Electromagnetic Theory I

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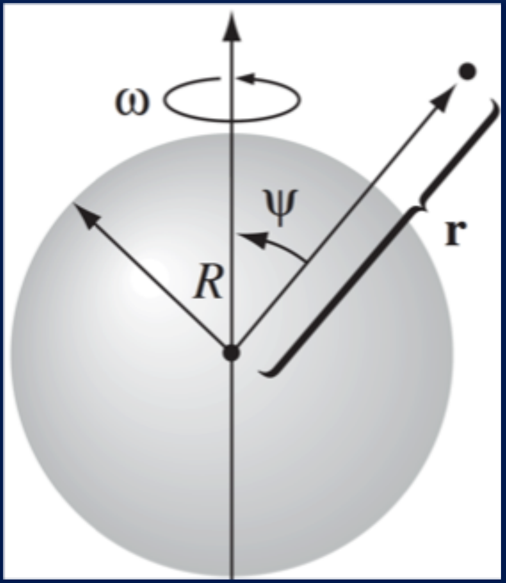
Chapter 5: Magnetostatics

- The Lorentz Force Law
- The Biot-Savart Law
- The Divergence and Curl of $\vec{B}$
- Magnetic Vector Potential ($\vec{A}$)

5.4 The Magnetic Vector Potential

Example: A spherical shell, of radius R, carrying a uniform surface charge σ, is set spinning at angular velocity ω. Find the vector potential it produces at point r

$$\vec{\kappa}(\vec{r}') = \sigma\vec{v} = \sigma\vec{\omega}\times\vec{r}' = \sigma\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ 0 & 0 & \omega \\ x' & y' & z'\end{vmatrix} = -y'\omega\sigma\hat{x} + x'\omega\sigma\hat{y}$$



$$\vec{r}' = R\hat{r}'$$
$$= R\sin\theta'\cos\phi'\ \hat{x} + R\sin\theta'\sin\phi'\ \hat{y} + R\cos\theta'\ \hat{z}$$

$$\mathscr{r} = \sqrt{r^2 + R^2 - 2rR\cos\gamma}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{\kappa}(\vec{r}')}{\mathscr{r}}da' = \frac{\mu_0\sigma\omega R^3}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{[-\sin\theta'\sin\phi'\,\hat{x} + \sin\theta'\cos\phi'\,\hat{y}]}{\sqrt{r^2+R^2-2rR\cos\gamma}}\sin\theta'\,d\theta' d\phi'$$

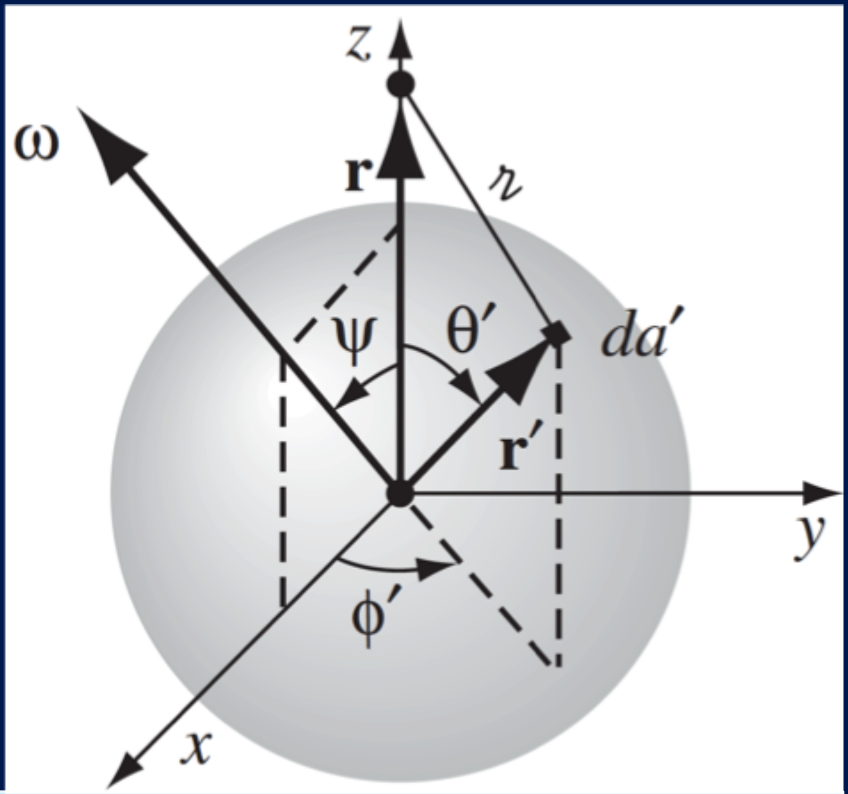
$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

