Electromagnetic Theory I

Abdallah Sayyed-Ahmad

Department of Physics

Birzeit University

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جامعة بيرزيت

BIRZEIT UNIVERSITY

Chapter 5: Magnetostatics

- The Lorentz Force Law
- The Biot-Savart Law
- The Divergence and Curl of $\vec{B}$
- Magnetic Vector Potential ($\vec{A}$)

5.2 The Biot-Savart Law

Steady Currents

A continuous flow of charges without charge piling up anywhere. Such current flow yield time independent magnetic field (Magnetostatics)

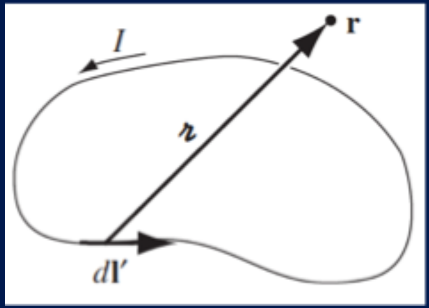
$$\frac{\partial \rho}{\partial t} = 0 \rightarrow \vec{\nabla} \cdot \vec{J} = 0$$

The magnetic field of a steady currents

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \vec{I} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} dl'$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \vec{\kappa}(\vec{r}') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} da'$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$$



Biot-Savart law play the same role as Coulomb law in electrostatics, $\mu_0 = 4\pi \times 10^{-7}\ N/A^2$ (free space permittivity)

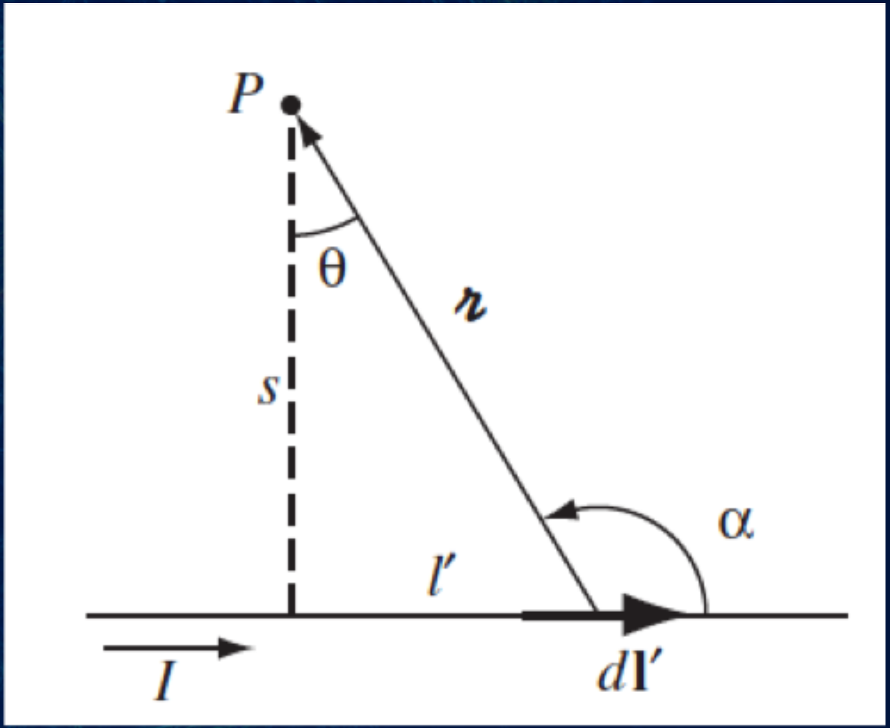
5.2 The Biot-Savart Law

Example: Find the magnetic field a distance s from a long straight wire carrying a steady current I

$$\vec{r} = s\hat{s}$$
$$\vec{r}' = z'\hat{z}$$
$$\vec{\mathscr{r}} = \vec{r} - \vec{r}' = s\hat{s} - z'\hat{z}$$

$$\mathscr{r}^2 = s^2 + z'^2$$

$$\vec{I}\times\hat{\mathscr{r}} = I d\vec{l'}\times\hat{\mathscr{r}}$$
$$= \frac{s\hat{\phi}dz'}{(s^2 + z'^2)^{1/2}}$$



