Electromagnetic Theory I

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Chapter 4: Electric Fields in Matter

- Polarization
- The Field of a Polarized Object
- The Electric Displacement
- Linear Dielectrics

4.4 Linear Dielectrics

Boundary value problems with linear dielectrics

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot\left(\epsilon_0\chi_e\vec{E}\right) = -\vec{\nabla}\cdot\left(\frac{\epsilon_0}{\epsilon}\chi_e\vec{D}\right) = -\frac{\chi_e}{1+\chi_e}\vec{\nabla}\cdot\vec{D} = -\frac{\chi_e}{1+\chi_e}\rho_f$$

*The bound charge density is proportional to the free charge density.

*In absence of free charge density, there will be no volume bound charge density. Any net charge must be resident on the surface and the electric potential obeys Laplace equation inside the linear dielectric.

Boundary conditions

$$D_{above}^{\perp} - D_{below}^{\perp} = \sigma_f \rightarrow \epsilon_{above}E_{above}^{\perp} - \epsilon_{below}E_{below}^{\perp} = \sigma_f$$

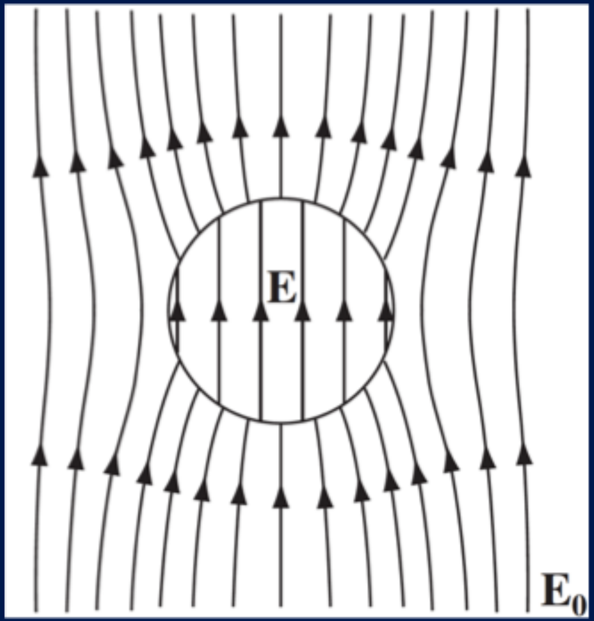
$$\epsilon_{above}\frac{\partial V_{above}}{\partial n} - \epsilon_{below}\frac{\partial V_{below}}{\partial n} = -\sigma_f$$

$$V_{above} - V_{below} = \int_{above}^{below}\vec{E}\cdot d\vec{l} \rightarrow V_{above} = V_{below}$$

4.4 Linear Dielectrics

Example: A sphere of homogeneous linear dielectric material is placed in an otherwise uniform electric field $\vec{E}_0$. Find the electric field inside the sphere.

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$



Boundary conditions

$$V(r,\theta) \to -E_0 z = -E_0 r\cos\theta \quad as\, r \to \infty$$

$$\epsilon_0 \frac{\partial V_{out}}{\partial r} - \epsilon \frac{\partial V_{in}}{\partial r} = -\sigma_f = 0 \quad at\, r = R$$

$$V_{out} = V_{in} \qquad at\, r = R$$

4.4 Linear Dielectrics

From $V(r,\theta) \rightarrow -E_0 z = -E_0 r\cos\theta \;\; as\, r \rightarrow \infty$, **we get**

$$V_{out}(\mathrm{r},\theta) = -E_0 r\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

From $V(r,\theta)\; has\ to\ be\ finite\;\; as\, r \rightarrow 0$, **we get**

$$V_{in}(\mathrm{r},\theta) = \sum_{l=0}^{\infty} A_l r^l\, P_l(\cos\theta)$$

4.4 Linear Dielectrics

From $\epsilon_0 \frac{\partial V_{out}}{\partial r} - \epsilon \frac{\partial V_{in}}{\partial r} = 0$ at r=R, we get

$$-\epsilon_0 E_0 \cos\theta - \epsilon_0 \sum_{l=0}^{\infty} (l+1) \frac{B_l}{R^{l+2}} P_l(\cos\theta) - \epsilon \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = 0$$

