Electromagnetic Theory I

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May 19, 2021

جامعة بيرزيت

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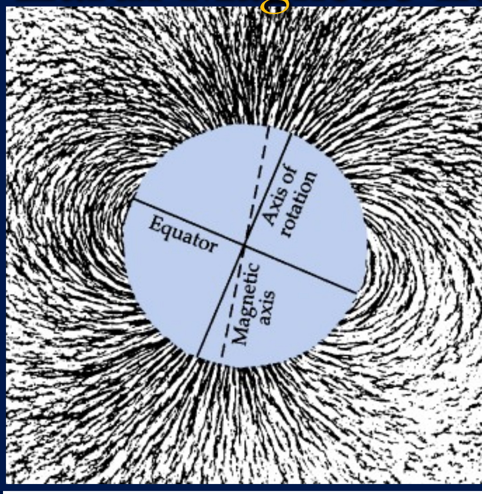
Chapter 5: Magnetostatics

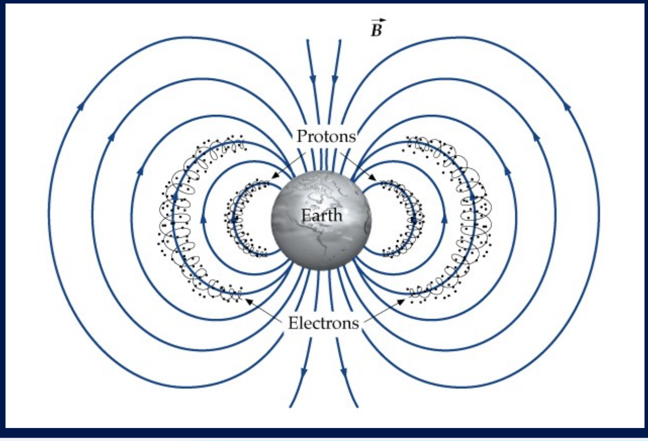
- The Lorentz Force Law
- The Biot-Savart Law
- The Divergence and Curl of $\vec{B}$
- Magnetic Vector Potential ($\vec{A}$)

