Electromagnetic Theory I

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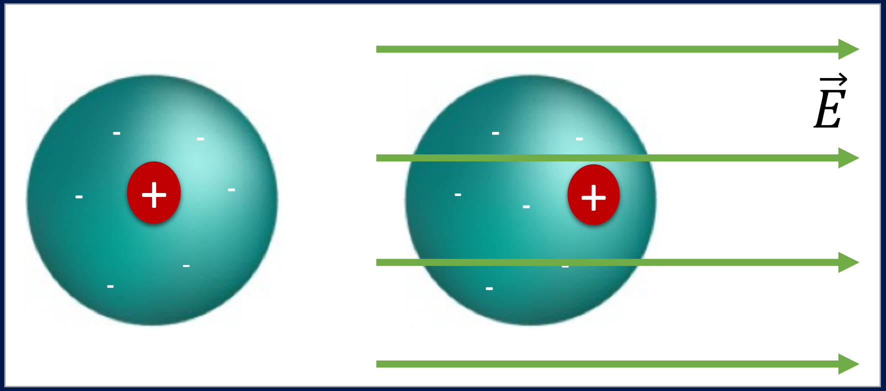
Chapter 4: Electric Fields in Matter

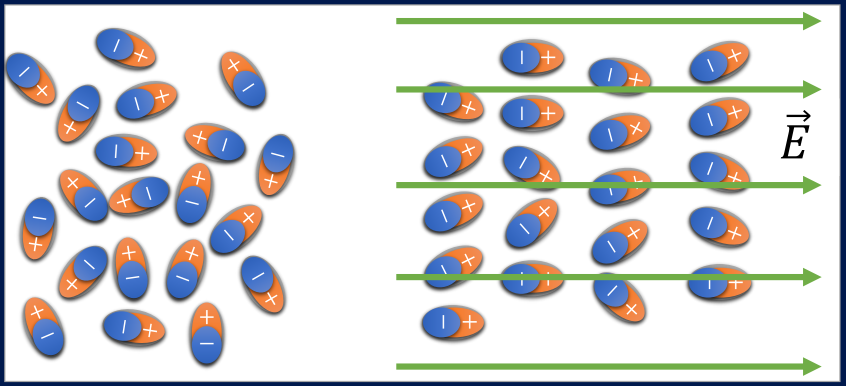
- Polarization
- The Field of a Polarized Object
- The Electric Displacement
- Linear Dielectrics

4.1 Polarization

Dielectrics (e.g. plastic, glass, …)

- In the most materials, charges are linked to atoms or molecules.
- They are considered bound and only can move within the atom or molecule.
- These restricted microscopic displacements give rise for the physical properties of dielectric materials.
- Electric fields can affect a charge distribution of a dielectric atom or molecule by stretching and rotating them.





4.1 Polarization

Induced Dipoles

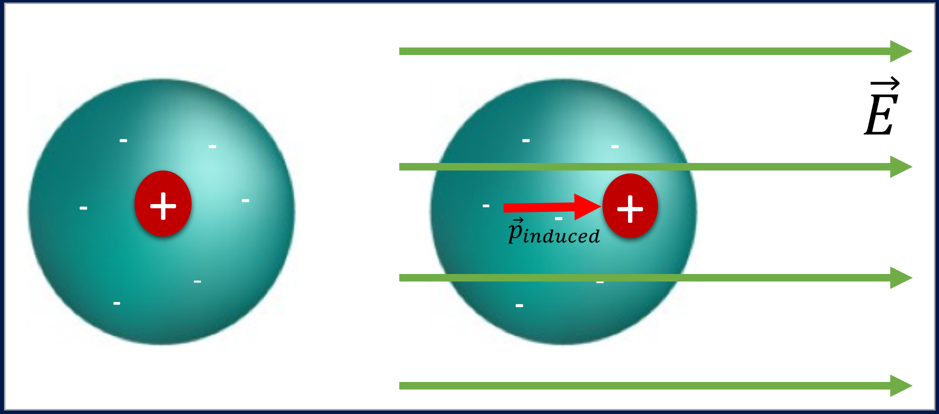
What happens to a neutral atom when it is placed in an electric field?