From $V_{out} = V_{in}$ at r=R, we get

$$-E_0 R \cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) - \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = 0$$

4.4 Linear Dielectrics

The solution of the later two equations is

$$A_l = B_l = 0 \qquad for\; l \neq 1$$

$$A_1 = -\frac{3}{\epsilon_r + 2} E_0 \; and \; B_1 = \frac{\epsilon_r - 1}{\epsilon_r + 2} R^3 E_0$$

$$V_{in}(r,\theta) = -\frac{3}{\epsilon_r + 2} E_0 r \cos\theta = -\frac{3}{\epsilon_r + 2} E_0 z$$

$$\vec{E}_{in} = -\vec{\nabla} V_{in} = \frac{3}{\epsilon_r + 2} E_0 \hat{z} = \frac{3}{\epsilon_r + 2} \vec{E}_0$$

4.4 Linear Dielectrics

Another interesting way to get the solution inside is as follows:

$$\vec{P}_0 = \epsilon_0 \chi_e \vec{E}_0 \rightarrow \vec{E}_1 = -\frac{\vec{P}_0}{3\epsilon_0} = -\frac{\chi_e}{3}\vec{E}_0$$

$$\vec{P}_1 = \epsilon_0 \chi_e \vec{E}_1 \rightarrow \vec{E}_2 = -\frac{\vec{P}_1}{3\epsilon_0} = \left(\frac{\chi_e}{3}\right)^2 \vec{E}_0$$

$$\vec{P}_2 = \epsilon_0 \chi_e \vec{E}_2 \rightarrow \vec{E}_3 = -\frac{\vec{P}_2}{3\epsilon_0} = -\left(\frac{\chi_e}{3}\right)^3 \vec{E}_0$$

$$\vec{E}_{in} = \vec{E}_0 + \vec{E}_1 + \cdots = \sum_{n=0}^{\infty} (-1)^n \left(\frac{\chi_e}{3}\right)^n \vec{E}_0 = \frac{1}{\left(1 + \frac{\chi_e}{3}\right)} \vec{E}_0 = \frac{1}{\left(1 + \frac{\epsilon_r - 1}{3}\right)} \vec{E}_0$$

$$\vec{E}_{in} = \frac{1}{\left(1 + \frac{\epsilon_r - 1}{3}\right)} \vec{E}_0 = \frac{3}{2 + \epsilon_r} \vec{E}_0$$