5.1 The Lorentz Force Law

Introduction

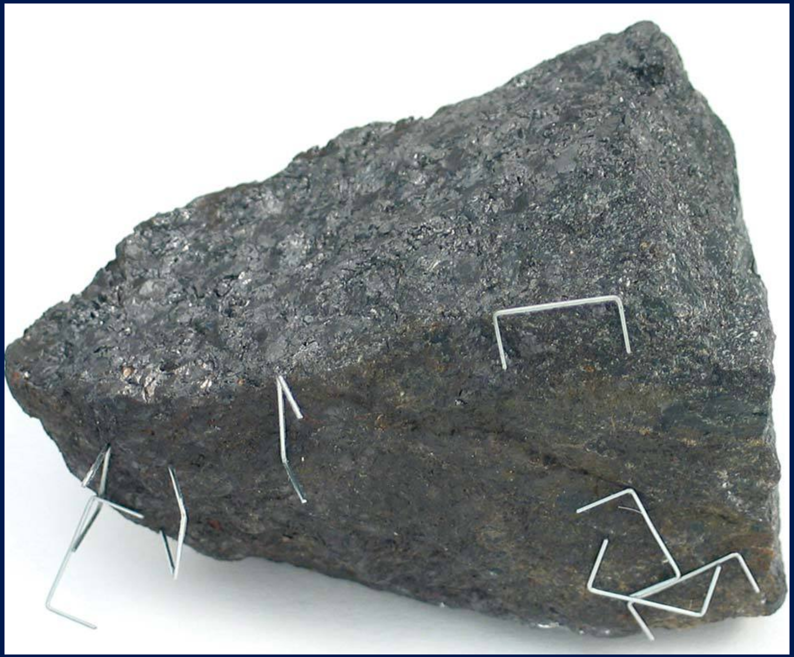
Earth Magnetic field





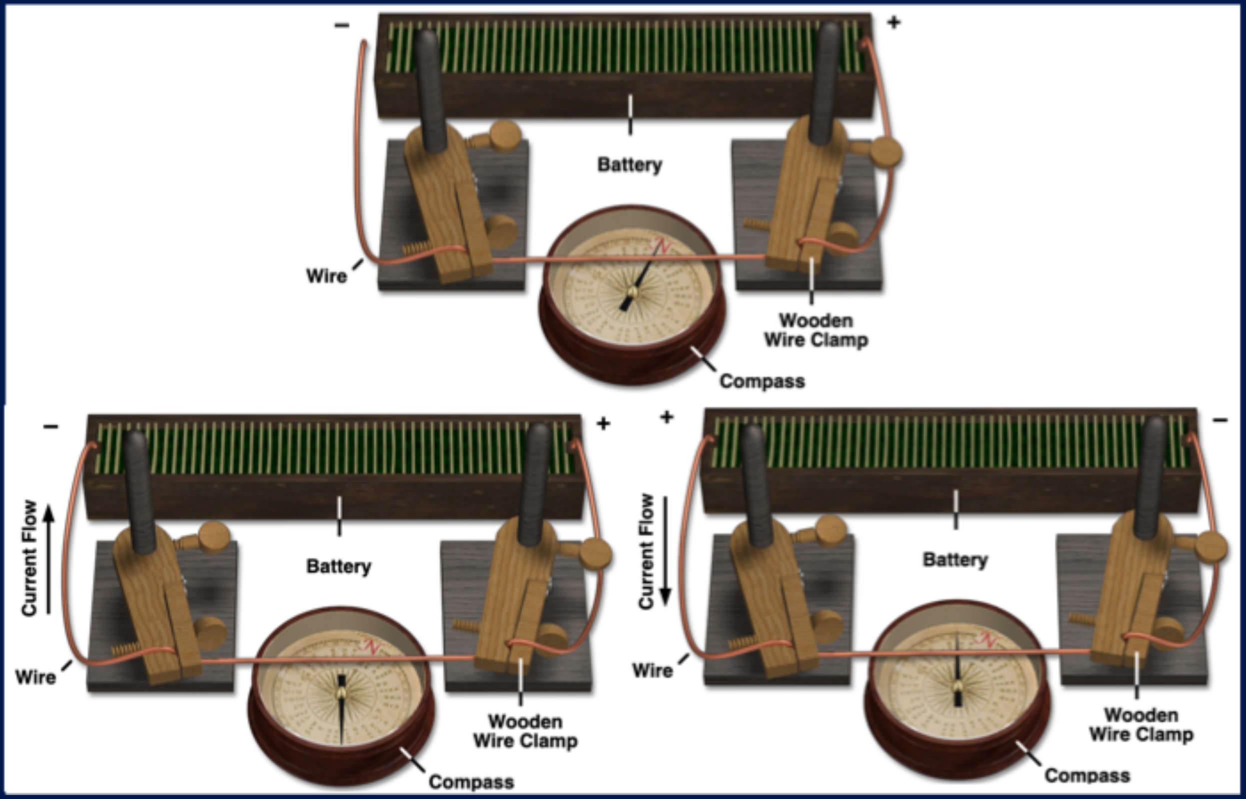
Lodestone (Magnetite Fe_3O_4)

