

Electromagnetic Theory I

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Chapter 5: Magnetostatics

- ✿ The Lorentz Force Law
- ✿ The Biot-Savart Law
- ✿ The Divergence and Curl of \vec{B}
- ✿ Magnetic Vector Potential (\vec{A})

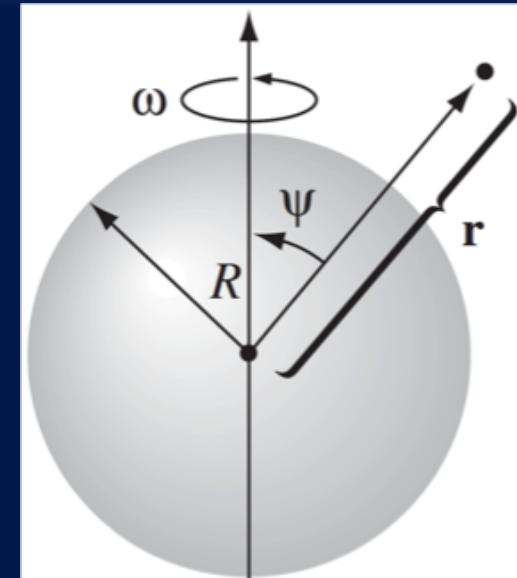
5.4 The Magnetic Vector Potential

Example: A spherical shell, of radius R , carrying a uniform surface charge σ , is set spinning at angular velocity ω . Find the vector potential it produces at point r

$$\vec{k}(\vec{r}') = \sigma \vec{v} = \sigma \vec{\omega} \times \vec{r}' = \sigma \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ x' & y' & z' \end{vmatrix} = -y' \omega \sigma \hat{x} + x' \omega \sigma \hat{y}$$

$$\begin{aligned} \vec{r}' &= R \hat{r}' \\ &= R \sin \theta' \cos \phi' \hat{x} + R \sin \theta' \sin \phi' \hat{y} + R \cos \theta' \hat{z} \end{aligned}$$

$$r = \sqrt{r^2 + R^2 - 2rR \cos \gamma}$$



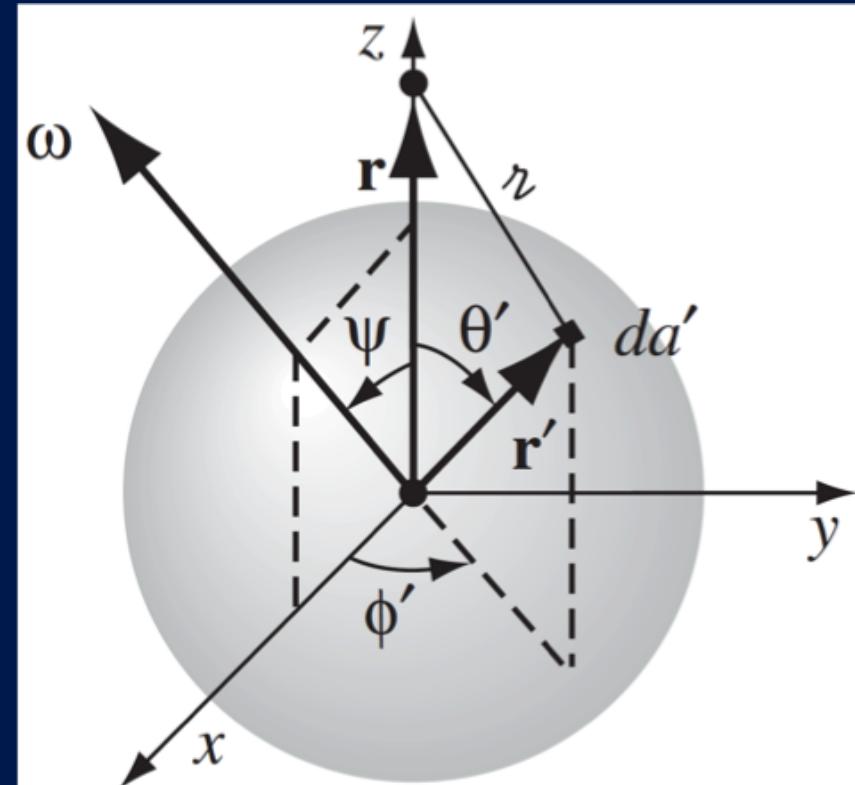
$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{k}(\vec{r}')}{r} da' = \frac{\mu_0 \sigma \omega R^3}{4\pi} \int_0^{2\pi} \int_0^\pi \frac{[-\sin \theta' \sin \phi' \hat{x} + \sin \theta' \cos \phi' \hat{y}]}{\sqrt{r^2 + R^2 - 2rR \cos \gamma}} \sin \theta' d\theta' d\phi'$$

