Electromagnetic Theory I

Abdallah Sayyed-Ahmad
Department of Physics
Birzeit University

April 7, 2021

جامعة بيرزيت
BIRZEIT UNIVERSITY

Chapter 3: Potentials

- Laplace Equation
- The Method of Images
- Separation of Variables
- Multipole Expansion

3.3 The Separation of Variables

We would like to solve Laplace equation directly using solutions that are products of functions each of which depends on one of the coordinates

Cartesian Coordinates: $V(x,y,z) = X(x)Y(y)Z(z)$

Cylindrical Coordinates: $V(s,\phi,z) = S(s)\Phi(\phi)Z(z)$

Spherical Coordinates: $V(r,\theta,\phi) = R(r)(\theta)\Phi(\phi)$

This approach is not always possible as it requires appropriate symmetry, boundary conditions and charge source distributions

3.3.1 Cartesian Coordinates

Example: Find the potential between two infinite parallel grounded conducting plates.

Two dimensional problem

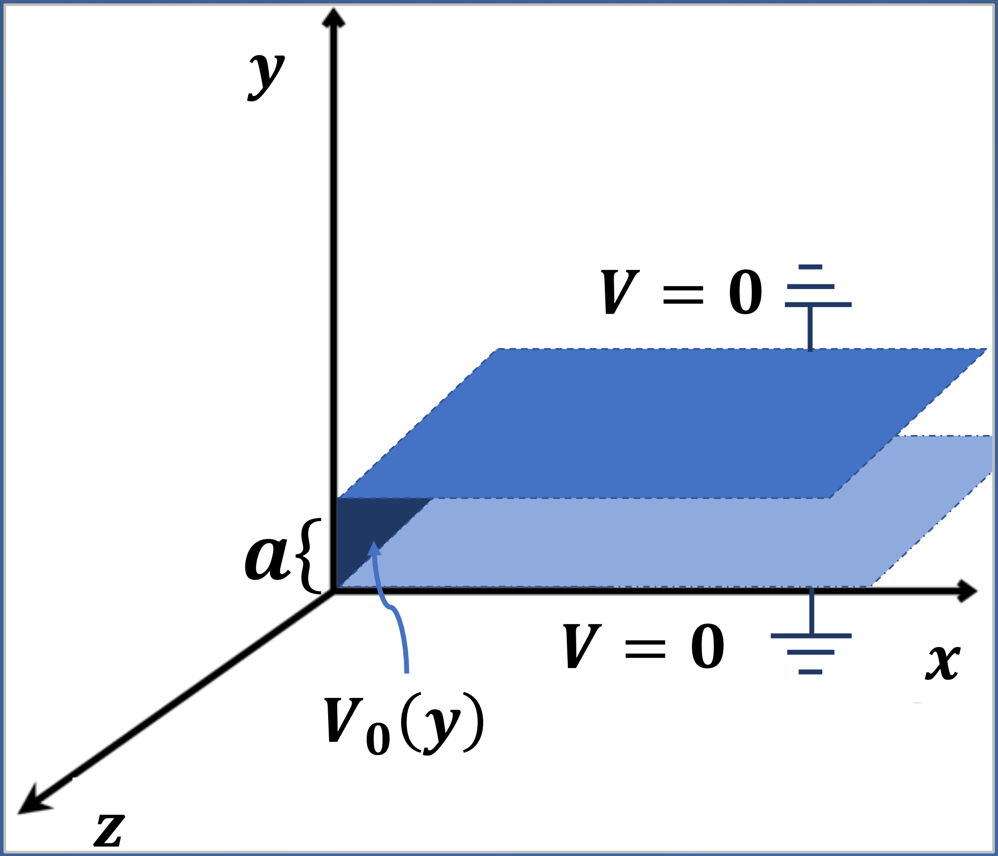
$$V(x,y,z) = V(x,y)$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Boundary Conditions

$V(x,0) = 0$ $\qquad V(x,a) = 0$

$V(0,y) = V_0(y)$ $\qquad V(\infty,y) = 0$



3.3.1 Cartesian Coordinates

We will assume that the solution is separable

$$V(x,y,z) = V(x,y) = X(x)Y(y)$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \rightarrow X''(x)Y(y) + X(x)Y''(y) = 0$$

Dividing both sides by $X(x)Y(y)$, we get

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

The first term $\left(\frac{X''}{X}\right)$depends on x only, while the second term $\left(\frac{Y''}{Y}\right)$ depends on y only, therefore both terms must be constant.