Known by Ancient civilizations



5.1 The Lorentz Force Law

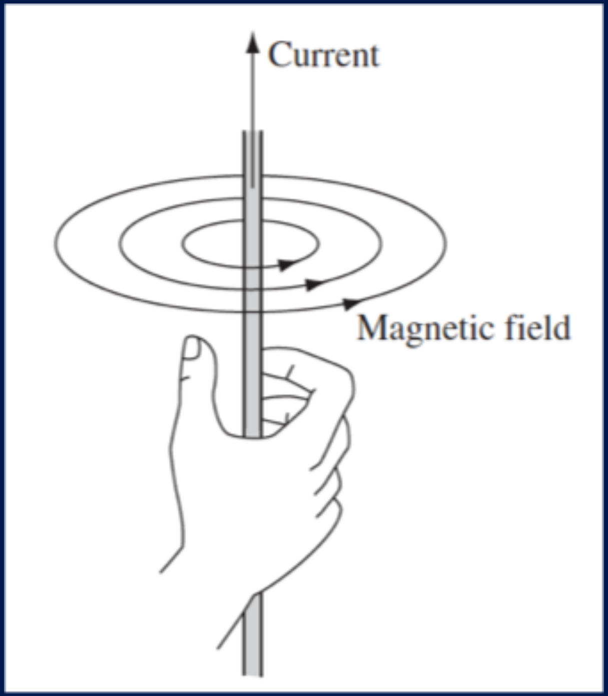
- Orsted 1820

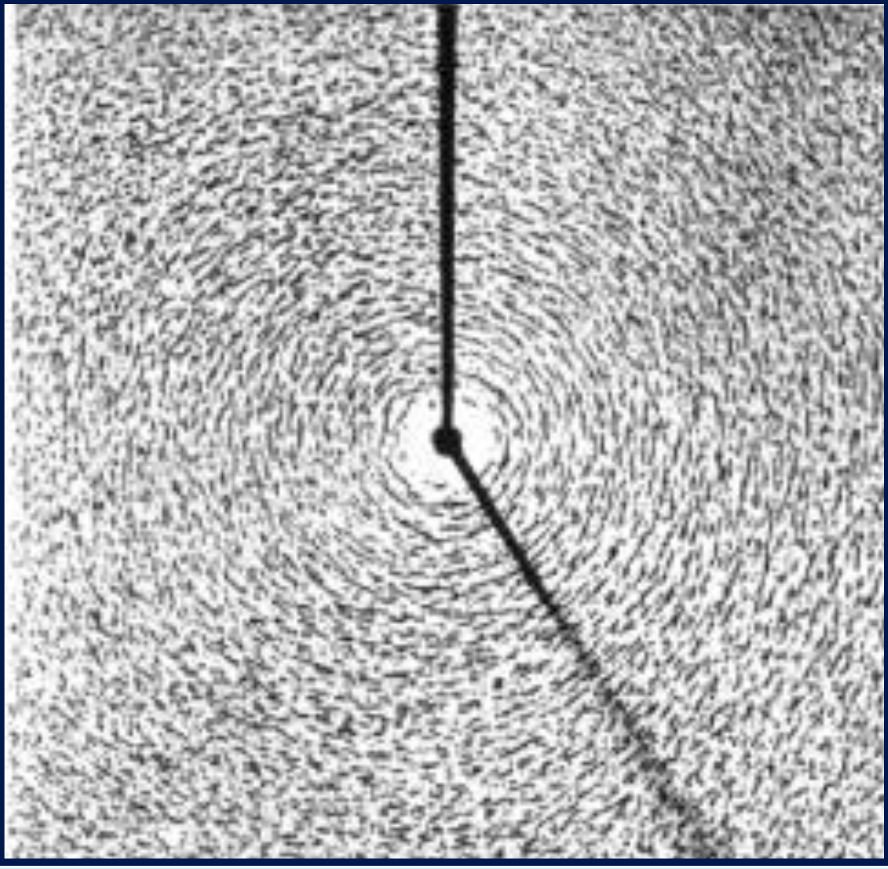


5.1 The Lorentz Force Law

Magnetic Fields due to currents

Moving charges generate magnetic fields



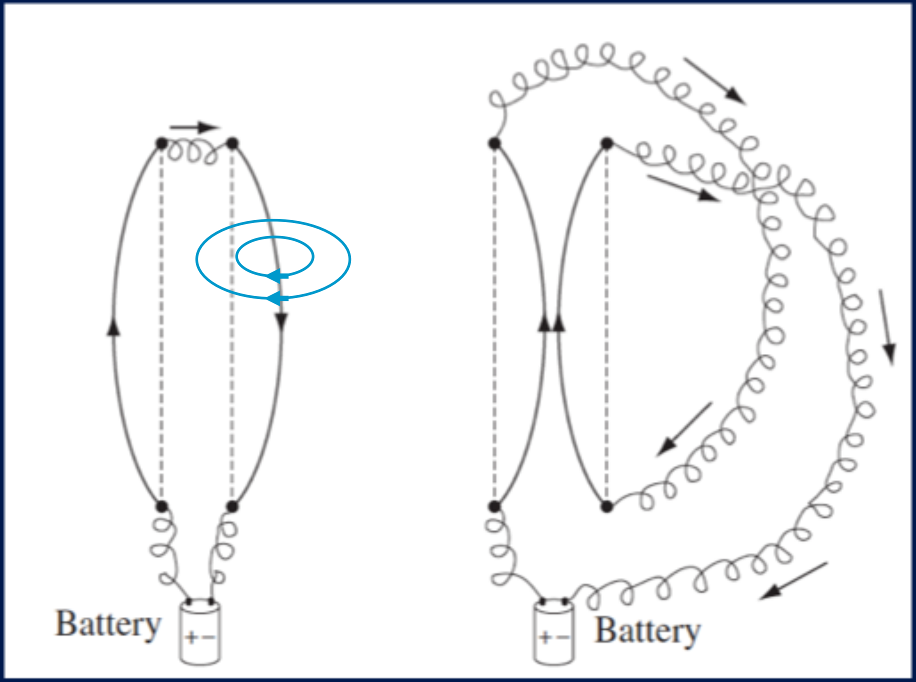


5.1 The Lorentz Force Law

Force between two wires

The forces between charges in motion:

- Attraction of parallel currents
- Repulsion of antiparallel currents
- Does charging up the wires make simply the electrical repulsion
- of like charges?
- The wires are electrically neutral.
- It is not electrostatic in nature.



5.1 The Lorentz Force Law

Lorentz Force Law

The net force on Q in the presence of both electric and magnetic fields:

$$\vec{F} = Q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

The magnetic force in a charge Q, moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$,