$$\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')$$

5.4 The Magnetic Vector Potential

$$\vec{k}(\vec{r}') = \sigma \vec{\omega} \times \vec{r}' = \sigma R \omega \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sin \psi & 0 & \cos \psi \\ \sin \theta' \cos \phi' & \sin \theta' \sin \phi' & \cos \theta' \end{vmatrix}$$
$$= \sigma R \omega [-\cos \psi \sin \theta' \sin \phi' \hat{x} + (\sin \theta' \cos \phi' \cos \psi - \cos \theta' \sin \psi) \hat{y} + \sin \theta' \sin \phi' \sin \psi \hat{z}]$$

$$r = \sqrt{r^2 + R^2 - 2rR \cos \theta'}$$



5.4 The Magnetic Vector Potential

$$\begin{aligned}
 \vec{A}(\vec{r}) &= \frac{\mu_0 \sigma}{4\pi} \int \frac{\vec{k}(\vec{r}')}{r'} da' \\
 &= \frac{\mu_0 \sigma \omega R^3}{4\pi} \int_0^{2\pi} \int_0^\pi \frac{[-\cos \psi \sin \theta' \sin \phi' \hat{x} + (\sin \theta' \cos \phi' \cos \psi - \cos \theta' \sin \psi) \hat{y} + \sin \theta' \sin \phi' \sin \psi \hat{z}]}{\sqrt{r^2 + R^2 - 2rR \cos \theta'}} \sin \theta' d\theta' d\phi' \\
 &= \frac{\mu_0 \sigma \omega R^3}{4\pi} \int_0^{2\pi} \int_0^\pi \frac{[(-\cos \theta' \sin \psi) \hat{y}]}{\sqrt{r^2 + R^2 - 2rR \cos \theta'}} \sin \theta' d\theta' d\phi' = -\frac{\mu_0 \sigma \omega R^3 \sin \psi}{2} \hat{y} \int_0^\pi \frac{\cos \theta' \sin \theta' d\theta'}{\sqrt{r^2 + R^2 - 2rR \cos \theta'}}
 \end{aligned}$$

$$\begin{aligned}
 \vec{A}(\vec{r}) &= \frac{\mu_0 \sigma \omega R^3 \sin \psi}{2} \hat{y} \int_1^{-1} \frac{udu}{\sqrt{r^2 + R^2 - 2rRu}} = \frac{\mu_0 \sigma \omega R^3 \sin \psi}{2} \hat{y} \left(\frac{(\sqrt{r^2 + R^2 + 2rR}) \sqrt{r^2 + R^2 - 2rRu}}{3R^2 r^2} \right) \Big|_{-1}^1 \\
 &= -\frac{\mu_0 \sigma \omega R^3 \sin \psi}{2} \hat{y} \begin{cases} \frac{2R}{3r^2} & r > R \\ \frac{2r}{3R^2} & r < R \end{cases} = -\hat{y} \begin{cases} \frac{\mu_0 \sigma \omega \sin \psi R^4}{3r^2} & r > R \\ \frac{\mu_0 \sigma \omega R \sin \psi r}{3} & r < R \end{cases} = \begin{cases} \frac{\mu_0 \sigma R^4}{3r^3} \vec{\omega} \times \vec{r} & r > R \\ \frac{\mu_0 \sigma R}{3} \vec{\omega} \times \vec{r} & r < R \end{cases}
 \end{aligned}$$

$$\vec{\omega} \times \vec{r} = -r\omega \sin \psi \hat{y}$$

5.4 The Magnetic Vector Potential

$$\vec{A}(\vec{r}) = \begin{cases} \frac{\mu_0 \sigma R^4}{3r^3} \vec{\omega} \times \vec{r} & r > R \\ \frac{\mu_0 \sigma R}{3} \vec{\omega} \times \vec{r} & r < R \end{cases} = \begin{cases} \frac{\mu_0 \sigma R^4 \omega}{3r^2} \sin \theta \hat{\phi} & r > R \\ \frac{\mu_0 \sigma R \omega}{3} r \sin \theta \hat{\phi} & r < R \end{cases}$$