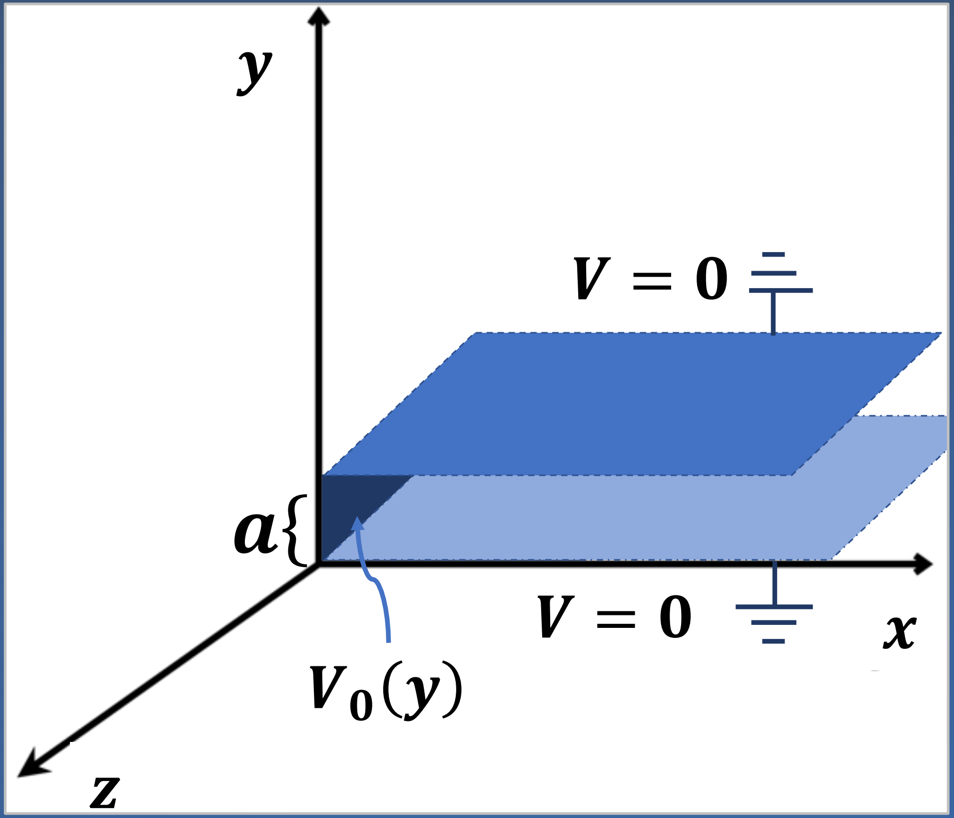
$$\frac{X''}{X} = c_1, \frac{Y''}{Y} = c_2 \rightarrow c_1 + c_2 = 0 \; or \; c_1 = -c_2$$

3.3.1 Cartesian Coordinates

From boundary conditions

$$\frac{1}{X}\frac{d^2X}{dx^2} = c_1 > 0 = k^2$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = c_2 = -c_1 < 0 = -k^2$$