$$\vec{F}_{mag} = Q\left(\vec{v}\times\vec{B}\right)$$

Magnetic forces do not work!

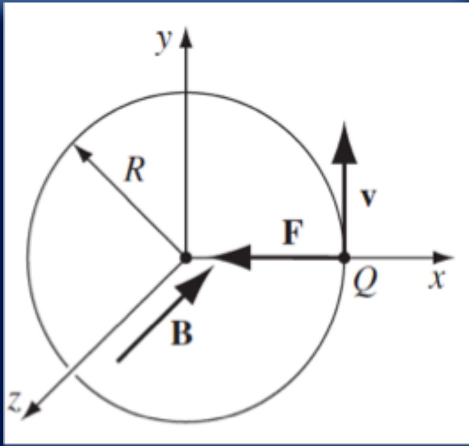
$$dW_{mag} = \vec{F}_{mag}\cdot d\vec{l} = Q\left(\vec{v}\times\vec{B}\right)\cdot\vec{v}dt = 0$$

Magnetic forces may change the direction in which a particle moves, but they cannot speed it up or slow it down.

5.1 The Lorentz Force Law

Example: Cyclotron motion?: If a charge Q moves counterclockwise, with speed v, around a circle of radius R, in a plane perpendicular to $\vec{B}$. what path will it follow?

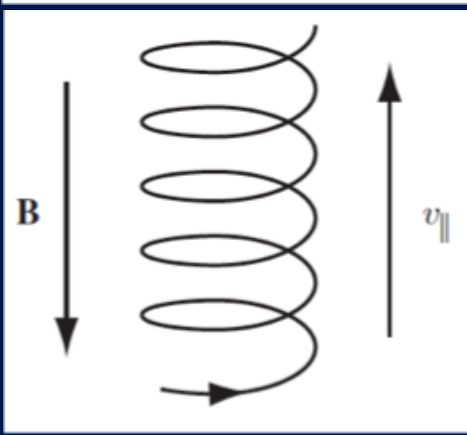
Cyclotrons represent the first model of modern particle accelerators. The magnetic force is inward, and has a fixed magnitude to sustain uniform circular motion:



$$\left|\vec{F}_{mag}\right| = \left|Q\left(\vec{v}\times\vec{B}\right)\right| = QvB = \frac{mv^2}{R} \rightarrow \omega = \frac{v}{R} = \frac{QB}{m}$$

