Electromagnetic Theory I

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جامعة بيرزيت

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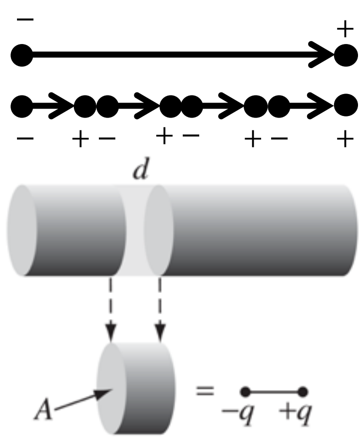
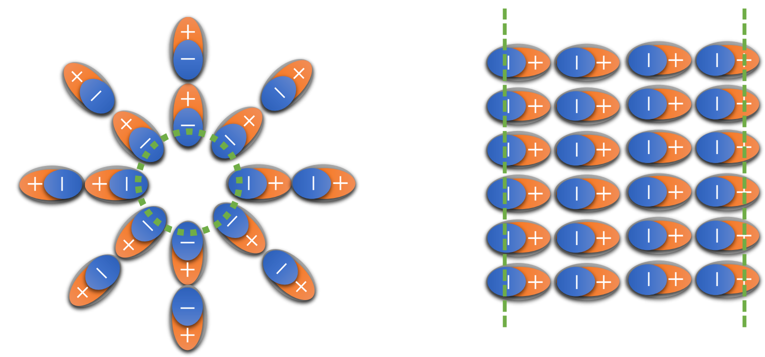
Chapter 4: Electric Fields in Matter

- Polarization
- The Field of a Polarized Object
- The Electric Displacement
- Linear Dielectrics

4.2 The Field of a Polarized Object

Physical interpterion of bound charges





$$\rho_b = -\vec{\nabla} \cdot \vec{P} \qquad \sigma_b = \vec{P} \cdot \hat{n}$$

$$\sigma_b = \frac{q}{A} = P$$

$$\sigma_b = \frac{q}{A_{end}} = P\cos\theta = \vec{P} \cdot \hat{n}$$

If the polarization is nonuniform, we get accumulations of bound charge within the material.

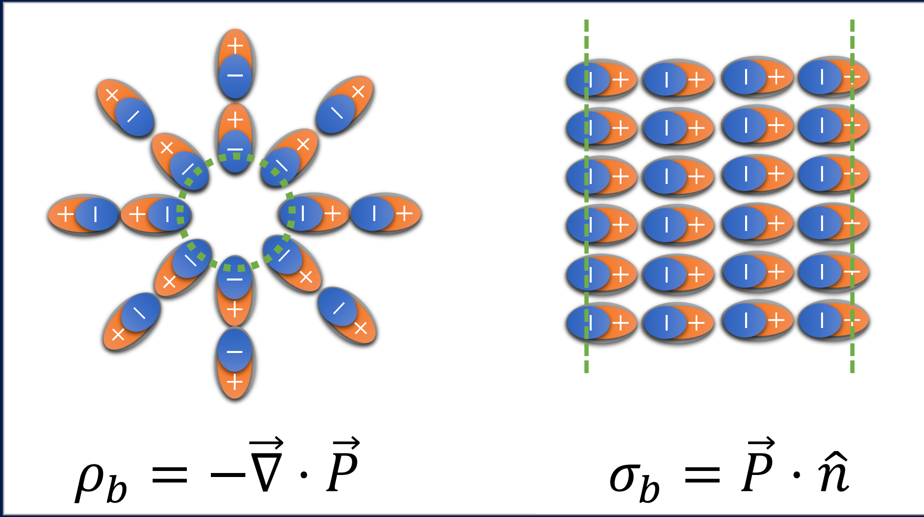
$$q_b = \int \rho_b d\tau = -\int \vec{\nabla} \cdot \vec{P} d\tau$$