The solution of the above

$$X(x) = A\exp(kx) + B\exp(-kx)$$

$$Y(y) = C\sin(ky) + D\cos(ky)$$

$$\rightarrow V(x,y) = \{A\exp(kx) + B\exp(-kx)\}\{C\sin(ky) + D\cos(ky)\}$$

3.3.1 Cartesian Coordinates

From boundary conditions

1) $V(x,0) = 0 = \{A\exp(kx) + B\exp(-kx)\}\{C\sin(k\,0) + D\cos(k\,0)\} = D\{A\exp(kx) + B\exp(-kx)\} = 0$

$$\rightarrow D = 0$$

2) $V(\infty, y) = \{A\exp(k\,\infty) + B\exp(-k\,\infty)\}\{C\sin(k\,y)\} = 0$

$$\rightarrow A = 0$$

3) $V(x,a) = BC\exp(-k\,x)\sin(k\,a) = \alpha\exp(-k\,x)\sin(k\,a) = 0$

$$\rightarrow k = \frac{n\pi}{a}, n = 1, 2, \cdots$$

3.3.1 Cartesian Coordinates

$$V_n(x,y) = \alpha_n \exp\left(-\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}\,y\right)$$

are solutions that satisfy the previous BCs.

To satisfy the last boundary condition, we note that the superposition of V_n

$$V(x,y) = \sum_{n=1}^{\infty} \alpha_n \exp\left(-\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}\,y\right)$$

Also satisfy Laplace equation

The last boundary condition, $V(0,y) = V_0(y)$, gives

$$V(0,y) = V_0(y) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi}{a}\,y\right)$$

3.3.1 Cartesian Coordinates

Note:

- This expansion is Fourier sine series.
- Sine/Cosine functions represent a complete orthogonal set.

$$\int_0^a V_0(y)\sin\left(\frac{m\pi}{a}y\right)dy = \sum_{n=1}^{\infty}\alpha_n \int_0^a \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{a}y\right)dy$$

Set of functions is called

Orthogonal-normal

$$\int_0^a f_n(x)f_m(x)dx = \delta_{nm}$$

Complete

$$g(x) = \sum_{n=0}^{\infty} a_n f_n(x)$$

3.3.1 Cartesian Coordinates

$$\int_0^a V_0(y)\sin\left(\frac{m\pi}{a}y\right)dy$$

$$= \alpha_m \int_0^a \sin^2\left(\frac{m\pi}{a}y\right)dy$$

$$+ \sum_{\substack{n=1\\ n\neq m}}^{\infty} \frac{\alpha_n}{2}\int_0^a \left[\cos\left((n+m)\frac{\pi}{a}y\right) - \cos\left((n-m)\frac{\pi}{a}y\right)\right]dy$$

$$= \frac{a}{2}\alpha_m$$

$$+ \sum_{\substack{n=1\\ n\neq m}}^{\infty} \frac{\alpha_n}{2}\left[-\frac{a}{(n+m)}\sin\left((n+m)\frac{\pi}{a}y\right) + \frac{a}{(n-m)}\sin\left((n-m)\frac{\pi}{a}y\right)\right]_0^a$$

$$= \frac{a}{2}\alpha_m \rightarrow \alpha_m = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{m\pi}{a}y\right)dy$$