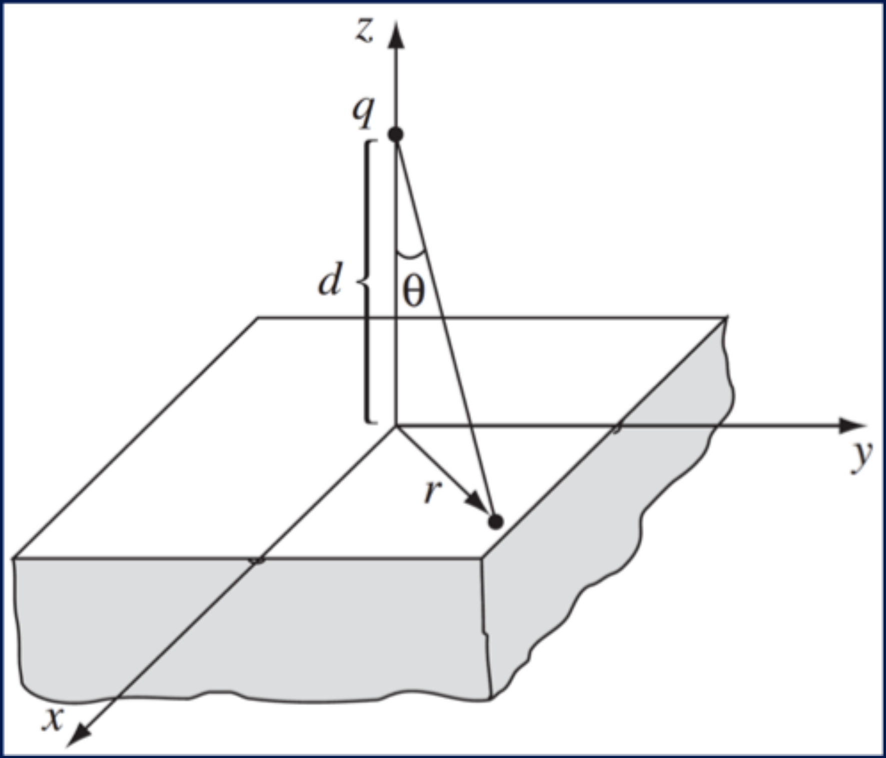
4.4 Linear Dielectrics

Example: Calculate the force on a point charge q situated a distance d above a uniform linear dielectric material of susceptibility χ_e

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\vec{\nabla} \cdot \left(\epsilon_0 \chi_e \vec{E}\right)$$
$$= -\vec{\nabla} \cdot \left(\frac{\epsilon_0}{\epsilon} \chi_e \vec{D}\right) = -\frac{\chi_e}{1+\chi_e} \vec{\nabla} \cdot \vec{D}$$
$$= -\frac{\chi_e}{1+\chi_e} \rho_f = 0$$

$$\sigma_b = \vec{P} \cdot \hat{z} = P_z = \epsilon_0 \chi_e E_z$$

$$E_z = E_{q_z} + E_{b_z}$$



4.4 Linear Dielectrics

$$E_{q_z} = -\frac{q}{4\pi\epsilon_0(d^2+r^2)}\cos\theta = -\frac{qd}{4\pi\epsilon_0(d^2+r^2)^{3/2}}$$

$$E_{b_z} = -\frac{\sigma_b}{2\epsilon_0}$$

$$\sigma_b = \epsilon_0\chi_e E_z = \epsilon_0\chi_e\left(-\frac{qd}{4\pi\epsilon_0(d^2+z^2)^{\frac{3}{2}}} - \frac{\sigma_b}{2\epsilon_0}\right)$$

$$\sigma_b = \frac{1}{1+\frac{\chi_e}{2}}\left(-\frac{q\chi_e d}{4\pi(d^2+r^2)^{\frac{3}{2}}}\right) = -\frac{1}{2\pi}\left(\frac{\chi_e}{2+\chi_e}\right)\left(\frac{qd}{(d^2+r^2)^{\frac{3}{2}}}\right)$$

4.4 Linear Dielectrics

Total surface bound charge

$$q_b = \int\int \sigma_b da = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -\frac{1}{2\pi}\left(\frac{\chi_e}{2+\chi_e}\right)\frac{qd\ dxdy}{(x^2+y^2+d^2)^{3/2}}$$

$$q_b = \int_0^{2\pi}\int_0^{\infty} -\frac{1}{2\pi}\left(\frac{\chi_e}{2+\chi_e}\right)\frac{qd\ sdsd\phi}{(s^2+d^2)^{3/2}}$$

$$= \left(\frac{\chi_e}{2+\chi_e}\right)\frac{qd}{(s^2+d^2)^{\frac{1}{2}}}\Bigg|_0^{\infty} = -q\left(\frac{\chi_e}{2+\chi_e}\right)$$

4.4 Linear Dielectrics

Force on q

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} q dQ = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} q\sigma_b da$$

$$\vec{F} = \frac{q}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(x^2+y^2+d^2)}\frac{(-x\hat{x}-y\hat{y}+d\hat{z})}{(x^2+y^2+d^2)^{1/2}}\left(-\frac{1}{2\pi}\left(\frac{\chi_e}{2+\chi_e}\right)\frac{qd}{(x^2+y^2+d^2)^{3/2}}\right)dxdy$$

$$\vec{F} = -\frac{q^2 d}{8\pi^2\epsilon_0}\left(\frac{\chi_e}{2+\chi_e}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(-x\hat{x}-y\hat{y}+d\hat{z})}{(x^2+y^2+d^2)^3}dxdy$$
$$= -\frac{q^2 d^2}{4\pi\epsilon_0}\left(\frac{\chi_e}{2+\chi_e}\right)\hat{\mathbf{z}}\int_0^{\infty}\frac{sds}{(s^2+d^2)^3}$$

$$\vec{F} = \frac{q^2 d^2}{16\pi\epsilon_0}\left(\frac{\chi_e}{2+\chi_e}\right)\hat{\mathbf{z}}\, u^{-2}\Big|_{d^2}^{\infty} = -\frac{q^2}{4\pi\epsilon_0(2d)^2}\left(\frac{\chi_e}{2+\chi_e}\right)\hat{\mathbf{z}}$$