- The nucleus is pushed in the direction of the field, and the electrons the opposite way.
- The two opposing forces on nucleolus reach a balance, leaving the atom polarized.
- The atom acquires a tiny dipole moment $\vec{p}$, which points in the same direction as $\vec{E}$.
- This induced dipole moment is approximately proportional to the field (as long as the latter is not too strong)

$$\vec{p} = \alpha \vec{E}$$

α is called atomic polarizability

Atomic Polarizabilities $(\alpha/4\pi\epsilon_0)$ in units of $10^{-30} m$									
H	He	Li	Be	C	Ne	Na	Ar	K	Cs
0.667	0.205	24.3	5.60	1.76	0.396	24.1	1.64	43.4	59.6



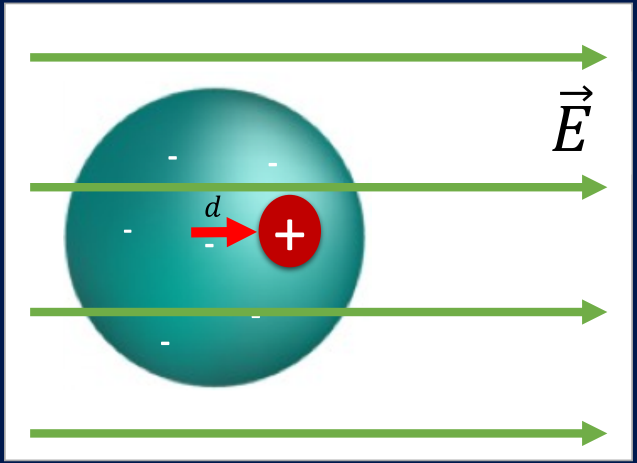
4.1 Polarization

Example: An atom consists of a point nucleus $(+Q)$ surrounded by a cloud $(-Q)$ of radius a. Calculate the atomic polarizability of such an atom.
uniformly charged spherical

- As simple model, we will assume
  - The electron cloud retains its spherical shape.
  - Equilibrium occurs when the nucleus is displaced a distance d from the center of the sphere and pulling by the internal field produced by the electron cloud.

At equilibrium,

$$E = E_e = \frac{1}{4\pi\epsilon_0}\frac{Qd}{a^3} = \frac{1}{4\pi\epsilon_0}\frac{p}{a^3}$$



$$\rightarrow p = 4\pi\epsilon_0 a^3 E$$

$$\rightarrow \alpha = 4\pi\epsilon_0 a^3 \rightarrow \alpha = 3\epsilon_0 v$$

4.1 Polarization

- **Example:** According to quantum mechanics, the electron cloud for a hydrogen atom in the ground state has a charge density:

$$\rho(\vec{r}) = \frac{q}{\pi a^3} e^{-2\frac{r}{a}}$$

where q is the charge of the electron and a is the Bohr radius. Find the atomic polarizability of such an atom.

- Using Gauss's Law, the electric field due to the electron cloud is given by

$$E_e = \frac{1}{4\pi\epsilon_0} \frac{Q_{enc}}{r^2}$$

$$Q_{enc} = \int_0^r \frac{q}{\pi a^3} e^{-2\frac{r'}{a}} 4\pi r'^2 dr' = q\left[1 - e^{-2\frac{r}{a}}\left(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2}\right)\right]$$

4.1 Polarization

- Similar to the previous example, the proton will be shifted a distance d at equilibrium, and hence

$$E = E_e = \frac{1}{4\pi\epsilon_0}\frac{q\left[1 - e^{-2\frac{d}{a}}\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right)\right]}{d^2}$$