5.2 The Biot-Savart Law

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{I}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}dl' = \frac{\mu_0 I\hat{\phi}}{4\pi}\int_{z_1}^{z_2}\frac{s\,dz'}{(s^2+z'^2)^{3/2}}$$

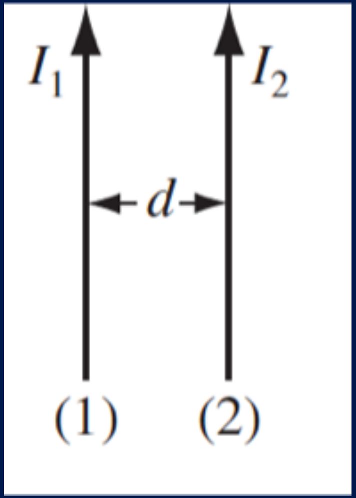
$$\vec{B}(\vec{r}) = \frac{\mu_0 I\hat{\phi}}{4\pi s}\int_{z_1}^{z_2}\frac{dz'/s}{(1+z'^2/s^2)^{3/2}}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0 I\hat{\phi}}{4\pi s}\left(\frac{\frac{z_2}{s}}{\sqrt{1+\frac{z_2^2}{s^2}}}-\frac{\frac{z_1}{s}}{\sqrt{1+\frac{z_1^2}{s^2}}}\right) = \frac{\mu_0 I\hat{\phi}}{4\pi s}\left(\frac{z_2}{\sqrt{s^2+z_2^2}}-\frac{z_1}{\sqrt{s^2+z_1^2}}\right)$$

$$= \frac{\mu_0 I\hat{\phi}}{4\pi s}(\sin\theta_2-\sin\theta_1)$$

5.2 The Biot-Savart Law

$$\vec{B}(\vec{r}) = \frac{\mu_0 I\hat{\phi}}{4\pi s}(\sin\frac{\pi}{2} - \sin(-\frac{\pi}{2})) = \frac{\mu_0 I\hat{\phi}}{2\pi s}$$



Let us find the force between two infinite straight wires

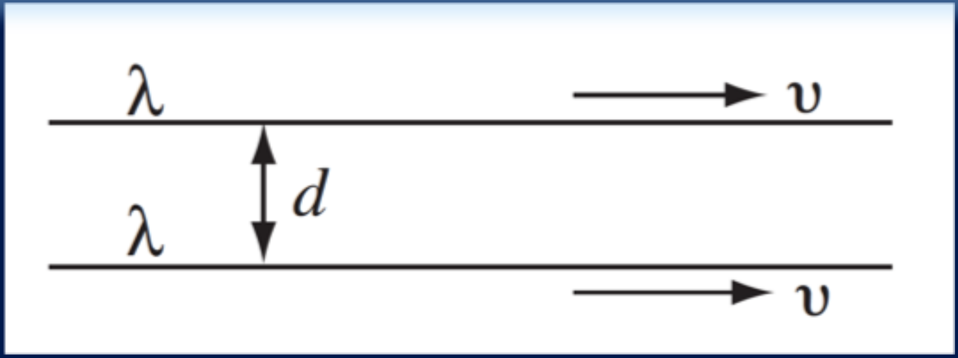
The magnetic field on (2) due to (1) is $\frac{\mu_0}{2\pi d}$ into the page

$$\vec{F}_{mag} = I\int d\vec{l}\times\vec{B} \rightarrow \vec{F}_{21} = I_2\frac{\mu_0 I_1}{2\pi d}\int dl\,, towards\ (1)$$

$$f = \frac{\mu_0 I_1 I_2}{2\pi d}\ (, attractive, force\ per\ unit\ length)$$

5.2 The Biot-Savart Law

Example: Two infinite straight line charges λ, a distance d apart, moving along at a constant speed v. How great would v have to be in order for the magnetic attraction to balance the electrical repulsion?



$$I_1 = I_2 = \lambda v$$

$$f_{mag} = \frac{\mu_0 I_1 I_2}{2\pi d} = \frac{\mu_0 \lambda^2 v^2}{2\pi d}$$

$$f_e = \lambda E = \lambda \frac{1}{2\pi\epsilon_0}\frac{\lambda}{d} = \frac{\lambda^2}{2\pi\epsilon_0 d}$$

$$\frac{\mu_0 \lambda^2 v^2}{2\pi d} = \frac{\lambda^2}{2\pi\epsilon_0 d} \rightarrow v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = c$$

Cannot be achieved! Therefore electric force always dominate!