4.4 Linear Dielectrics

Energy in dielectric systems

In general,

$$\vec{\nabla} \cdot \Delta\vec{D} = \Delta\rho_f$$

$$\Delta W = \int \Delta\rho_f V d\tau = \int \vec{\nabla} \cdot \Delta\vec{D} V d\tau = \int \vec{\nabla} \cdot (\Delta\vec{D} V) d\tau + \int \Delta\vec{D} \cdot \vec{E} d\tau$$

$$\Delta W = \int \Delta\vec{D} \cdot \vec{E} d\tau + \int \left(\Delta\vec{D} V\right) \cdot d\vec{a} = \int \Delta\vec{D} \cdot \vec{E} d\tau$$

For linear dielectrics

$$\Delta W = \int \Delta\vec{D} \cdot \vec{E} d\tau = \int \Delta(\epsilon\vec{E}) \cdot \vec{E} d\tau = \frac{1}{2} \int \Delta\left(\epsilon\vec{E} \cdot \vec{E}\right) d\tau = \Delta \frac{1}{2} \int \left(\vec{E} \cdot \vec{D}\right) d\tau$$

$$\textit{The stored energy} = \textit{total work done} \rightarrow W = \frac{1}{2} \int \left(\vec{E} \cdot \vec{D}\right) d\tau$$

4.4 Linear Dielectrics

Example: A sphere of radius R is filled with material of dielectric constant ϵ_r and uniform embedded free charge ρ_f. What is the energy of this configuration?

Using Gauss law

$$\vec{D}(\vec{r}) = \begin{cases} \frac{1}{3}\rho_f r\hat{r} & r < R \\ \frac{1}{3}\rho_f \frac{R^3}{r^2}\hat{r} & r \geq R \end{cases}$$

$$\vec{E}(\vec{r}) = \frac{1}{\epsilon}\vec{D}(\vec{r}) = \begin{cases} \frac{1}{3\epsilon_0\epsilon_r}\rho_f r\hat{r} & r < R \\ \frac{1}{3\epsilon_0}\rho_f \frac{R^3}{r^2}\hat{r} & r \geq R \end{cases}$$

$$W = \frac{1}{2}\int_0^\infty \vec{E}\cdot\vec{D}\,4\pi r^2 dr = \frac{2\pi\rho_f^2}{9\epsilon_0\epsilon_r}\int_0^R r^4\,dr + \frac{2\pi\rho_f^2 R^6}{9\epsilon_0}\int_R^\infty \frac{dr}{r^2}$$

$$= \frac{2\pi\rho_f^2}{9\epsilon_0\epsilon_r}\frac{R^5}{5} + \frac{2\pi\rho_f^2 R^6}{9\epsilon_0}\frac{1}{R} = \frac{2\pi\rho_f^2 R^5}{9\epsilon_0}\left(\frac{1}{5\epsilon_r} + 1\right)$$