You can now calculate the magnetic field, we will just do it inside (Uniform!!)

$$\begin{aligned}\vec{B} &= \vec{\nabla} \times \vec{A} = \frac{2}{3} \mu_0 R \omega \sigma (\cos \theta \hat{r} - \sin \theta \hat{\theta}) \\ &= \frac{2}{3} \mu_0 R \omega \sigma \hat{z} = \frac{2}{3} \mu_0 R \sigma \vec{\omega}\end{aligned}$$

5.4 The Magnetic Vector Potential

Example: Find the vector potential of an infinite solenoid with n turns per unit length, radius R , and current I

Since the current is not localized and extends to infinity, we cannot use

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I}}{r} dl'$$

$$\oint \vec{A} \cdot d\vec{l} = 2\pi s A = \int \vec{\nabla} \times \vec{A} \cdot d\vec{a}$$

$$= \int \vec{B} \cdot d\vec{a} = \text{flux through loop} = \begin{cases} \pi s^2 \mu_0 n I & \text{inside} \\ \pi R^2 \mu_0 n I & \text{outside} \end{cases}$$

5.4 The Magnetic Vector Potential

$$\vec{A} = \begin{cases} \frac{s\mu_0 n I}{2} \hat{\phi} & \text{inside} \\ \frac{R^2 \mu_0 n I}{2s} \hat{\phi} & \text{outside} \end{cases}$$

5.4 The Magnetic Vector Potential

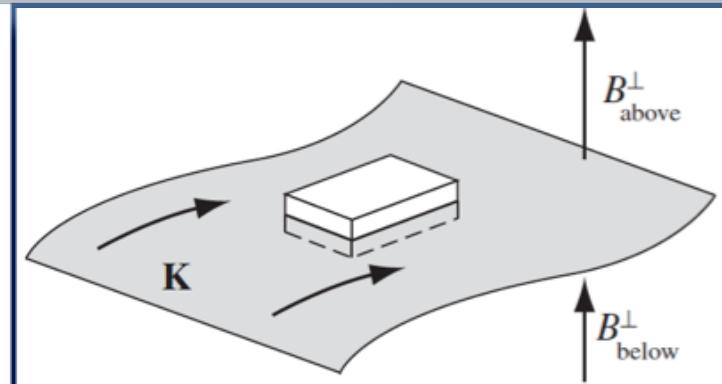
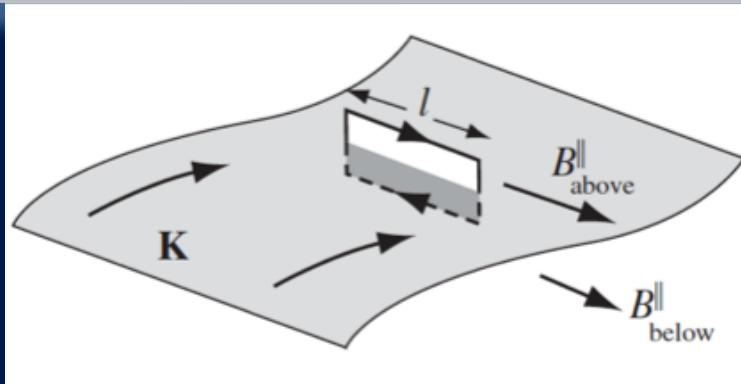
Summary of Magnetostatics