For $d << a$

$$\left[1 - e^{-2\frac{d}{a}}\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right)\right] = 1 - \left(1 - 2\frac{d}{a} + 2\frac{d^2}{a^2} - \frac{4}{3}\frac{d^3}{a^3} + \cdots\right)\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right)$$

$$= \frac{4}{3}\frac{d^3}{a^3} + HOT$$

$$E = \frac{1}{3\pi\epsilon_0}\frac{qd}{a^3} = \frac{1}{3\pi\epsilon_0}\frac{p}{a^3}$$

$$\rightarrow p = 3\pi\epsilon_0 a^3 E$$

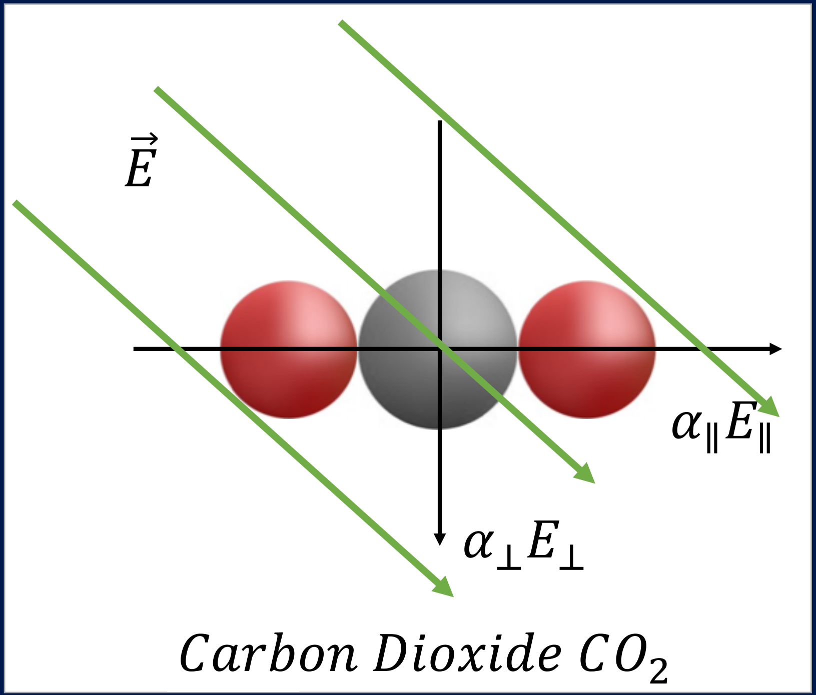
$$\rightarrow \alpha = 3\pi\epsilon_0 a^3$$

4.1 Polarization

Polarizability Tensor

- For example in Carbon dioxide , when the field is at some angle to the axis, it will have it parallel and perpendicular components,

$$\vec{p} = \alpha_{\parallel}\vec{E}_{\parallel} + \alpha_{\perp}\vec{E}_{\perp}$$



in general, linear response theory gives

$$\vec{p} = \overline{\overline{\alpha}}\vec{E}$$

$$p_x = \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z$$
$$p_y = \alpha_{yx}E_x + \alpha_{yy}E_y + \alpha_{yz}E_z$$
$$p_z = \alpha_{zx}E_x + \alpha_{zy}E_y + \alpha_{zz}E_z$$

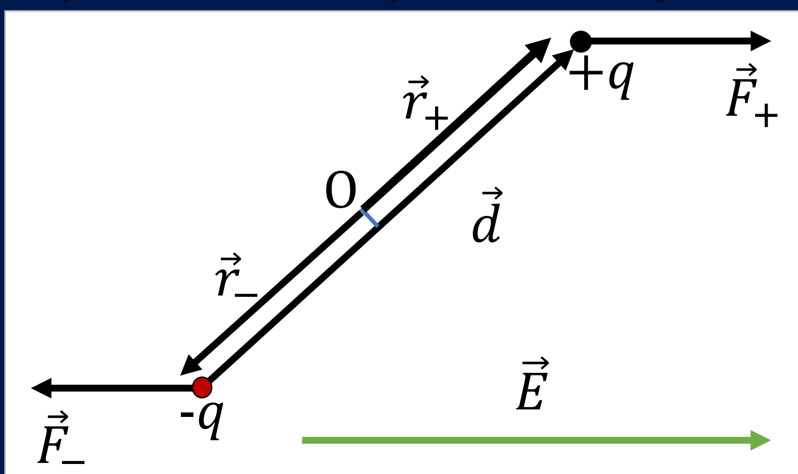
$\overline{\overline{\alpha}}$ is called polarizability tensor

