in which we do not know the *potential* at the boundary, but rather know the *charges* on various conducting surfaces.
  - If you put charge Q_1 on the first conductor, Q_2 on the second, … Charges moves around resulting in some specified charge density in the region between the conductors.
  - Is the electric field now uniquely determined?
  - Or are there perhaps a number of different ways the charges could arrange themselves on their respective conductors, each leading to a different field?



In a volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density . the electric field is uniquely determined if the *total charge* on each conductor is given.

3.1.6 Boundary Conditions and Uniqueness Theorem

Proof:

$$\vec{\nabla}\cdot\vec{E}_1 = \frac{\rho}{\epsilon_0}$$

$$\oint_{ith\ conducting\ surface} \vec{E}_1 \cdot d\vec{a} = \frac{Q_i}{\epsilon_0} \qquad \oint_{outer\ boundary} \vec{E}_1 \cdot d\vec{a} = \frac{Q_{tot}}{\epsilon_0}$$

$$\vec{\nabla}\cdot\vec{E}_2 = \rho/\epsilon_0$$

$$\oint_{ith\ conducting\ surface} \vec{E}_2 \cdot d\vec{a} = \frac{Q_i}{\epsilon_0} \qquad \oint_{outer\ boundary} \vec{E}_2 \cdot d\vec{a} = \frac{Q_{tot}}{\epsilon_0}$$

$$\vec{E}_3 = \vec{E}_2 - \vec{E}_1 \qquad \vec{\nabla}\cdot\vec{E}_3 = 0 \qquad \oint \vec{E}_3 \cdot d\vec{a} = 0$$

3.1.6 Boundary Conditions and Uniqueness Theorem

Proof:

$$\vec{\nabla}\cdot\left(V_3\vec{E}_3\right) = V_3\vec{\nabla}\cdot\vec{E}_3 + \vec{E}_3\cdot\vec{\nabla}V_3 = -E_3^2$$

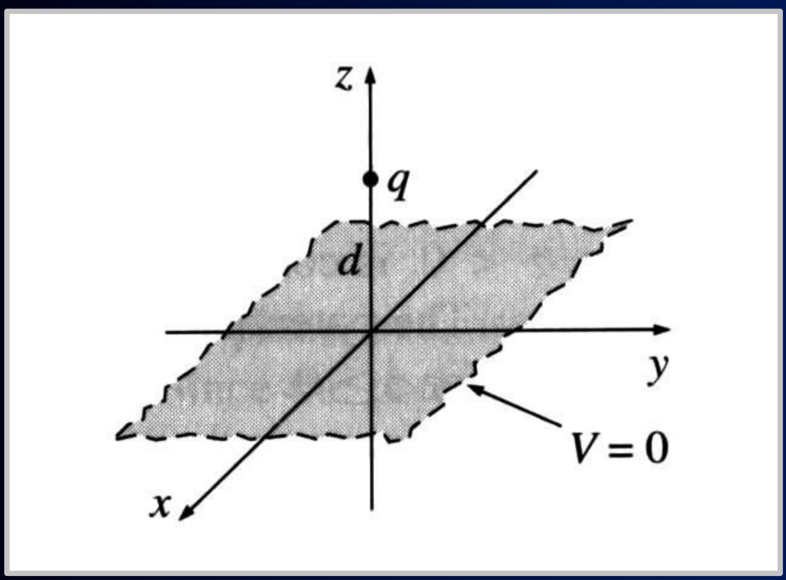
$$\int_\Omega \vec{\nabla}\cdot\left(V_3\vec{E}_3\right)d\tau = \oint \mathrm{V}_3\vec{E}_3\cdot d\vec{a} = 0 = -\int_\Omega E_3^2 d\tau$$