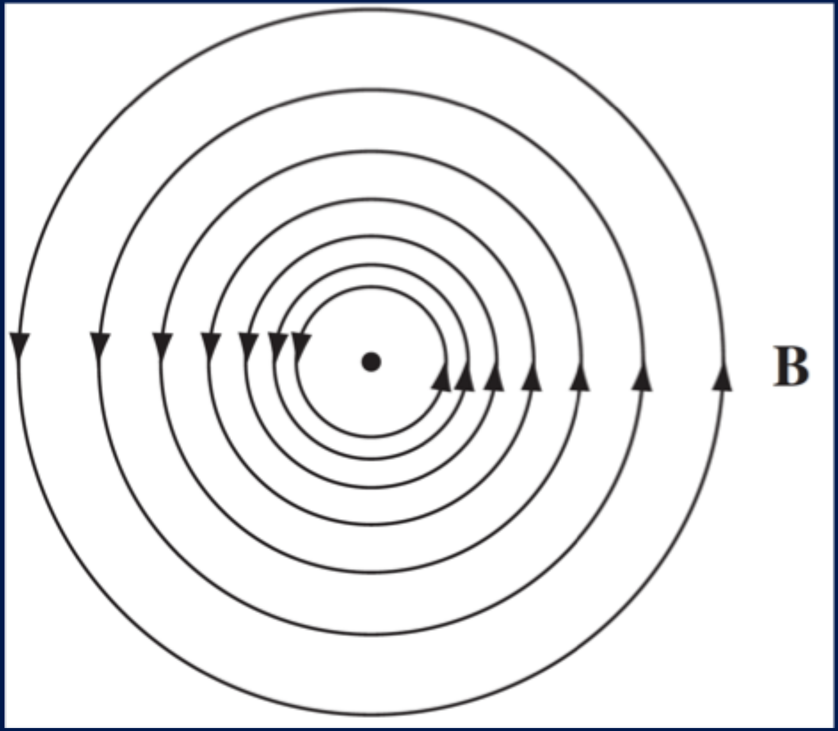
5.3 The Divergence and Curl of $\vec{B}$

Straight line currents

$$d\vec{l} = ds\hat{s} + s\,d\phi\hat{\phi} + dz\hat{z}$$

$$\vec{B} = \frac{\mu_0 I\hat{\phi}}{2\pi s}$$

$$\oint \vec{B}\cdot d\vec{l} = \int_0^{2\pi} \frac{\mu_0 I}{2\pi} d\phi = \mu_0 I$$