3.3.1 Cartesian Coordinates

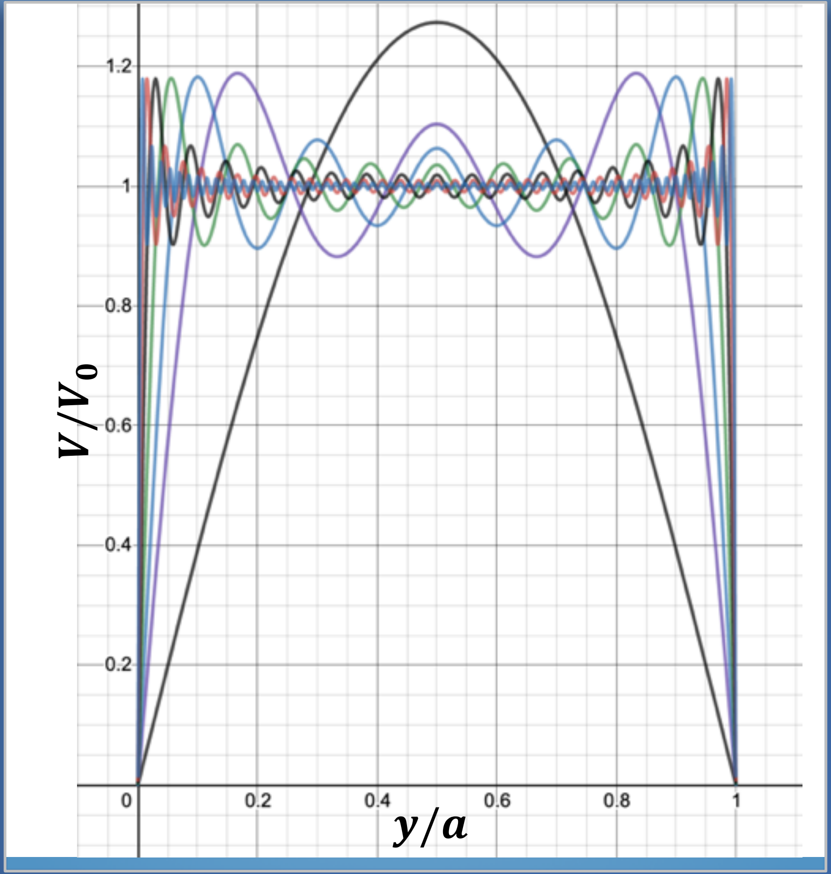
Example: $V_0(y) = V_0 = constant$

$$\alpha_m = \frac{2}{a}\int_0^a V_0 \sin\left(\frac{m\pi}{a}\, y\right) dy$$

$$= -\frac{2V_0}{a}\cos\left(\frac{m\pi}{a}\, y\right)\frac{a}{m\pi}\Bigg|_0^a$$

$$= -\frac{2V_0}{m\pi}\left(-1^m - 1\right)$$

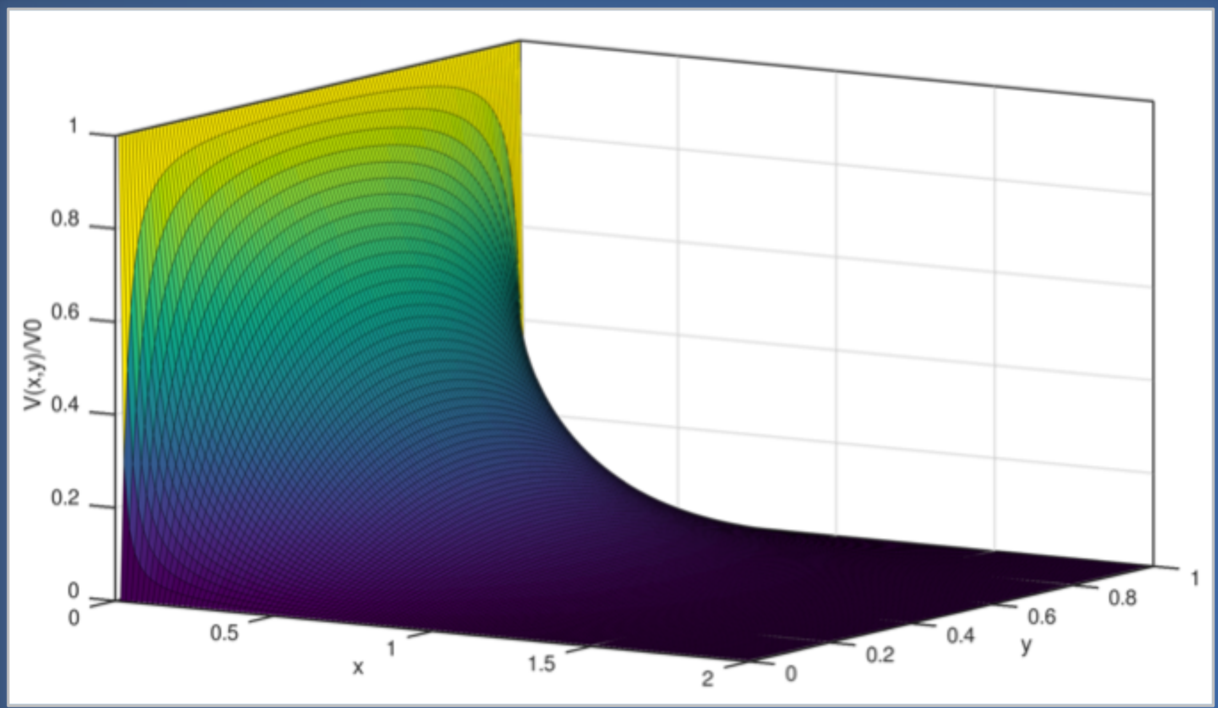
$$= \begin{cases} 0 & m\ is\ even \\ \dfrac{4V_0}{m\pi} & m\ is\ odd \end{cases}$$