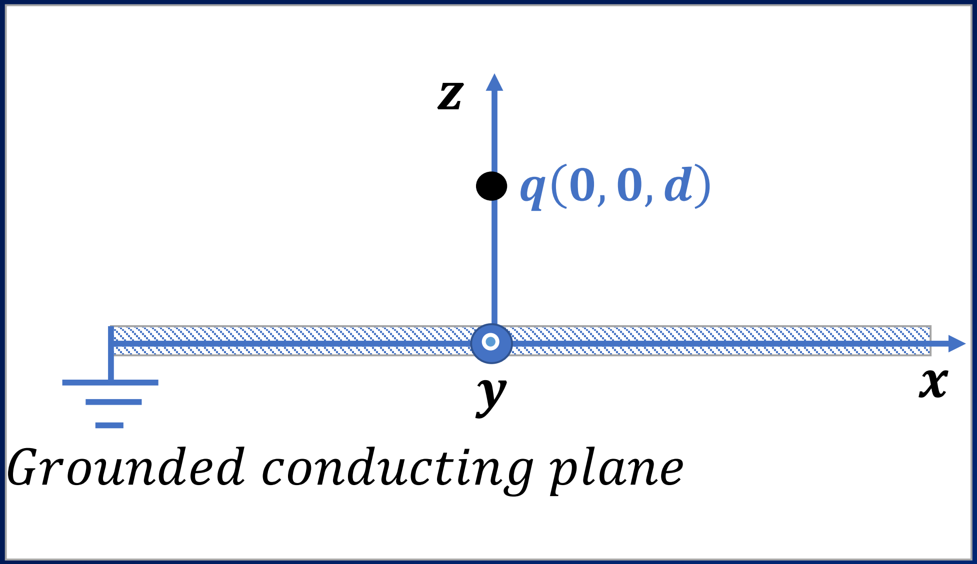
$$\vec{E}_3 = 0$$

3.2 The Method of Images

It is a method that replaces the original boundary by appropriate image charges so simplify the formal solution of Poisson equation of the original problem.

Point charge above a conducting grounded plane:

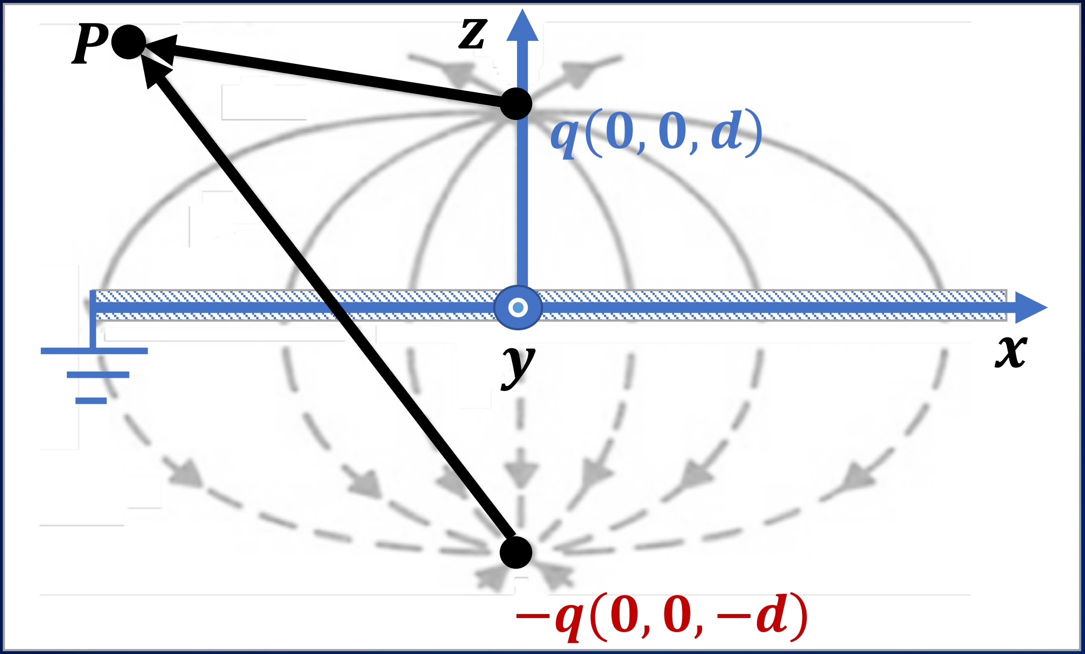




3.2 The Method of Images

The equation above the grounded plane

$$\nabla^2 V = -q\delta(x)\delta(y)\delta(z-d)/\epsilon_0$$



Boundary condition

$$V(x, y, 0) = 0$$

Solution

$$V(x,y,z) = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\left[\dfrac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \dfrac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right] & z \geq 0 \\ 0 & z < 0 \end{cases}$$

3.2.2 Induced Surface Charge

Induced charge density

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial z}\bigg|_{z=0} = -\frac{1}{2\pi}\frac{qd}{(x^2+y^2+d^2)^{3/2}}$$