4.2 The Field of a Polarized Object

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \vec{\nabla}' \cdot \left(\frac{\vec{P}(\vec{r}')}{\mathscr{r}}\right) - \left(\frac{\vec{\nabla}' \cdot \vec{P}(\vec{r}')}{\mathscr{r}}\right) d\tau'$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\oint \frac{\vec{P}(\vec{r}')}{\mathscr{r}} \cdot d\vec{a}' + \frac{1}{4\pi\epsilon_0}\int -\left(\frac{\vec{\nabla}' \cdot \vec{P}(\vec{r}')}{\mathscr{r}}\right) d\tau'$$

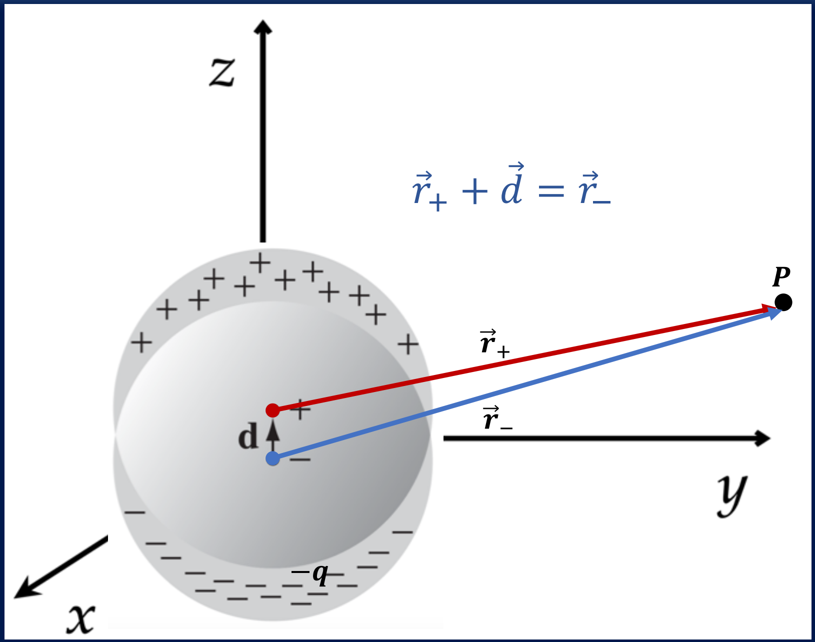
$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\oint \frac{\sigma_b}{\mathscr{r}} da' + \frac{1}{4\pi\epsilon_0}\int \frac{\rho_b}{\mathscr{r}} d\tau'$$



4.2 The Field of a Polarized Object

Example: Find the electric field produced by a uniformly polarized sphere of radius R. (revisited)

Let us consider a positive and negative uniformly charged spheres with small center mismatch d



Recall, inside a uniformly charged sphere,

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}\hat{r} = \frac{\rho r}{3\epsilon_0}\hat{r} = \frac{\rho}{3\epsilon_0}\vec{r}$$

Hence, in the region of overlap between the two spheres,

$$\vec{E} = \vec{E}_+ + \vec{E}_- = \frac{\rho}{3\epsilon_0}(\vec{r}_+ - \vec{r}_-) = -\frac{\rho}{3\epsilon_0}\vec{d}$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{Q\vec{d}}{R^3} = -\frac{1}{4\pi\epsilon_0}\frac{\vec{p}}{R^3} = -\frac{\vec{P}}{3\epsilon_0}$$

4.2 The Field of a Polarized Object

Example: A sphere of radius R carries a frozen-in polarization $\vec{P}(\vec{r}) = k\vec{r}$. Find The electric field everywhere

We have a spherically symmetric system so we can use Gauss's Law

$$\vec{E} = \frac{Q_{enc}}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$$

For r<R,

$$Q_{enc} = \int_0^r \rho_b\, 4\pi r^2 dr = -4\pi \int_0^r \vec{\nabla}\cdot\vec{P}\, r^2 dr = -4\pi \int_0^r 3k\, r^2 dr = -4\pi k r^3 \rightarrow \vec{E} = -\frac{kr\hat{r}}{\epsilon_0} = -\frac{\vec{P}}{\epsilon_0}$$