3.3.1 Cartesian Coordinates

$$V(x,y)=\frac{4V_0}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\exp\left(-\frac{(2n+1)\pi}{a}x\right)\sin\left(\frac{(2n+1)\pi}{a}y\right)$$

$$V(x,y)=\frac{2V_0}{\pi}\tan^{-1}\left(\sin\left(\frac{\pi}{a}y\right)/\sinh\left(\frac{\pi}{a}x\right)\right)$$



3.3.1 Cartesian Coordinates

Example: Two infinitely long grounded metal plates, again at $y = 0$ and $y = a$, are connected at $x = \pm b$ by metal strips maintained at a constant potential V_0. Find the potential inside the resulting rectangular pipe

Two dimensional problem

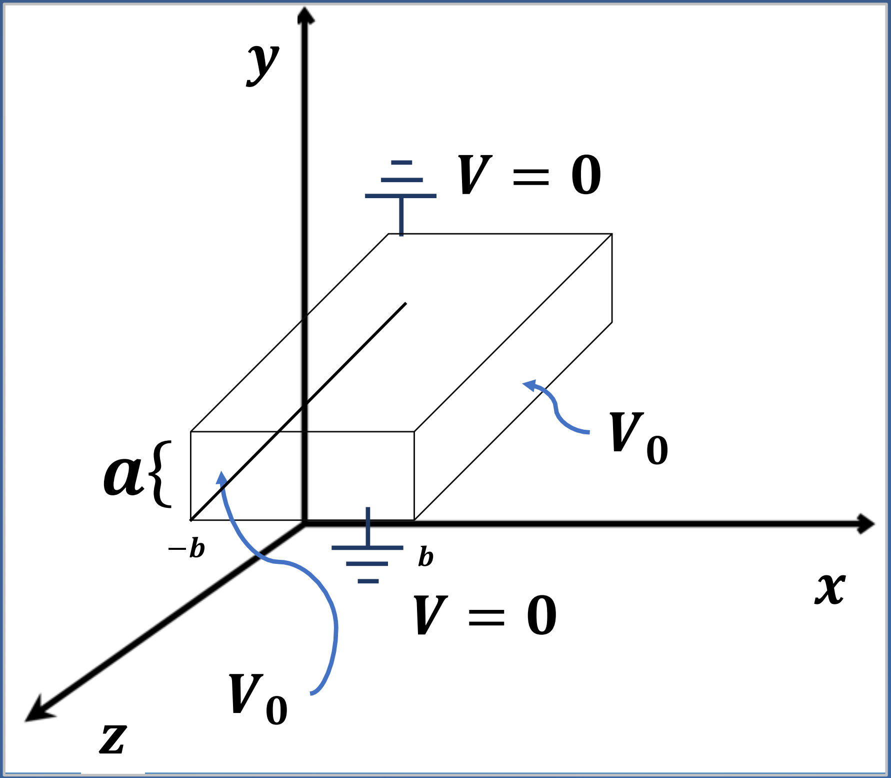
$$V(x, y, z) = V(x, y)$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Boundary Conditions

$$V(x, 0) = 0 \qquad V(x, a) = 0$$

$$V(-b, y) = V_0 \qquad V(b, y) = V_0$$



3.3.1 Cartesian Coordinates

We will assume that the solution is separable

$$V(x,y,z) = V(x,y) = X(x)Y(y)$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \rightarrow X''(x)Y(y) + X(x)Y''(y) = 0$$

Dividing both sides by $X(x)Y(y)$, we get

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

The first term $\left(\frac{X''}{X}\right)$ depends on x only, while the second term $\left(\frac{Y''}{Y}\right)$ depends on y only, therefore both terms must be constant.