Total induced charge

$$Q = \int\int \sigma da = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -\frac{1}{2\pi}\frac{qd\,dxdy}{(x^2+y^2+d^2)^{3/2}}$$

$$Q = \int_0^{2\pi}\int_0^{\infty} -\frac{1}{2\pi}\frac{qd\,sdsd\phi}{(s^2+d^2)^{3/2}} = \frac{qd}{(s^2+d^2)^{\frac{1}{2}}}\bigg|_0^{\infty} = -q$$

3.2.3 Force

Force on q

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} q dQ = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} q\sigma da$$

$$\vec{F} = \frac{q}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(x^2+y^2+d^2)} \frac{(-x\hat{x} - y\hat{y} + d\hat{z})}{(x^2+y^2+d^2)^{1/2}} \left(-\frac{1}{2\pi}\frac{qd}{(x^2+y^2+d^2)^{3/2}}\right) dxdy$$

$$\vec{F} = -\frac{q^2 d}{8\pi^2\epsilon_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(-x\hat{x} - y\hat{y} + d\hat{z})}{(x^2+y^2+d^2)^3} dxdy = -\frac{q^2 d^2}{4\pi\epsilon_0}\hat{\mathbf{z}}\int_0^{\infty}\frac{sds}{(s^2+d^2)^3}$$

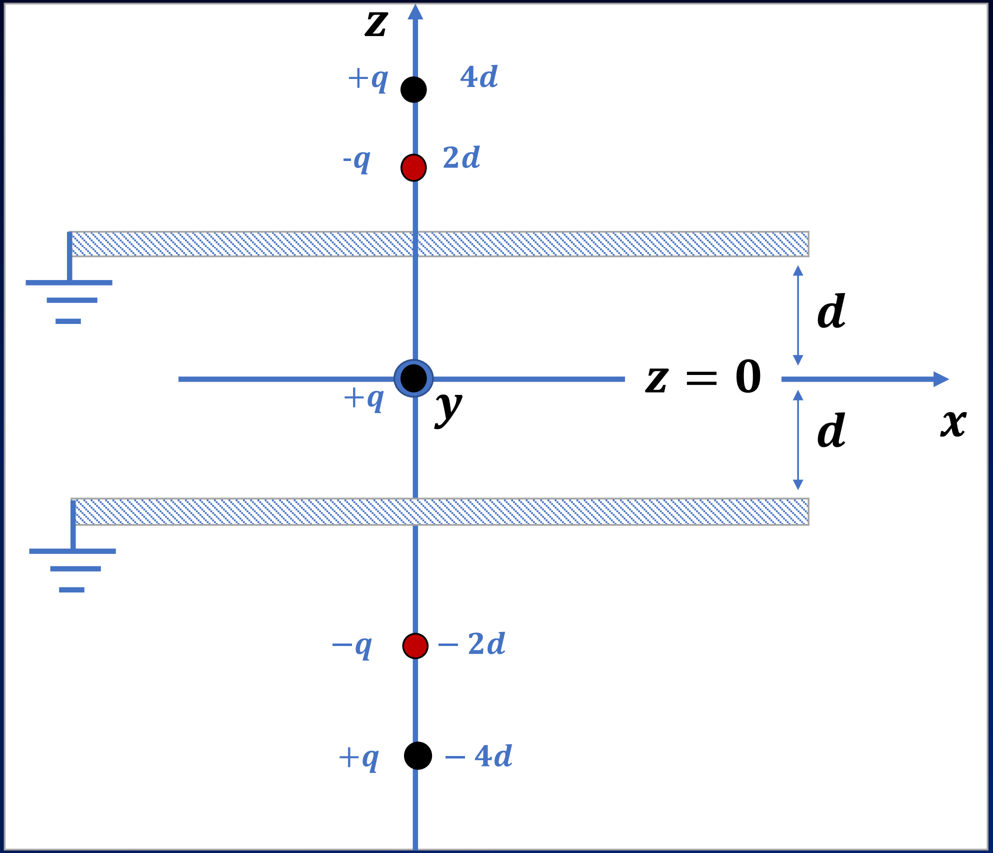
$$\vec{F} = \frac{q^2 d^2}{16\pi\epsilon_0}\hat{\mathbf{z}}\, u^{-2}\Big|_{d^2}^{\infty} = -\frac{q^2}{4\pi\epsilon_0(2d)^2}\hat{\mathbf{z}}$$