For r>R,

$$Q_{enc} = \int_0^R \rho_b\, 4\pi r^2 dr + 4\pi R^2 \sigma_b = -4\pi \int_0^R \vec{\nabla}\cdot\vec{P}\, r^2 dr + 4\pi R^2 \vec{P}(R\hat{r})\cdot\hat{r}$$

$$= -4\pi \int_0^R 3k\, r^2 dr + 4\pi R^2 (kR\hat{r})\cdot\hat{r} = -4\pi k R^3 + 4\pi k R^3 = 0 \rightarrow \vec{E} = 0$$

4.3 The Electric Displacement

Gauss's Law in the presence of dielectrics

If any free charge exists within a dielectric, the total charge density can be written as

$$\rho = \rho_f + \rho_b \rightarrow \epsilon_0 \vec{\nabla} \cdot \vec{E} = \rho_f - \vec{\nabla} \cdot \vec{P}$$

We usually know the free energy charge density,

Electric Displacement Vector

$$\rho_f = \epsilon_0 \vec{\nabla} \cdot \vec{E} + \vec{\nabla} \cdot \vec{P} = \vec{\nabla} \cdot \left(\epsilon_0 \vec{E} + \vec{P}\right) = \vec{\nabla} \cdot \vec{\mathbf{D}}$$

If symmetry is present, we can calculate $\vec{D}$ by the integral form of Gauss's law,

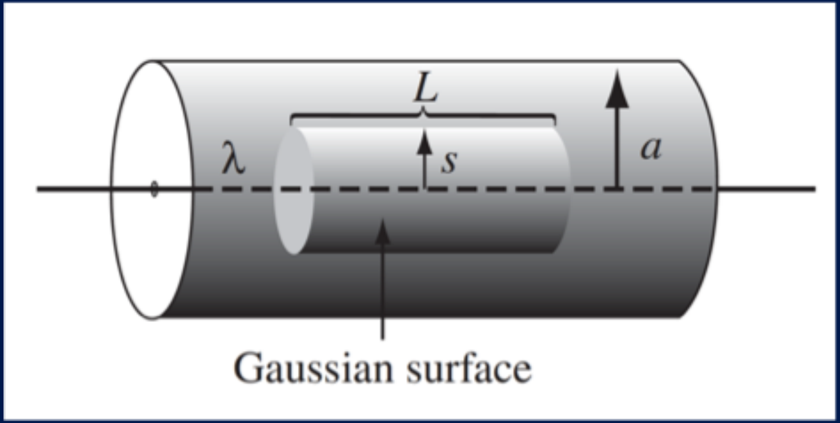
$$\int \vec{\mathbf{D}} \cdot d\vec{a} = \boldsymbol{Q}_{f\,enc}$$

4.2 The Field of a Polarized Object

- **Example:** A long straight wire, carrying uniform line charge λ, is surrounded by rubber insulation out to a radius a. Find the electric displacement $\vec{D}$ and the field $\vec{E}$.

We have a cylindrically symmetric system so we can use Gauss's Law

$$\int \vec{D} \cdot d\vec{a} = Q_{f_{enc}}$$



For $s < a$,

$$\int \vec{D} \cdot d\vec{a} = D2\pi sL = Q_{f_{enc}} = \lambda L \rightarrow \vec{D}(\vec{s}) = \frac{\lambda}{2\pi s}\hat{s}$$

$$\vec{E} = \frac{1}{\epsilon_0}(\vec{D} - \vec{P})$$

Inside the rubber the electric field cannot be determined, since we do not know $\vec{P}$

For $s > a$,

$$\vec{D}(\vec{s}) = \frac{\lambda}{2\pi s}\hat{s} \;\&\; \vec{E} = \frac{1}{\epsilon_0}(\vec{D} - \vec{P}) = \vec{E} = \frac{1}{\epsilon_0}(\vec{D} - 0) = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s}$$