5.3 The Divergence and Curl of $\vec{B}$

For a bundle of wires

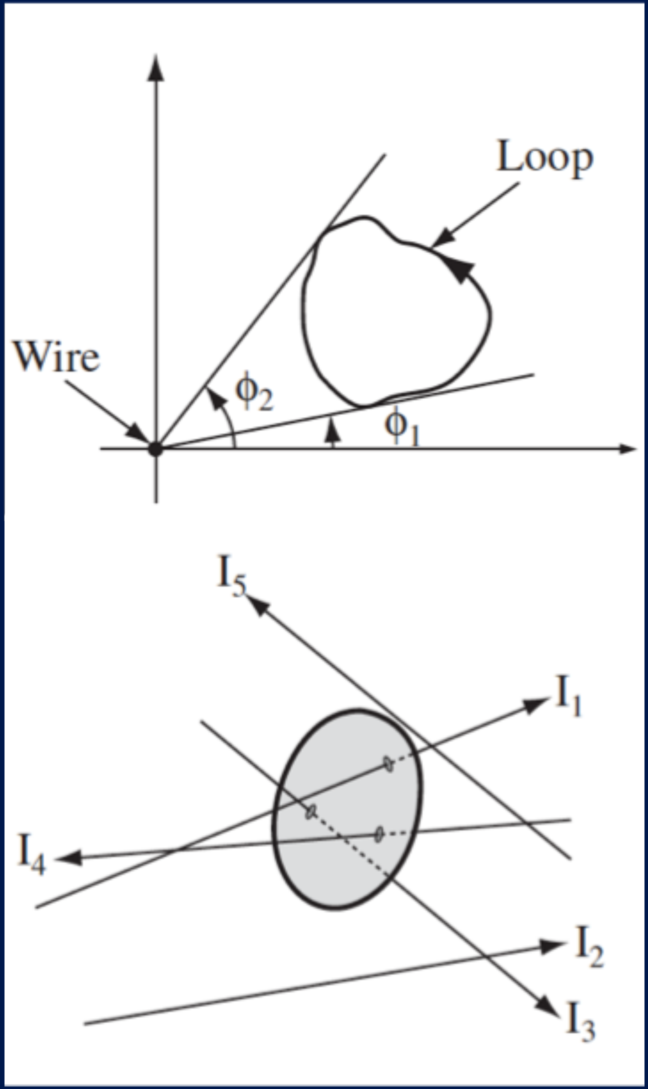
$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$\oint \vec{B} \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} = \mu_0 I_{enc}$$

$$= \mu_0 \int \vec{J} \cdot d\vec{a}$$

$$(\vec{\nabla} \times \vec{B}) = \mu_0 \vec{J}$$

But, this is not general and only valid for infinite straight line currents.



5.3 The Divergence and Curl of $\vec{B}$

The divergence of $\vec{B}$

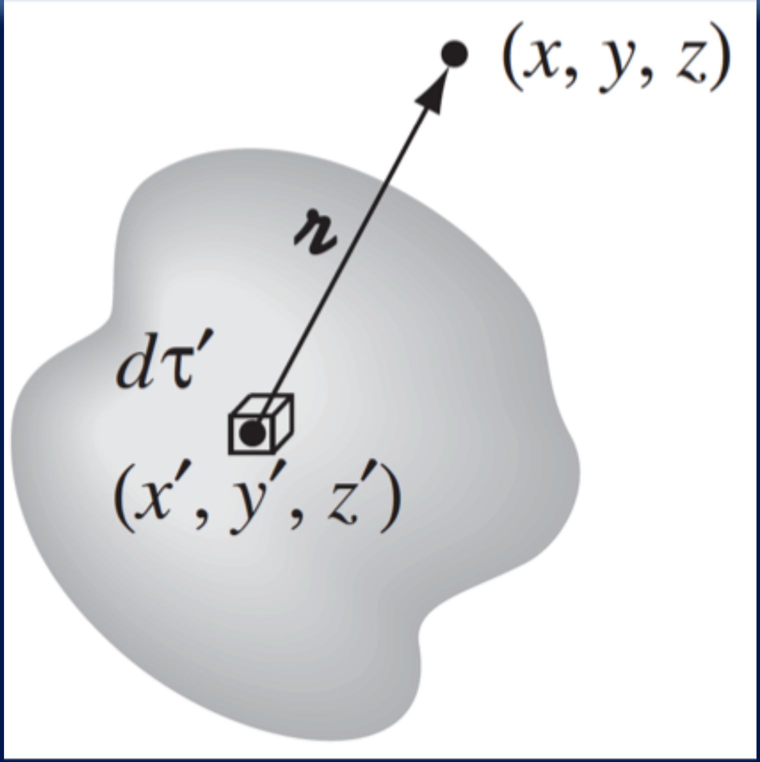
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

$$\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot\frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

$$= \frac{\mu_0}{4\pi}\int \vec{\nabla}\cdot\left[\vec{J}(\vec{r}')\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right]d\tau' =$$