4.1 Polarization

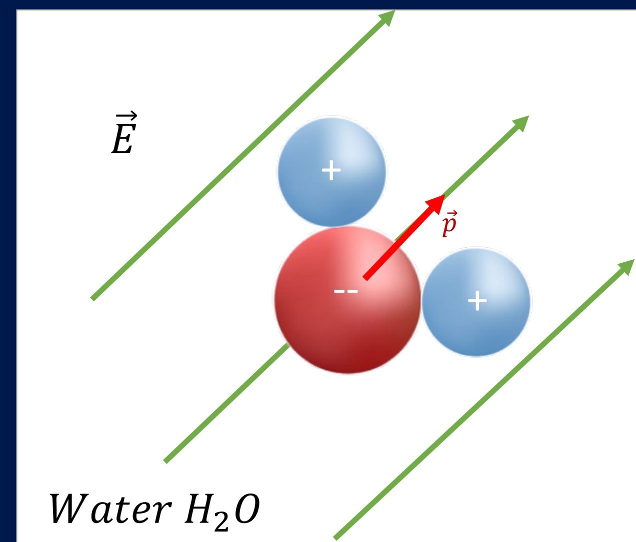
Alignment of Polar Molecules

- Polar molecules such as water have intrinsic dipole moment, hence when an external field is applied, there will be a net torque on the molecule. It will be free to rotate and swing around until it points in the direction of the applied field.

If the electric field is uniform



$$\vec{F}_{net} = 0$$



$$\vec{N} = \vec{r}_+ \times \vec{F}_+ + \vec{r}_- \times \vec{F}_- = \frac{\vec{d}}{2} \times q\vec{E} + \frac{-\vec{d}}{2} \times -q\vec{E} = q\vec{d} \times \vec{E} = \vec{\mathrm{p}} \times \vec{\mathrm{E}}$$

4.1 Polarization

If the electric field is not uniform

$$\vec{F}_{net} = \vec{F}_{+} + \vec{F}_{-} = q\left(\vec{E}_{+} - \vec{E}_{-}\right) = q\Delta\vec{\mathrm{E}} \approx q\vec{\nabla}\vec{E} \cdot \vec{d}$$
$$= q\left(\vec{d} \cdot \vec{\nabla}\right)\vec{E} = \left(\vec{p} \cdot \vec{\nabla}\right)\vec{E}$$

$$\vec{N} = \vec{\mathbf{p}}\times\vec{\mathrm{E}} + \vec{\mathbf{r}}\times\vec{F}_{net} = \vec{\mathbf{p}}\times\vec{\mathrm{E}} + \vec{r}\times\left(\vec{p} \cdot \vec{\nabla}\right)\vec{E}$$

4.1 Polarization

Energy of an Ideal Dipole

- For a physical dipole,

$$U = \sum_{n=1}^{2} q_i V(\vec{r}_i) = qV(\vec{r}_+) - qV(\vec{r}_-) = qV\left(\vec{r} + \frac{\vec{d}}{2}\right) - qV\left(\vec{r} - \frac{\vec{d}}{2}\right) = q\left(-\int_{\vec{r}-\frac{\vec{d}}{2}}^{\vec{r}+\frac{\vec{d}}{2}} \vec{E} \cdot d\vec{l}\right)$$