Experimentally the momentum of a particle can be found using

$$p = mv = QBR$$

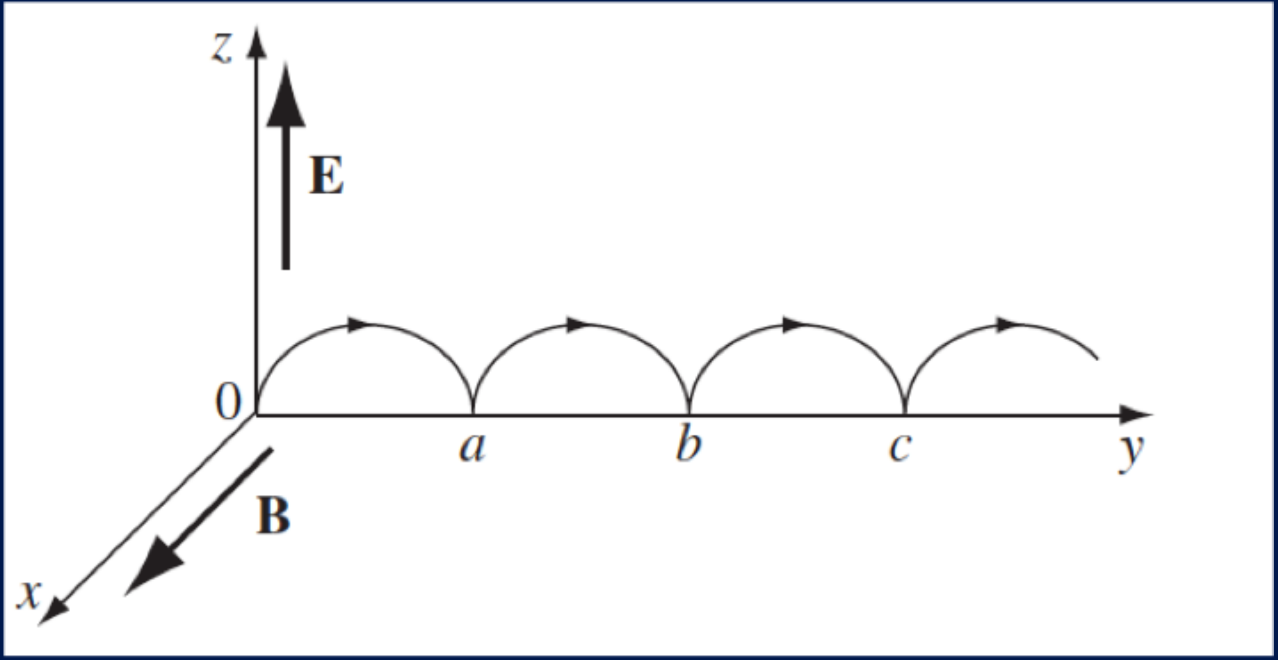


If the velocity has a parallel component to the magnetic field, we get helical motion

5.1 The Lorentz Force Law

Example: Cycloid motion?: If a charge Q at rest is released from the origin, what path it will follow?

$$\vec{E} = E\hat{z}$$
$$\vec{B} = B\hat{x}$$
$$\vec{v} = \dot{y}\hat{y} + \dot{z}\hat{z}$$
$$\vec{v}_0 = 0, \vec{r}_0 = 0$$



$$\vec{F} = Q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

$$\vec{F} = QE\hat{z} + Q\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \dot{y} & \dot{z} \\ B & 0 & 0 \end{vmatrix}$$
$$\vec{F} = Q(B\dot{z}\hat{y} + (E - B\dot{y})\hat{z})$$