$$= \frac{\mu_0}{4\pi}\int \left[\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot(\vec{\nabla}\times\vec{J}) - \vec{J}\cdot\vec{\nabla}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right]d\tau'$$

$$= 0$$



5.3 The Divergence and Curl of $\vec{B}$

The curl of $\vec{B}$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times\frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau' = \frac{\mu_0}{4\pi}\int \vec{\nabla}\times\left[\vec{J}(\vec{r}')\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right]d\tau'$$

$$= \frac{\mu_0}{4\pi}\int\left[\vec{J}\left(\vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)-(\vec{J}\cdot\vec{\nabla})\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right]d\tau' = \mu_0\vec{J}-\frac{\mu_0}{4\pi}\int\left[(\vec{J}\cdot\vec{\nabla})\frac{\hat{\mathscr{r}}}{\mathscr{r}^2})\right]d\tau'$$

$$But!!, -(\vec{J}\cdot\vec{\nabla})\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = (\vec{J}\cdot\vec{\nabla}')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

5.3 The Divergence and Curl of $\vec{B}$

The divergence and curl of $\vec{B}$

$$\left[\left((\vec{J}\cdot\vec{\nabla}')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]_x = (\vec{J}\cdot\vec{\nabla}')\left(\frac{x-x'}{\mathscr{r}^3}\right) = \vec{\nabla}'\cdot\left(\vec{J}\left(\frac{x-x'}{\mathscr{r}^3}\right)\right) - \left(\frac{x-x'}{\mathscr{r}^3}\right)\vec{\nabla}'\cdot\vec{J}$$

$$= \vec{\nabla}'\cdot\left(\vec{J}\left(\frac{x-x'}{\mathscr{r}^3}\right)\right)$$

$$\int\left[(\vec{J}\cdot\vec{\nabla})\frac{\hat{\mathscr{r}}}{\mathscr{r}^2})\right]d\tau' = \hat{x}\int\vec{\nabla}'\cdot\left(\vec{J}\left(\frac{x-x'}{\mathscr{r}^3}\right)\right)d\tau' + \hat{y}\int\vec{\nabla}'\cdot\left(\vec{J}\left(\frac{y-y'}{\mathscr{r}^3}\right)\right)d\tau' + \hat{z}\int\vec{\nabla}'\cdot\left(\vec{J}\left(\frac{z-z'}{\mathscr{r}^3}\right)\right)d\tau'$$

$$\int\left[(\vec{J}\cdot\vec{\nabla})\frac{\hat{\mathscr{r}}}{\mathscr{r}^2})\right]d\tau' = \hat{x}\int\left(\vec{J}\left(\frac{x-x'}{\mathscr{r}^3}\right)\right)\cdot d\vec{a}' + \hat{y}\int\left(\vec{J}\left(\frac{y-y'}{\mathscr{r}^3}\right)\right)\cdot d\vec{a}' + \hat{z}\int\left(\vec{J}\left(\frac{z-z'}{\mathscr{r}^3}\right)\right)\cdot d\vec{a} = 0$$