4.2 The Field of a Polarized Object

Example: A thick spherical shell (inner radius a, outer radius b) is made of dielectric material with a "frozen-in" polarization $\vec{P}(\vec{r}) = k\frac{\hat{r}}{r}$ (No free charge inside). Find the electric field in all three regions

We have a spherically symmetric system so we use Gauss's law

$$\vec{D} = \frac{Q_{f enc}}{4\pi}\frac{\hat{r}}{r^2}$$

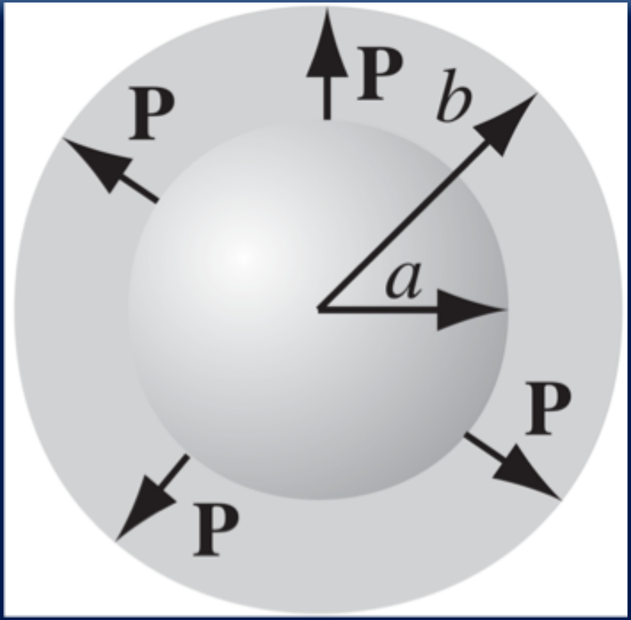
But no free charges anywhere, so always $Q_{f enc} = 0$

$$\vec{D} = 0$$

For r $< a$, $\vec{E} = \frac{1}{\epsilon_0}(\vec{D} - \vec{P}) = \frac{1}{\epsilon_0}(\vec{0} - \vec{0}) = 0$

For b $>$ **r** $> a$, $\vec{E} = \frac{1}{\epsilon_0}(\vec{D} - \vec{P}) = \frac{1}{\epsilon_0}\left(\vec{0} - k\frac{\hat{r}}{r}\right) = -\frac{k\hat{r}}{\epsilon_0 r}$

For r $>$ **b,** $\vec{E} = \frac{1}{\epsilon_0}(\vec{D} - \vec{P}) = \frac{1}{\epsilon_0}(\vec{0} - \vec{0}) = 0$



4.3 The Electric Displacement

A deceptive parallel: misleading comparison between $\vec{E}$ and $\vec{D}$

$\vec{\nabla}\cdot\vec{\mathbf{D}} = \rho_f$ ***is similar to*** $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$

"to solve problems involving dielectrics, you just forget all about the bound charge, calculate the field as you ordinarily would using Coulomb law, only call the answer $\vec{D}$ instead of $\vec{E}$ "

$$\vec{D}(\vec{r}) = \frac{1}{4\pi}\int \frac{\rho_f}{\mathscr{r}^2}\hat{\mathscr{r}}\, d\tau$$

This is because

$$\vec{\nabla}\times\vec{D} = \vec{\nabla}\times\left(\epsilon_0\vec{E} + \vec{P}\right) = \vec{\nabla}\times\vec{P} \neq 0$$

4.3 The Electric Displacement

Boundary Conditions

$\vec{\nabla} \cdot \vec{\mathbf{D}} = \rho_f \rightarrow$

$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$

The *normal component of* $\vec{D}$ is discontinuous

$\vec{\nabla} \cdot \vec{\mathbf{E}} = \rho/\epsilon_0 \rightarrow$

$E^{\perp}_{above} - E^{\perp}_{below} = 1/\epsilon_0(\sigma_f + \sigma_b)$

$\vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P} \rightarrow$

$D^{\parallel}_{above} - D^{\parallel}_{below} = P^{\parallel}_{above} - P^{\parallel}_{below}$