Just from two experimental observations: Biot-Savart Law and Superposition

The diagram illustrates the relationships between magnetic fields and vector potential. It features three circular regions: a grey region at the top with current density \vec{J} , a black region at the bottom left with vector potential \vec{A} , and an orange region at the bottom right with magnetic field \vec{B} . A central blue triangle connects these regions. The left side of the triangle shows the relationship $\vec{A}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \vec{J}(\vec{r}') \frac{d\vec{r}'}{r}$. The right side shows the Biot-Savart Law $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}') \times \frac{\hat{r}}{r^2} d\vec{r}'$. The bottom side of the triangle shows the vector identity $\nabla \times \vec{B} = \mu_0 \vec{J}$ and the divergence condition $\nabla \cdot \vec{B} = 0$. A green arrow points from the bottom side to the right side of the triangle. A blue arrow points from the left side to the right side. A horizontal blue arrow at the bottom indicates the relationship $\vec{B} = \nabla \times \vec{A}; \nabla \cdot \vec{A} = 0$. Below the triangle, a line integral is shown: $\oint \vec{A} \cdot d\vec{l} = \int \vec{B} \cdot d\vec{a}$.

5.4 The Magnetic Vector Potential

Magnetostatics Boundary Conditions



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I = \mu_0 \kappa l$$

$$B_{\text{above}}^{\parallel} - B_{\text{below}}^{\parallel} = \mu_0 \kappa$$

The **tangential component of \vec{B}** is discontinuous

$$\nabla \cdot \vec{B} = 0$$

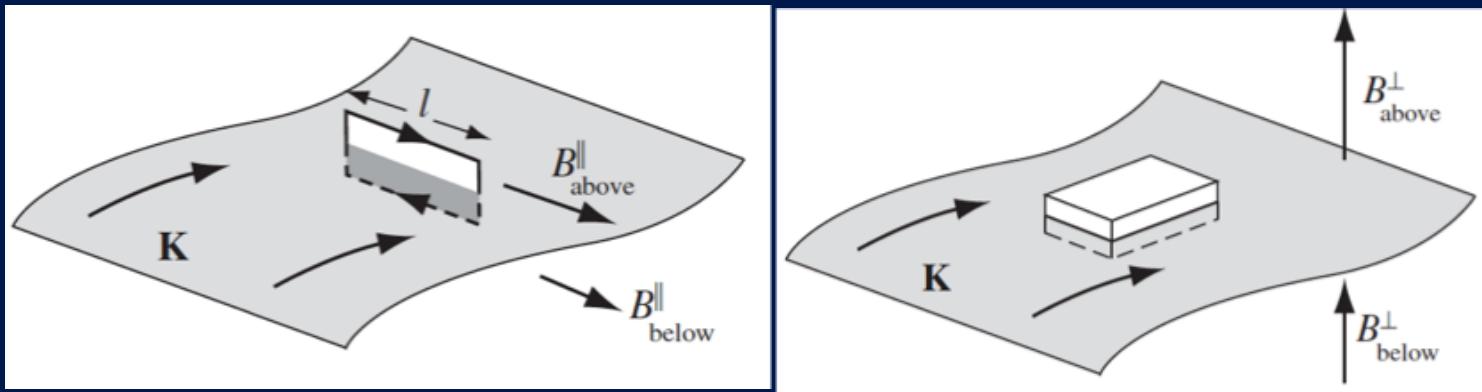
$$\oint \vec{B} \cdot d\vec{a} = 0$$

$$B_{\text{above}}^{\perp} - B_{\text{below}}^{\perp} = 0$$

The **normal component of \vec{B}** is continuous

$$\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \mu_0 \vec{\kappa} \times \hat{n}$$

5.4 The Magnetic Vector Potential



$$\oint \vec{A} \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \int \vec{B} \cdot d\vec{a} = 0$$

$$A_{\text{above}}^{\parallel} - A_{\text{below}}^{\parallel} = 0$$

$$\vec{\nabla} \cdot \vec{A} = 0$$

$$\oint \vec{A} \cdot d\vec{a} = 0$$

$$A_{\text{above}}^{\perp} - A_{\text{below}}^{\perp} = 0$$

$$\vec{A}_{\text{above}} = \vec{A}_{\text{below}}$$

5.4 The Magnetic Vector Potential

$$\vec{B}_{above} - \vec{B}_{below} = \mu_0 \vec{\kappa} \times \hat{n} \quad \vec{B} = \vec{\nabla} \times \vec{A}$$



$$\frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{\kappa}$$