3.2.3 Energy

$$W = \int \vec{F} \cdot d\vec{l} = \int_{\infty}^{d} \frac{q^2}{4\pi\epsilon_0 (2z)^2} dz = -\frac{q^2}{4\pi\epsilon_0} \frac{1}{4z}\Big|_{\infty}^{d} = -\frac{q^2}{4\pi\epsilon_0} \frac{1}{4d}$$

Different from the energy for assembly of two charges

3.2.4 Other Image Problems



The equation

$$\nabla^2 V = -q\delta(x)\delta(y)\delta(z)/\epsilon_0$$

Boundary condition

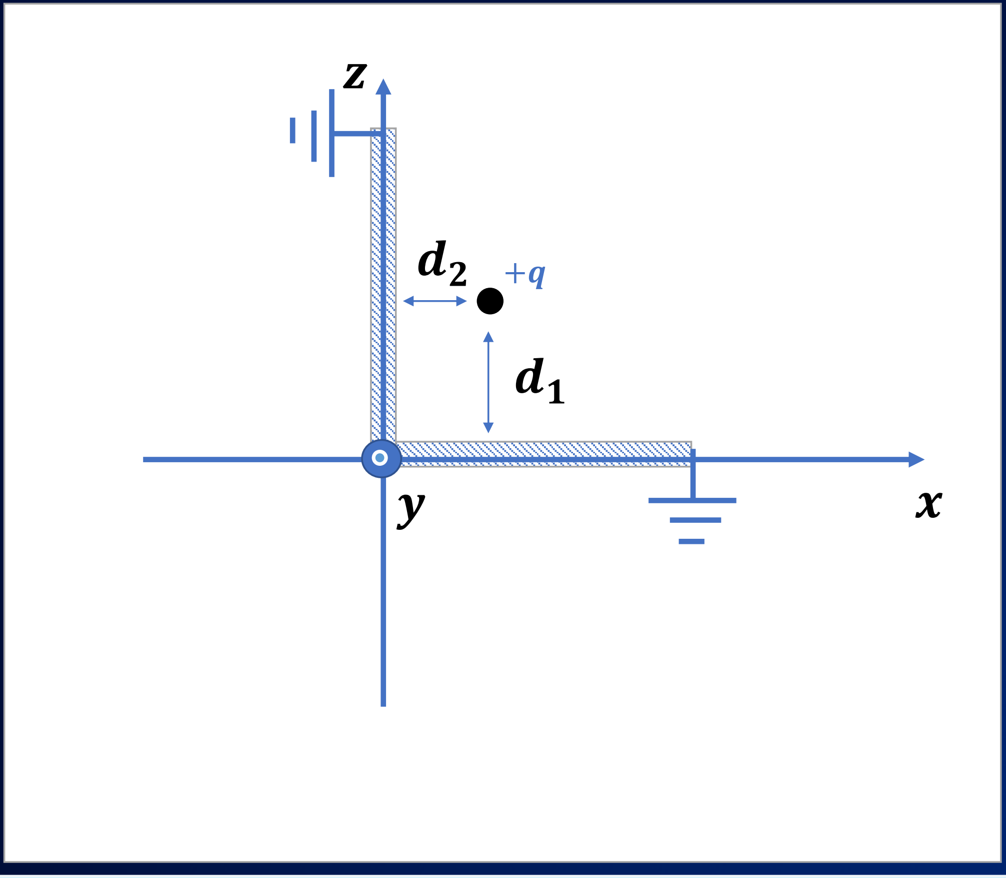
$$V(x, y, d) = 0$$
$$V(x, y, -d) = 0$$

Solution

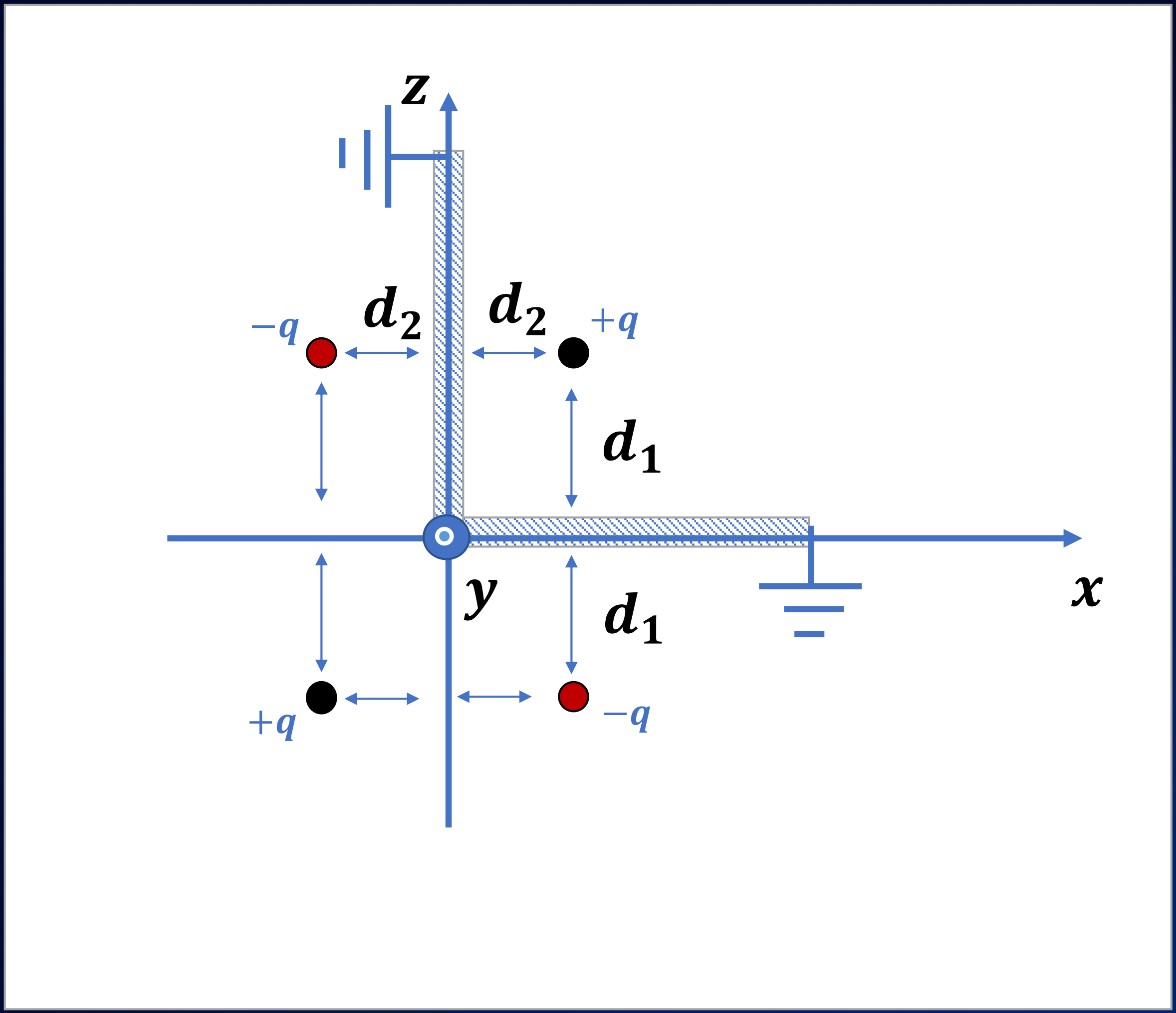
$$V(x,y,z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n}{\sqrt{x^2+y^2+(z+2nd)^2}} + \frac{(-1)^n}{\sqrt{x^2+y^2+(z-2nd)^2}}\right)\right]$$

3.2.4 Other Image Problems

Example: A positive point charge q is located at distances d_1 and d_2, respectively, from two grounded perpendicular conducting half-planes, as shown in the figure. Determine the force on q caused by the charges induced on the planes.



3.2.4 Other Image Problems



3.2.4 Other Image Problems

Example: A positive point charge q is located at distances $\boldsymbol{a}$ out side of a metallic grounded sphere. Determine the force on q caused by the charges induced on the sphere.