5.3 The Divergence and Curl of $\vec{B}$

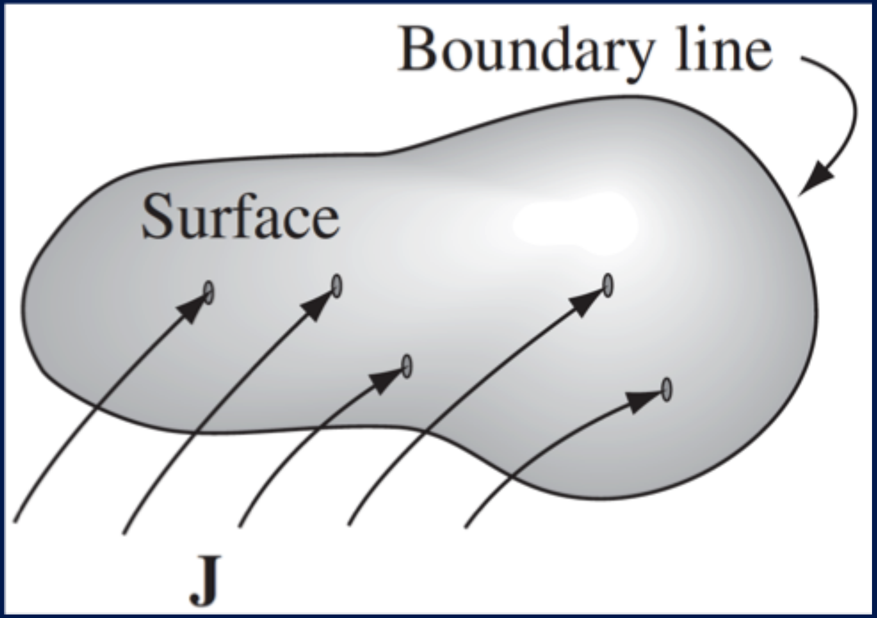
Ampere's Law, The curl of $\vec{B}$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} - \frac{\mu_0}{4\pi}\int\left[\left(\vec{J}\cdot\vec{\nabla}\right)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right]d\tau'$$

$$= \mu_0\vec{J} - 0 = \mu_0\vec{J}$$

$$\int\vec{B}\cdot d\vec{l} = \int\left(\vec{\nabla}\times\vec{B}\right)\cdot d\vec{a}$$