$$\frac{X''}{X} = c_1, \frac{Y''}{Y} = c_2 \rightarrow c_1 + c_2 = 0 \; or \; c_1 = -c_2$$

3.3.1 Cartesian Coordinates

From boundary conditions

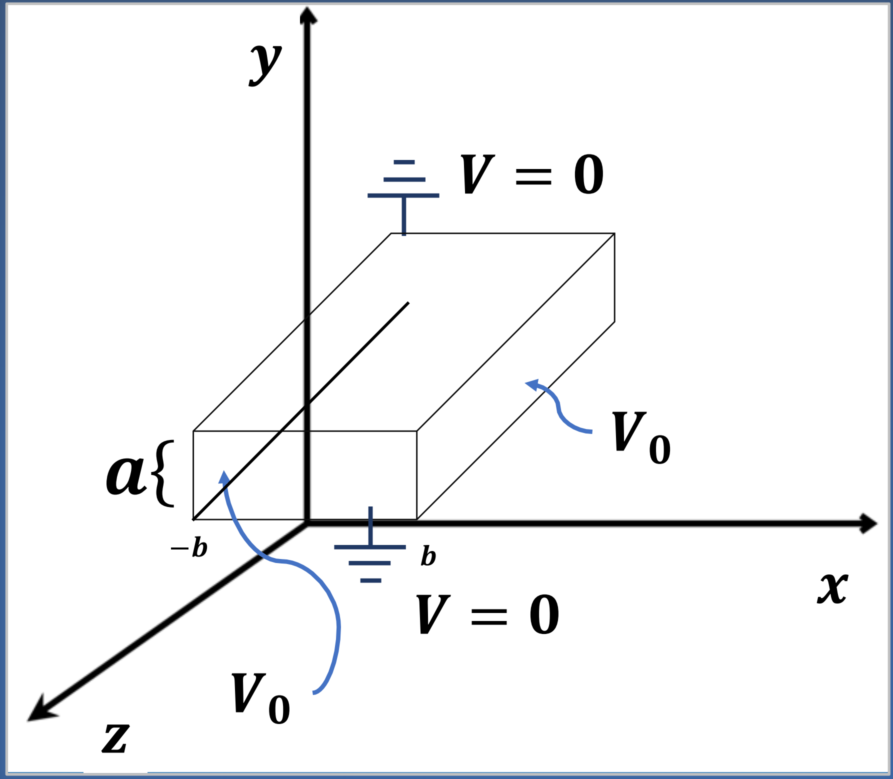
$$\frac{1}{X}\frac{d^2X}{dx^2} = c_1 > 0 = k^2$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = c_2 = -c_1 < 0 = -k^2$$

The solution of the above

$$X(x) = A\exp(kx) + B\exp(-kx)$$

$$Y(y) = C\sin(ky) + D\cos(ky)$$



$$\rightarrow V(x,y) = \{A\exp(kx) + B\exp(-kx)\}\{C\sin(ky) + D\cos(ky)\}$$

3.3.1 Cartesian Coordinates

From boundary conditions

1) $V(x,0) = 0 = \{A\exp(kx) + B\exp(-kx)\}\{C\sin(k\,0) + D\cos(k\,0)\} = D\{A\exp(kx) + B\exp(-kx)\} = 0$

$$\rightarrow D = 0$$

2) $V(b,y) = \{A\exp(k\,b) + B\exp(-k\,b)\}\{C\sin(k\,y)\}$

3) $V(-b,y) = \{A\exp(-k\,b) + B\exp(k\,b)\}\{C\sin(k\,y)\}$

$even\ function \rightarrow A = B$

4) $V(x,a) = \alpha\cosh kx\sin(k\,a) = 0$

$$\rightarrow k = \frac{n\pi}{a}, n = 1, 2, \cdots$$

3.3.1 Cartesian Coordinates

To satisfy the last boundary condition, we note that the superposition of V_n

$$V(x,y) = \sum_{n=1}^{\infty} \alpha_n \cosh\left(-\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}\,y\right)$$