- For an ideal dipole, $\vec{\mathrm{d}} \rightarrow 0$

$$U = \lim_{\vec{d}\rightarrow 0} q\left(-\int_{\vec{r}-\frac{\vec{d}}{2}}^{\vec{r}+\frac{\vec{d}}{2}} \vec{E} \cdot d\vec{l}\right) = -q\vec{E} \cdot \left[\left(\vec{r} + \frac{\vec{d}}{2}\right) - \left(\vec{r} - \frac{\vec{d}}{2}\right)\right] = -q\vec{E} \cdot \vec{d} = -\vec{p} \cdot \vec{E}$$

4.1 Polarization

Example: A dipole $\vec{p}$ is a distance $\vec{r}$ from a point charge q , and oriented so that $\vec{p}$ makes an angle θ_0 with the vector $\vec{r}$ from q to $\vec{p}$. What is the force on $\vec{p}$?

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{(x^2+y^2+z^2)^{\frac{3}{2}}}(x\hat{x}+y\hat{y}+z\hat{z}) =$$

$$\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E}$$

$$F_x = \left(p_x\frac{\partial}{\partial x}+p_y\frac{\partial}{\partial y}+p_z\frac{\partial}{\partial z}\right)E_x$$

$$= \frac{q}{4\pi\epsilon_0}\left[p_x\frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} - p_y\frac{-3xy}{(x^2+y^2+z^2)^{\frac{3}{2}}} - p_z\frac{-3xz}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right] = \frac{q}{4\pi\epsilon_0}\left[\frac{\vec{p}}{r^3}-\frac{3\vec{r}(\vec{p}\cdot\vec{r})}{r^5}\right]_x$$

$$\vec{F} = \frac{q}{4\pi\epsilon_0}\left[\frac{\vec{p}}{r^3}-\frac{3\vec{r}(\vec{p}\cdot\vec{r})}{r^5}\right] = \frac{q}{4\pi\epsilon_0 r^3}[\vec{p}-3\hat{r}(\vec{p}\cdot\hat{r})] = \frac{q}{4\pi\epsilon_0 r^3}[\vec{p}-3\hat{r}\ p\cos\theta_0]$$

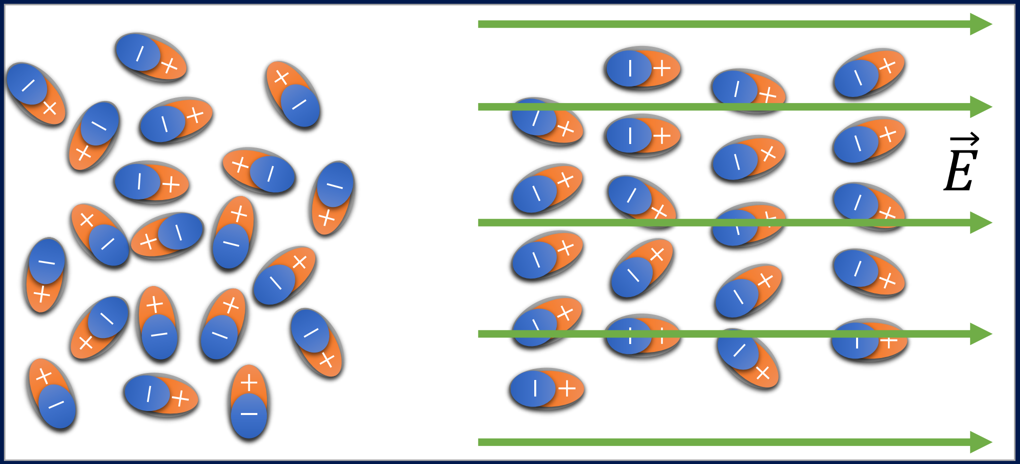
4.1 Polarization

When a piece of dielectric material is placed in an electric field, its constituent dipoles tend to point along the direction of the field and the material become **polarized**

This Polarization will change the electric strength in a dielectric medium which can be quantified through the polarization vector $(\vec{P})$