$$= \mu_0\int\vec{J}\cdot d\vec{a} = \mu_0 I_{enc}$$

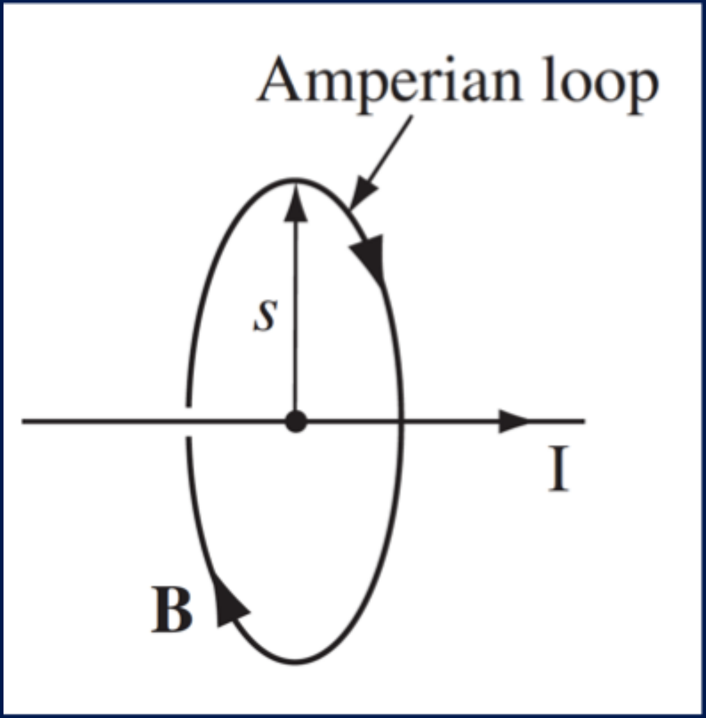


5.3 The Divergence and Curl of $\vec{B}$

Ampere's Law, Examples

Example: Find the magnetic field at distance s from a long straight wire

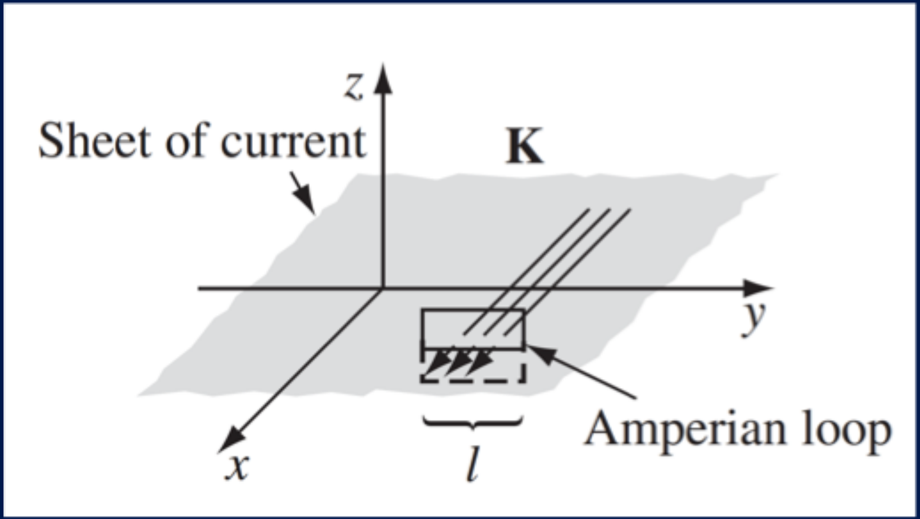
$$\oint \vec{B} \cdot d\vec{l} = 2\pi s B = \mu_0 I_{enc} = \mu_0 I \rightarrow B = \frac{\mu_0 I}{2\pi s}$$



5.3 The Divergence and Curl of $\vec{B}$

Example: Find the magnetic field of an infinite uniform surface current K, flowing over the xy-plane

$$\oint \vec{B}\cdot d\vec{l} = 2lB = \mu_0 I_{enc} = \mu_0 lK \rightarrow B = \frac{\mu_0 K}{2}$$

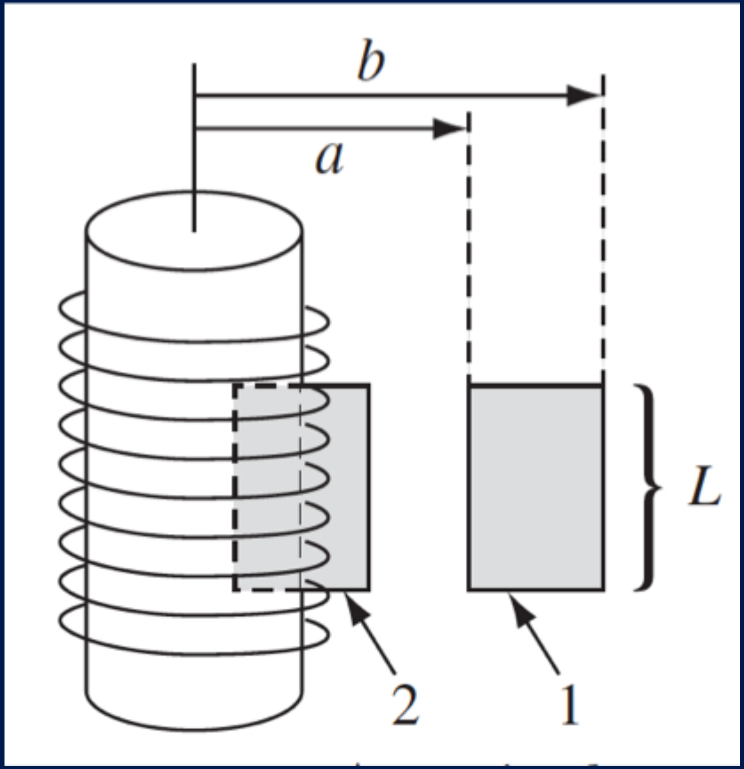


$$\vec{B} = \begin{cases} -\dfrac{\mu_0 K}{2}\hat{y} & z > 0 \\ \dfrac{\mu_0 K}{2}\hat{y} & z < 0 \end{cases}$$

5.3 The Divergence and Curl of $\vec{B}$

Example: Find the magnetic field of a very long solenoid, consisting of n closely wound turns per unit length on a cylinder of radius R and carrying a steady current I.