5.1 The Lorentz Force Law

$$\vec{F} = Q(B\dot{z}\hat{y} + (E - B\dot{y})\hat{z}) = m\vec{a} = m(\ddot{y}\hat{y} + \ddot{z}\hat{z})$$

$$Q(E - B\dot{y}) = m\ddot{z}$$

$$QB\dot{z} = m\ddot{y}$$

$$\ddot{y} = \frac{QB}{m}\ddot{z} = \frac{Q^2B^2}{m^2}\left(\frac{E}{B} - \dot{y}\right) = \omega^2\left(\frac{E}{B} - \dot{y}\right)$$

$$\ddot{z} = -\frac{QB}{m}\ddot{y} = -\frac{Q^2B^2}{m^2}\dot{z} = -\omega^2\dot{z}$$

$$y(t) = C_1\cos\omega t + C_2\sin\omega t + \frac{E}{B}t + C_3$$

$$z(t) = -C_1\sin\omega t + C_2\cos\omega t + C_4$$

$\vec{v}_0 = 0, \vec{r}_0 = 0 \rightarrow$

$$y(t) = -\frac{E}{\omega B}\sin\omega t + \frac{E}{B}t$$

$$z(t) = -\frac{E}{\omega B}\cos\omega t + \frac{E}{\omega B}$$

5.1 The Lorentz Force Law

$$let\ R = \frac{E}{\omega B}$$

$$(z - R)^2 = (-R\cos\omega t + R - R)^2 = R^2\cos^2\omega t$$

$$(y - R\omega t)^2 = (R\sin\omega t + R\omega t - R\,\omega t)^2 = R^2\sin^2\omega t$$

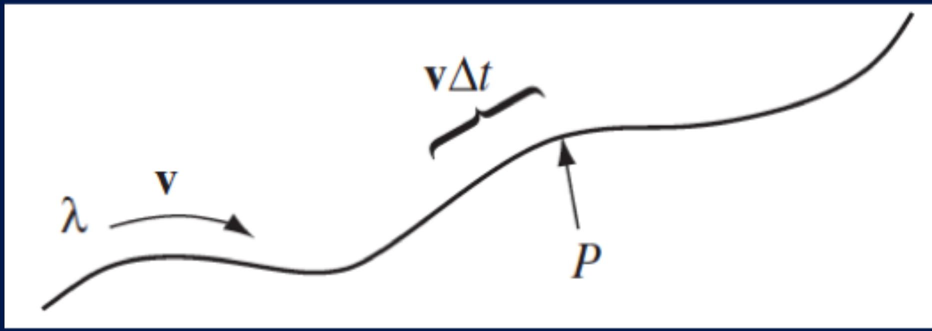
$$(y - R\omega t)^2 + (z - R)^2 = R^2\sin^2\omega t + R^2\cos^2\omega t = R^2$$

Cycloid motion: a circle, of radius R, with its center $(0, R\omega t, R)$ travels in the y-direction at a constant speed, $v = E/B$

5.1 The Lorentz Force Law

Currents

A line charge λ traveling down at velocity $\vec{v}$



$$\vec{I} = \lambda\vec{v}$$

The magnetic force on a segment of current-carrying wire is

$$\vec{F}_{mag} = \int \vec{v}\times\vec{B}\,dq = \int \vec{v}\times\vec{B}\,\lambda dl = \int \vec{I}\times\vec{B}\,dl$$