$\vec{P}$ is the total dipole moment per unit volume

$$\vec{P} = \lim_{v \to 0} \frac{1}{v} \sum_{\vec{p}_i \in v} \vec{p}_i$$



4.2 The Field of a Polarized Object

Bound Charges

For a single dipole, $V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}$

For a collection of dipoles represented in the continuous limit by the polarization vector, the potential generated by the microscopic dipoles is given by

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\vec{P}(\vec{r}\,')\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau' \rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \vec{P}(\vec{r}\,')\cdot\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right)d\tau'$$

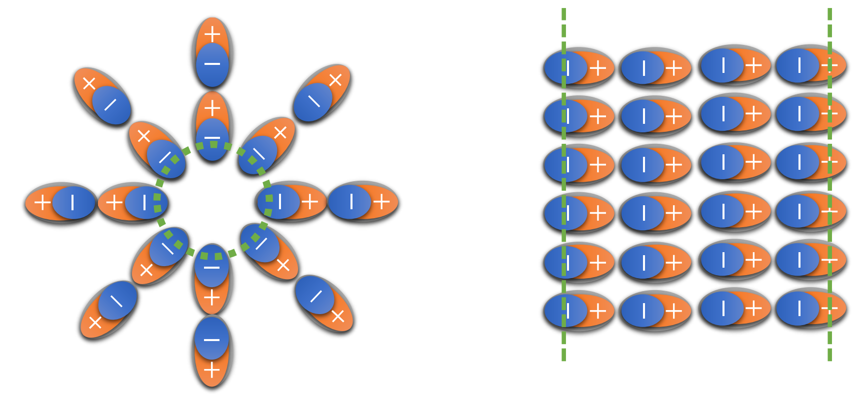
$$\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

4.2 The Field of a Polarized Object

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \vec{\nabla}'\cdot\left(\frac{\vec{P}(\vec{r}')}{\mathscr{r}}\right) - \left(\frac{\vec{\nabla}'\cdot\vec{P}(\vec{r}')}{\mathscr{r}}\right) d\tau'$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\oint \frac{\vec{P}(\vec{r}')}{\mathscr{r}}\cdot d\vec{a}' + \frac{1}{4\pi\epsilon_0}\int -\left(\frac{\vec{\nabla}'\cdot\vec{P}(\vec{r}')}{\mathscr{r}}\right) d\tau'$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\oint \frac{\sigma_b}{\mathscr{r}}\cdot da' + \frac{1}{4\pi\epsilon_0}\int \frac{\rho_b}{\mathscr{r}} d\tau'$$



$$\rho_b = -\vec{\nabla}\cdot\vec{P} \qquad \sigma_b = \vec{P}\cdot\hat{n}$$

4.2 The Field of a Polarized Object

Example: Find the electric field produced by a uniformly polarized sphere of radius R.

$$P(\vec{r}) = P_0\hat{z}$$
$$\rho_b = -\vec{\nabla}\cdot\vec{P} = 0 \qquad \rightarrow \qquad V(\mathrm{r},\theta) = \begin{cases} \dfrac{P_0 r}{3\epsilon_0}\cos\theta & r \le R \\ \dfrac{P_0 R^3}{3\epsilon_0 r^2}\cos\theta & r \ge R \end{cases}$$
$$\sigma_b = \vec{P}\cdot\hat{r} = P_0\cos\theta$$

Outside the sphere the potential is identical to that of a perfect dipole at the origin,

$$\vec{E}(\mathrm{r},\theta) = -\vec{\nabla}V = \begin{cases} -\dfrac{\vec{p}}{4\pi\epsilon_0 R^3} & r \le R \\ \dfrac{1}{4\pi\epsilon_0 r^3}(3\vec{p}\cdot\hat{r}\hat{r} - \vec{p}) & r \ge R \end{cases}$$

$$\vec{p} = \frac{4}{3}\pi R^3\vec{P}$$