$$\oint \vec{B} \cdot d\vec{l} = lB = \mu_0 I_{enc} = \mu_0 n l I \rightarrow B = n\mu_0 I$$



$$\vec{B} = \begin{cases} n\mu_0 I\hat{z} & R < 0 \\ 0 & z > 0 \end{cases}$$

5.3 The Divergence and Curl of $\vec{B}$

Maxwell equations for static electromagnetism

Gauss's Law $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$

$\vec{\nabla} \times \vec{E} = 0$

$\vec{E} \to 0$

Electrostatics

$\vec{F} = Q\left(\vec{E} + \vec{v} \times \vec{B}\right)$ Lorentz Force

$\vec{\nabla} \cdot \vec{B} = 0$

Magnetostatics

Ampere's Law $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

$\vec{B} \to 0$

5.4 The Magnetic Vector Potential

The Vector Potential

In electrostatics $\vec{\nabla}\times\vec{E} = 0 \rightarrow \vec{E} = -\vec{\nabla}V$ **Scalar Potential**

In magnetostatics $\vec{\nabla}\cdot\vec{B} = 0 \rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$ **Vector Potential**

But the vector potential is not uniquely determined as

$$\vec{A} = \vec{A}' + \vec{\nabla}\lambda$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times\left(\vec{A}' + \vec{\nabla}\lambda\right) = \vec{\nabla}\times\vec{A}'$$

$$since\ \vec{\nabla}\times\vec{\nabla}\lambda = 0$$

5.4 The Magnetic Vector Potential

But we are interested in using Ampere's law to find the magnetic field

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}$$

* If we choose $\vec{A}$ such that $\vec{\nabla}\cdot\vec{A} = 0$ (Coulomb Gauge) , we get

$$-\nabla^2\vec{A} = \mu_0\vec{J}$$

Which is similar to Poisson equation used to find the electrostatic potential

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{I}}{r}dl' \quad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{\kappa}(\vec{r}')}{r}da' \quad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}')}{r}da'$$

5.4 The Magnetic Vector Potential

* If we choose $\vec{A}_0$ such that $\vec{\nabla} \cdot \vec{A}_0 \neq 0$ (Coulomb Gauge) , we get

$$\vec{A} = \vec{A}_0 + \vec{\nabla}\lambda$$

$$\vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot \vec{A}_0 + \nabla^2\lambda = 0 \rightarrow \nabla^2\lambda = -\vec{\nabla} \cdot \vec{A}_0$$

$$\lambda(\vec{r}) = \int \frac{-\vec{\nabla} \cdot \vec{A}_0}{\mathscr{r}} dl'$$

Therefore, we can always find λ to make $\vec{\nabla} \cdot \vec{A} = 0$

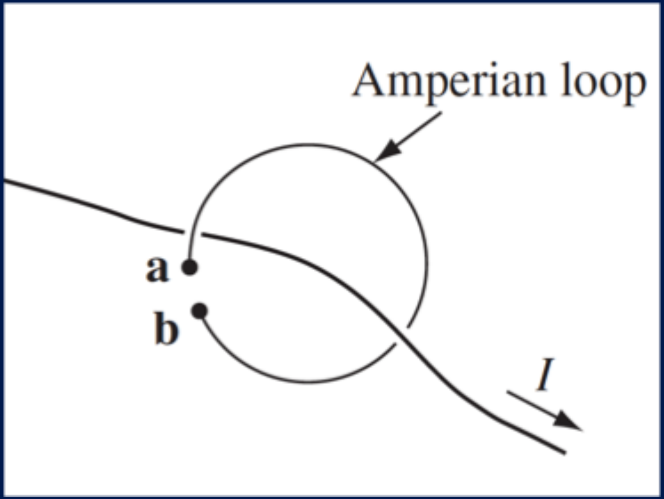
Introducing the vector potential is in general less than the electrostatic potential as we still need to solve for three components. However, it is important for theoretical reasons.

5.4 The Magnetic Vector Potential

The magnetic field cannot be written as $\vec{B} = -\vec{\nabla}U$

$$\int_a^b B \cdot d\vec{l} = \mu_0 I = -\int_a^b \vec{\nabla}U \cdot d\vec{l}$$

$$= U(b) - U(a)$$



Therefore,

$$a \to b, U(b) \neq U(a)$$

The defined potential is not a single valued function