$\vec{\nabla} \times \vec{E} = 0 \rightarrow$

$E^{\parallel}_{above} - E^{\parallel}_{below} = 0$

4.4 Linear Dielectrics

Susceptibility, Permittivity and Dielectric constant

For linear dielectrics, if the electric field is weak, $\vec{P} \propto \vec{E}$

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

Electric susceptibility

*The electric field is the total field from free charges and the polarization itself.

$$\vec{D} = \left(\epsilon_0 \vec{E} + \vec{P}\right) = \left(\epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E}\right)$$
$$= \epsilon_0 (1 + \chi_e) \vec{E} = \epsilon \vec{E}$$

Permittivity

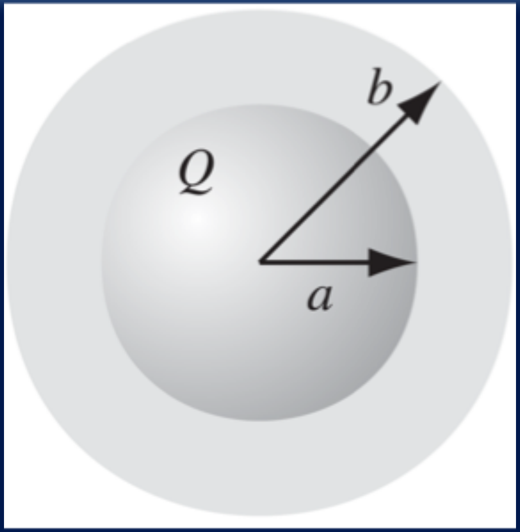
Relative permittivity or dielectric constant

$$\epsilon_r = \frac{\epsilon}{\epsilon_0}$$

Material	Dielectric Constant
Vacuum	1
Helium	1.000065
Neon	1.00013
Hydrogen	1.00025
Argon	1.00052
Air (dry)	1.00054
Nitrogen	1.00055
Water vapor (100° C)	1.00587
Diamond	5.7
Salt	5.9
Silicon	11.8

4.4 Linear Dielectrics

Example: A metal sphere of radius a carries a charge Q. It is surrounded, out to radius b, by linear dielectric material of permittivity ϵ. Find the potential V at the center (relative to infinity).



For r $< a$, $\vec{E} = \vec{P} = \vec{D} = 0$

For b $>$ **r** $> a$,

$$\vec{D} = \frac{1}{4\pi}\frac{Q\hat{r}}{r^2} \rightarrow \vec{E} = \frac{1}{\epsilon}\vec{D} = \frac{1}{4\pi\epsilon}\frac{Q\hat{r}}{r^2} \ \& \vec{P} = \epsilon_0\chi_e\vec{E} = \epsilon_0\left(\frac{\epsilon}{\epsilon_0} - 1\right)\vec{E}$$

For r $> b$,

$$\vec{D} = \frac{1}{4\pi}\frac{Q\hat{r}}{r^2} \rightarrow \vec{E} = \frac{1}{\epsilon_0}\vec{D} = \frac{1}{4\pi\epsilon_0}\frac{Q\hat{r}}{r^2} \ \& \ \vec{P} = 0$$

$$V(0) = -\int_\infty^0 \vec{E}\cdot d\vec{l} = -\int_\infty^b \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}dr - \int_b^a \frac{1}{4\pi\epsilon}\frac{Q}{r^2}dr - \int_b^a 0\,dr$$

$$V(0) = \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right)$$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = 0$$

$$\sigma_b = \vec{P}\cdot\hat{n} = \begin{cases} -\epsilon_0\chi_e Q/4\pi\epsilon a^2 & r = a \\ \epsilon_0\chi_e Q/4\pi\epsilon b^2 & r = b \end{cases}$$