The last boundary condition, $V(0,y) = V_0(y)$, gives

$$V(b,y) = V_0 = \sum_{n=1}^{\infty} \alpha_n \cosh\left(-\frac{n\pi}{a}b\right)\sin\left(\frac{n\pi}{a}\,y\right)$$

3.3.1 Cartesian Coordinates

$$\int_0^a V_0 \sin\left(\frac{m\pi}{a} y\right) dy = \sum_{n=1}^{\infty} \alpha_n \cosh\left(-\frac{n\pi}{a} b\right) \int_0^a \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{m\pi}{a} y\right) dy$$

$$\alpha_m = \frac{2}{a \cosh\left(-\frac{n\pi}{a} b\right)} \int_0^a V_0(y) \sin\left(\frac{m\pi}{a} y\right) dy$$

$$\alpha_m = \begin{cases} 0 & m \text{ is even} \\ \dfrac{4V_0}{m\pi \cosh\left(-\frac{m\pi}{a} b\right)} & m \text{ is odd} \end{cases}$$

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{m\pi \cosh\left(-\frac{m\pi}{a} x\right)}{m\pi \cosh\left(-\frac{m\pi}{a} b\right)} \sin\left(\frac{(2n+1)\pi}{a} y\right)$$

3.3.1 Cartesian Coordinates

Example: infinitely long rectangular metal pipe is grounded, but one end, at $x = 0$, is maintained at a specified potential $V_0(y,z)$, Find the potential inside the pipe.

Three dimensional problem

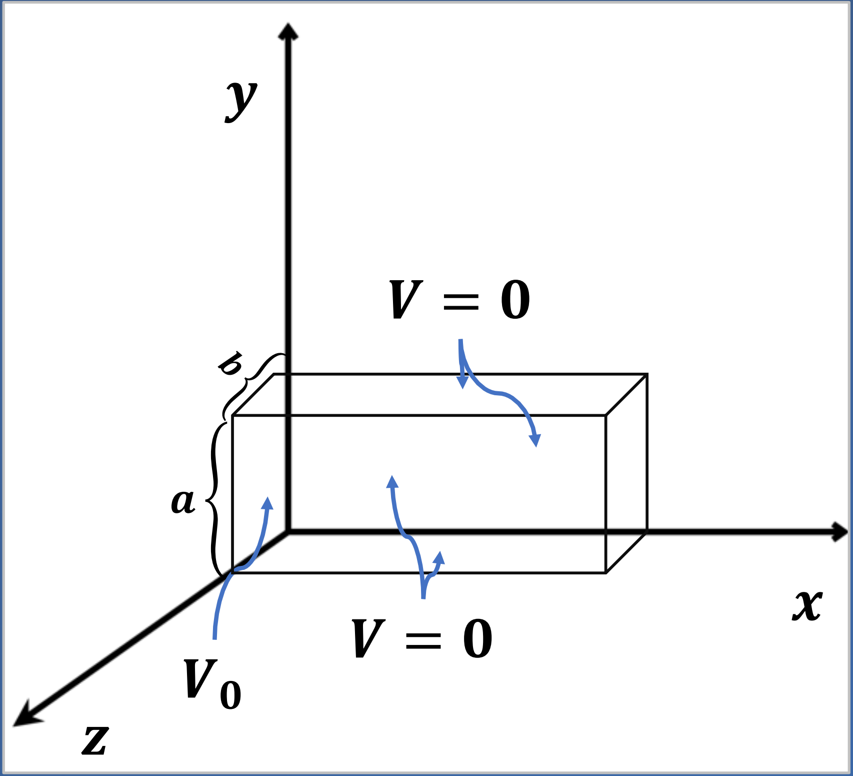
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Boundary Conditions

$$V(x,0,z) = 0 \qquad V(x,a,z) = 0$$

$$V(0,y,z) = V_0(y,z) \qquad V(\infty,y,z) = 0$$

$$V(x,y,0) = 0 \qquad V(x,y,b) = 0$$



3.3.1 Cartesian Coordinates

We will assume that the solution is separable

$$V(x,y,z) = X(x)Y(y)Z(z)$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \to X''(x)Y(y)Z(z) + X(x)Y''(y)Z(z) + X(x)Y(y)Z''(z) = 0$$

Dividing both sides by $X(x)Y(y)Z(z)$, we get

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

The first term $\left(\frac{X''}{X}\right)$ depends on x only, while the second term $\left(\frac{Y''}{Y}\right)$ depends on y and the third term depends z on only, therefore all terms must be constant.