4.4 Linear Dielectrics

Forces on Dielectrics

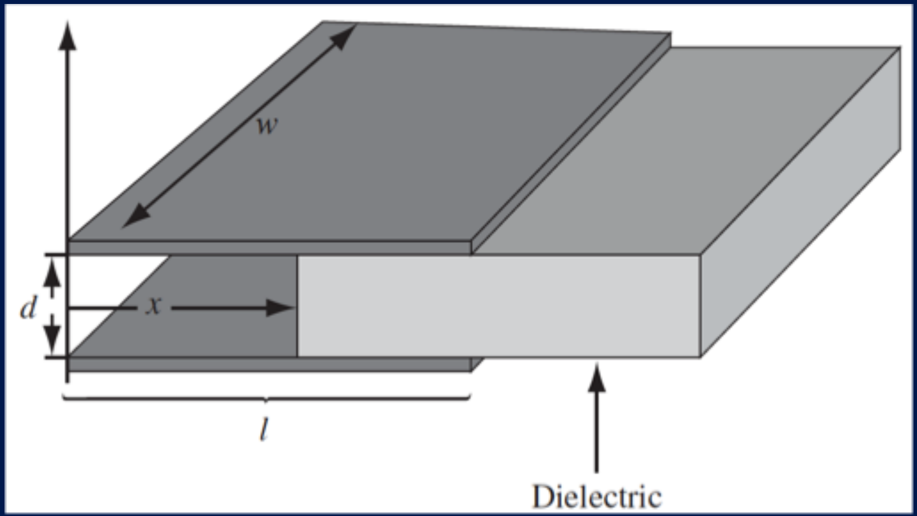
$$\sigma = \epsilon_0 E = \epsilon_0 \frac{V}{d}$$

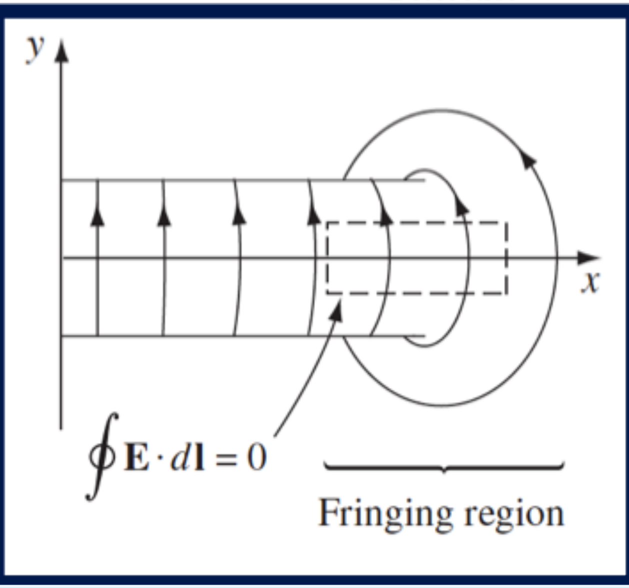
$$\sigma_f = \epsilon E = \epsilon_0 \epsilon_r \frac{V}{d} = \epsilon_r \sigma$$

$$C = \frac{Q}{V} = \frac{wx\sigma + w(l-x)\sigma_f}{V}$$

$$C = \frac{1}{V}\left(wx\epsilon_0 \frac{V}{d} + w(l-x)\epsilon_0\epsilon_r \frac{V}{d}\right)$$

$$C = \frac{w\epsilon_0}{d}\left(x\,(1-\epsilon_r) + l\epsilon_r\right)$$
$$= \frac{w\epsilon_0}{d}\left(-\chi_e x + l\epsilon_r\right)$$





4.4 Linear Dielectrics

At constant Q (isolated)

$$W = \frac{1}{2}\frac{Q^2}{C} \rightarrow F_x = -\frac{dW}{dx} = -\frac{dW}{dC}\frac{dC}{dx} = -\frac{1}{2}\frac{Q^2}{C^2}\frac{w\epsilon_0\chi_e}{d} = -\frac{w\epsilon_0\chi_e}{2d}V^2$$

At constant V

$$W = \frac{1}{2}CV^2 \rightarrow F_x = -\frac{dW}{dx} = \frac{w\epsilon_0\chi_e}{2d}V^2$$

Wrong sign!! To maintain the capacitor at a fixed potential V, we need to connect it up to a battery. But in that case the battery also does work as the dielectric moves;

$$F_x = -\frac{dW}{dx} + V\frac{dQ}{dx} = -\frac{dW}{dC}\frac{dC}{dx} + V\frac{dQ}{dC}\frac{dC}{dx} = -\frac{1}{2}V^2\frac{dC}{dx} + V^2\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx}$$
$$= -\frac{w\epsilon_0\chi_e}{2d}V^2$$