The equation

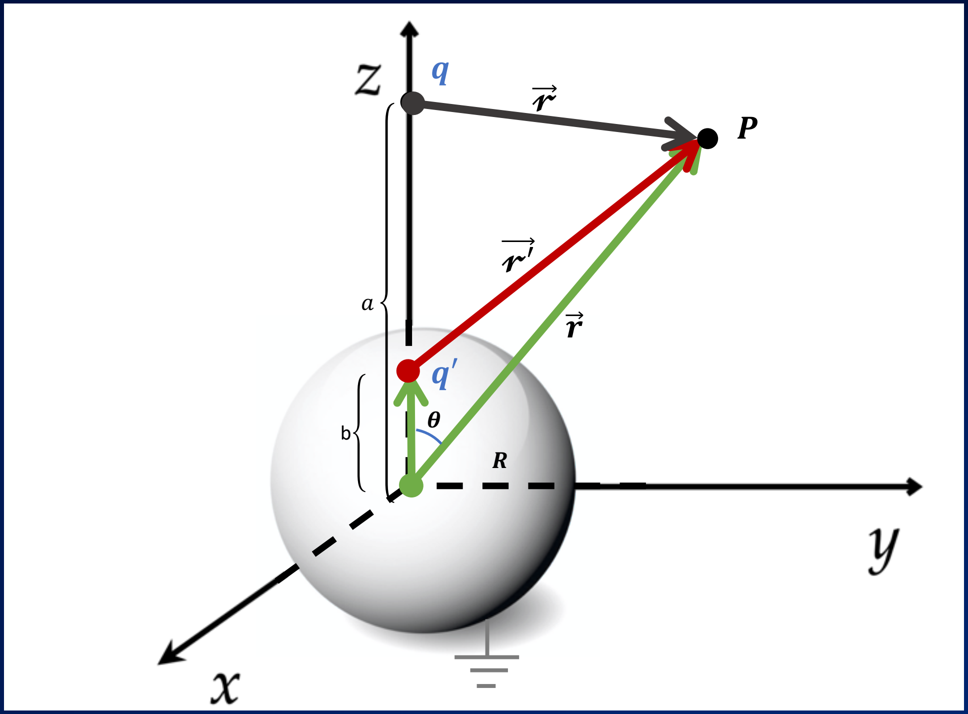
$$\nabla^2 V = -q\delta(x)\delta(y)\delta(z-a)/\epsilon_0$$

Boundary condition

$$V(R\hat{r}) \quad = 0$$

Solution

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{q'}{r'}\right]$$



3.2.4 Other Image Problems

$$V(R\hat{z}) = 0 \rightarrow \frac{1}{4\pi\epsilon_0}\left(\frac{q}{a-R}+\frac{q'}{R-b}\right) = 0$$

$$V(-R\hat{z}) = 0 \rightarrow \frac{1}{4\pi\epsilon_0}\left(\frac{q}{a+R}+\frac{q'}{R+b}\right) = 0$$

or

$$\left(\frac{q}{a+R}+\frac{q'}{R+b}\right) = 0$$

$$\left(\frac{q}{a-R}+\frac{q'}{R-b}\right) = 0$$

$$q' = -q\frac{R+b}{a+R}$$

$$\left(\frac{q}{a-R}+\frac{-q\frac{R+b}{a+R}}{R-b}\right) = 0$$

3.2.4 Other Image Problems

$$\frac{R-b}{a-R}=\frac{R+b}{a+R}$$

$$(R-b)(a+R)=(R+b)(a-R)$$

$$R^2-bR+aR-ab=R^2-bR+aR+ab$$

$$2aR=2ab\rightarrow b=\frac{R^2}{a}$$

$$q'=-q\frac{R+b}{a+R}\rightarrow q'=-q\frac{R+\frac{R^2}{a}}{a+R}=-q\frac{R}{a}$$

3.2.4 Other Image Problems

Solution

$$V(x,y,z)=\frac{1}{4\pi\epsilon_0}\left[\frac{q}{r}+\frac{q'}{r'}\right]=\frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\vec{r}-a\hat{z}|}-\frac{R}{a}\frac{1}{\left|\vec{r}-\frac{R^2}{a}\hat{z}\right|}\right]$$

$$\vec{F}=\frac{1}{4\pi\epsilon_0}\frac{qq'}{(a-b)^2}\hat{z}=-\frac{q^2}{4\pi\epsilon_0}\frac{R}{a}\frac{1}{\left(a-\frac{R^2}{a}\right)^2}\hat{z}$$

$$=-\frac{q^2}{4\pi\epsilon_0}\frac{Ra}{(a^2-R^2)^2}\hat{z}$$

3.2.4 Other Image Problems

Induced Charge density

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} - \frac{R}{a}\frac{1}{\sqrt{r^2 + \frac{R^4}{a^2} - 2\frac{R^2}{a}r\cos\theta}}\right]$$

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} - \frac{1}{\sqrt{\frac{a^2r^2}{R^2} + R^2 - 2ar\cos\theta}}\right]$$

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial r}\bigg|_{r=R} = -\frac{1}{4\pi}\frac{q(R^2 - a^2)}{R(R^2 + a^2 - 2aR\cos\theta^{3/2})}$$