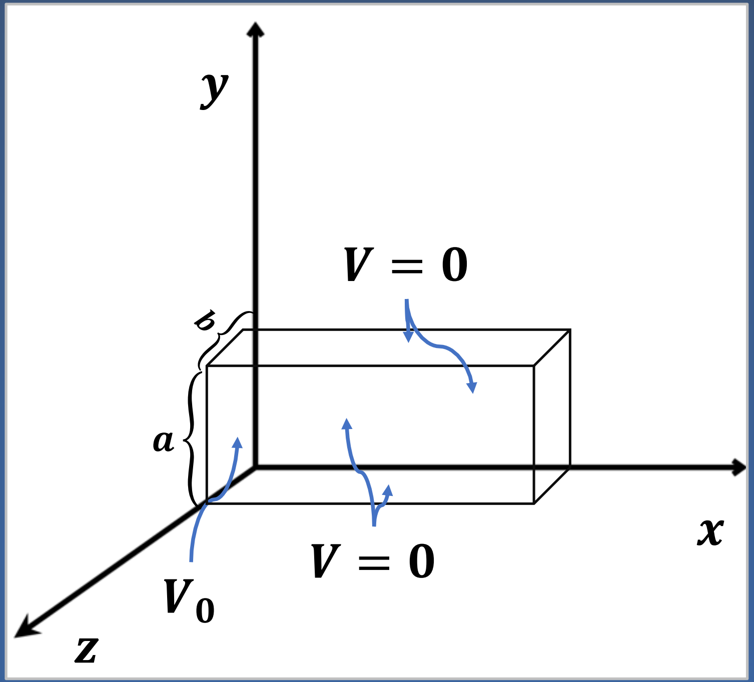
$$\frac{X''}{X} = c_1, \frac{Y''}{Y} = c_2, \frac{Z''}{Z} = -c_1 - c_2$$

3.3.1 Cartesian Coordinates

From boundary conditions

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -k^2, \frac{1}{Z}\frac{d^2z}{dz^2} = -l^2$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = k^2 + l^2$$



The solution of the above

$$X(x) = A\exp\left(\sqrt{k^2+l^2}x\right) + B\exp\left(-\sqrt{k^2+l^2}x\right)$$

$$Y(y) = C\sin(ky) + D\cos(ky)$$

$$Z(z) = E\sin(lz) + F\cos(lz)$$

3.3.1 Cartesian Coordinates

To satisfy the last boundary condition, we note that the superposition of V_{nm}

$$V(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \alpha_{nm} \exp\left(-\pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}x\right) \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{b}z\right)$$

The last boundary condition, $V(0,y) = V_0(y)$, gives

$$V(0,y,z) = V_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \alpha_{nm} \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{b}z\right)$$

3.3.1 Cartesian Coordinates

$$\int_0^b \int_0^a V_0 \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{m\pi}{a} y\right) dydz$$

$$= \sum_{m\prime=1}^{\infty} \sum_{n\prime=1}^{\infty} \alpha_{n\prime m\prime} \int_0^b \int_0^a V_0 \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{m\pi}{a} y\right) \sin\left(\frac{n'\pi}{a} y\right) \sin\left(\frac{m'\pi}{b} z\right) dydz$$

$$\alpha_{nm} = \begin{cases} 0 & n \; or \; m \; is \; even \\ \dfrac{16V_0}{nm\pi^2} & n \; and \; m \; are \; odd \end{cases}$$

$$V(x,y,z) = \frac{16V_0}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(2n+1)(2m+1)} \exp\left(-\pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}} x\right) \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{m\pi}{b} z\right)$$