$\vec{I}$ and $d\vec{l}$ are in the same direction

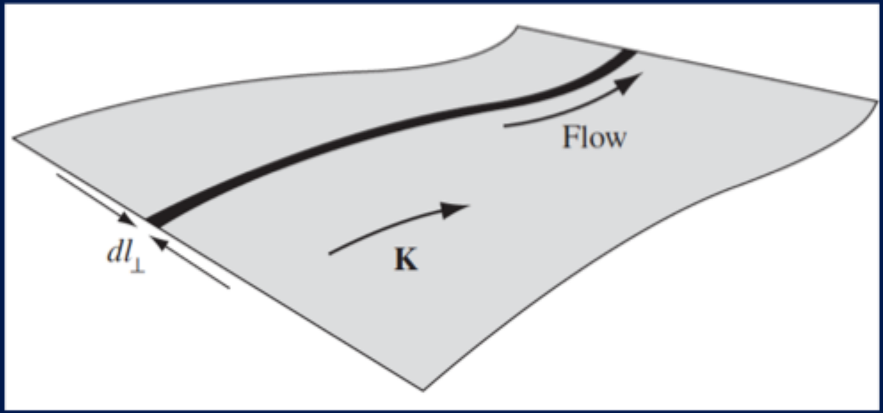
For a steady current along the wire

$$\vec{F}_{mag} = I\int d\vec{l}\times\vec{B}$$

5.1 The Lorentz Force Law

When charge flows over surface with surface current density $\vec{\kappa}$

$$\vec{\kappa} = \frac{d\vec{I}}{dl^{\perp}} = \sigma\vec{v}$$



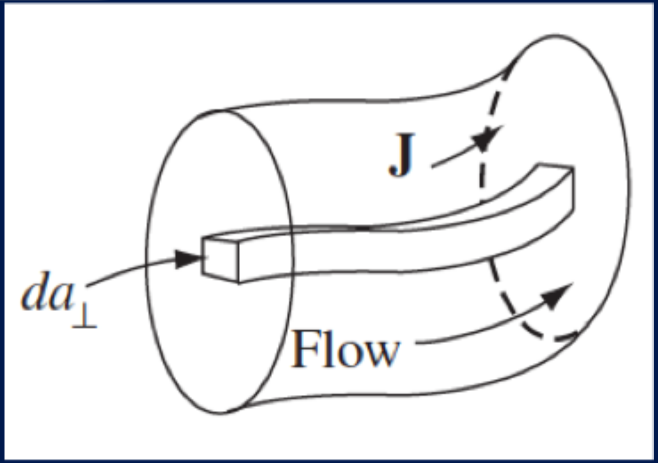
$\vec{\kappa}$:current per unit width-perpendicular-to-flow

The magnetic force on the sheet

$$\vec{F}_{mag} = \int \vec{v}\times\vec{B}\, dq = \int \vec{v}\times\vec{B}\,\sigma da = \int \vec{\kappa}\times\vec{B} da$$

5.1 The Lorentz Force Law

When charge flows through a volume with volume current density $\vec{J}$



$$\vec{J} = \frac{d\vec{I}}{da^{\perp}} = \rho\vec{v}$$

$\vec{J}$:current per unit area-perpendicular-to-flow

The magnetic force on the sheet

$$\vec{F}_{mag} = \int \vec{v}\times\vec{B}\, dq = \int \vec{v}\times\vec{B}\, \rho d\tau = \int \vec{J}\times\vec{B} d\tau$$

5.1 The Lorentz Force Law

Continuity equation

$$\vec{J} = \frac{d\vec{I}}{da^{\perp}}$$

The current crossing a surface S can be written as

$\rightarrow$

$$I = \int_S J da_{\perp} = \int_S \vec{J} \cdot d\vec{a}$$

But, the total charge per unit time leaving a volume V is

Conservation of charge

$$\int_S \vec{J} \cdot d\vec{a} = \int_V \vec{\nabla} \cdot \vec{J} d\tau = -\frac{d}{dt} \int_V \rho d\tau = -\int_V \frac{\partial \rho}{\partial t} d\tau$$

5.1 The Lorentz Force Law

Continuity equation

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

(Local charge conservation)

$$\frac{\partial \rho}{\partial t} = 0 \rightarrow \vec{\nabla} \cdot \vec{J} = 0 \rightarrow \int_S \vec{J} \cdot d\vec{a} = 0 \rightarrow \sum_k I_k = 0$$

(Kirchhoff's current law)

5.1 The Lorentz Force Law

Example: Suppose the current density distributed over a wire of circular cross section, with radius a is given by $J = ks$. Find the total current in the wire

$$I = \int_S \vec{J} \cdot d\vec{a} = \int_0^{2\pi} \int_0^{a} ks\, s ds\, d\phi = 2\pi k \int_0^{a} s^2 ds = \frac{2\pi k a^3}{3}$$