5.4 The Magnetic Vector Potential

$$\vec{\kappa}(\vec{r}') = \sigma\vec{\omega}\times\vec{r}' = \sigma R\omega\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sin\psi & 0 & \cos\psi \\ \sin\theta'\cos\phi' & \sin\theta'\sin\phi' & \cos\theta' \end{vmatrix}$$

$$= \sigma R\omega[-\cos\psi\sin\theta'\sin\phi'\,\hat{x} + (\sin\theta'\cos\phi'\cos\psi - \cos\theta'\sin\psi)\hat{y} + \sin\theta'\sin\phi'\ \sin\psi\ \hat{z}]$$

$$\mathscr{r} = \sqrt{r^2 + R^2 - 2rR\cos\theta'}$$



5.4 The Magnetic Vector Potential

$$\vec{A}(\vec{r}) = \frac{\mu_0\sigma}{4\pi}\int \frac{\vec{\kappa}(\vec{r}')}{\mathscr{r}}da'$$

$$= \frac{\mu_0\sigma\omega R^3}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{[-\cos\psi\sin\theta'\sin\phi'\,\hat{x} + (\sin\theta'\cos\phi'\cos\psi - \cos\theta'\sin\psi)\hat{y} + \sin\theta'\sin\phi'\sin\psi\,\hat{z}]}{\sqrt{r^2+R^2-2rR\cos\theta'}}\sin\theta'\,d\theta' d\phi'$$

$$= \frac{\mu_0\sigma\omega R^3}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{[(-\cos\theta'\sin\psi)\hat{y}]}{\sqrt{r^2+R^2-2rR\cos\theta'}}\sin\theta'\,d\theta' d\phi' = -\frac{\mu_0\sigma\omega R^3\sin\psi}{2}\hat{y}\int_0^{\pi}\frac{\cos\theta'\sin\theta'\,d\theta'}{\sqrt{r^2+R^2-2rR\cos\theta'}}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0\sigma\omega R^3\sin\psi}{2}\hat{y}\int_1^{-1}\frac{u\,du}{\sqrt{r^2+R^2-2rRu}} = \frac{\mu_0\sigma\omega R^3\sin\psi}{2}\hat{y}\left(\frac{\left(\sqrt{r^2+R^2+2rR}\right)\sqrt{r^2+R^2-2rRu}}{3R^2r^2}\right)\Bigg|_{-1}^{1}$$

$$= -\frac{\mu_0\sigma\omega R^3\sin\psi}{2}\hat{y}\begin{cases}\dfrac{2R}{3r^2} & r>R\\ \dfrac{2r}{3R^2} & r<R\end{cases} = -\hat{y}\begin{cases}\dfrac{\mu_0\sigma\omega\sin\psi\,R^4}{3r^2} & r>R\\ \dfrac{\mu_0\sigma\omega R\sin\psi\,r}{3} & r<R\end{cases} = \begin{cases}\dfrac{\mu_0\sigma R^4}{3r^3}\vec{\omega}\times\vec{r} & r>R\\ \dfrac{\mu_0\sigma R}{3}\vec{\omega}\times\vec{r} & r<R\end{cases}$$

$$\vec{\omega}\times\vec{r} = -r\omega\sin\psi\,\hat{y}$$

5.4 The Magnetic Vector Potential

$$\vec{A}(\vec{r}) = \begin{cases} \dfrac{\mu_0 \sigma R^4}{3r^3} \vec{\omega}\times\vec{r} & r > R \\ \dfrac{\mu_0 \sigma R}{3} \vec{\omega}\times\vec{r} & r < R \end{cases} = \begin{cases} \dfrac{\mu_0 \sigma R^4 \omega}{3r^2} \sin\theta\, \hat{\phi} & r > R \\ \dfrac{\mu_0 \sigma R \omega}{3} r \sin\theta\, \hat{\phi} & r < R \end{cases}$$

You can now calculate the magnetic field, we will just do it inside (Uniform!!)

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{2}{3}\mu_0 R\omega\sigma\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)$$

$$= \frac{2}{3}\mu_0 R\omega\sigma\hat{z} = \frac{2}{3}\mu_0 R\sigma\vec{\omega}$$

5.4 The Magnetic Vector Potential

Example: Find the vector potential of an infinite solenoid with n turns per unit length, radius R, and current I

Since the current is not localized and extends to infinity, we cannot use

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}}{r}\, dl'$$

$$\oint \vec{A}\cdot d\vec{l} = 2\pi s A = \int \vec{\nabla}\times\vec{A}\cdot d\vec{a}$$

$$= \int \vec{B}\cdot d\vec{a} = flux\ through\ \ loop = \begin{cases} \pi s^2\mu_0 nI & inside \\ \pi R^2\mu_0 nI & outside \end{cases}$$

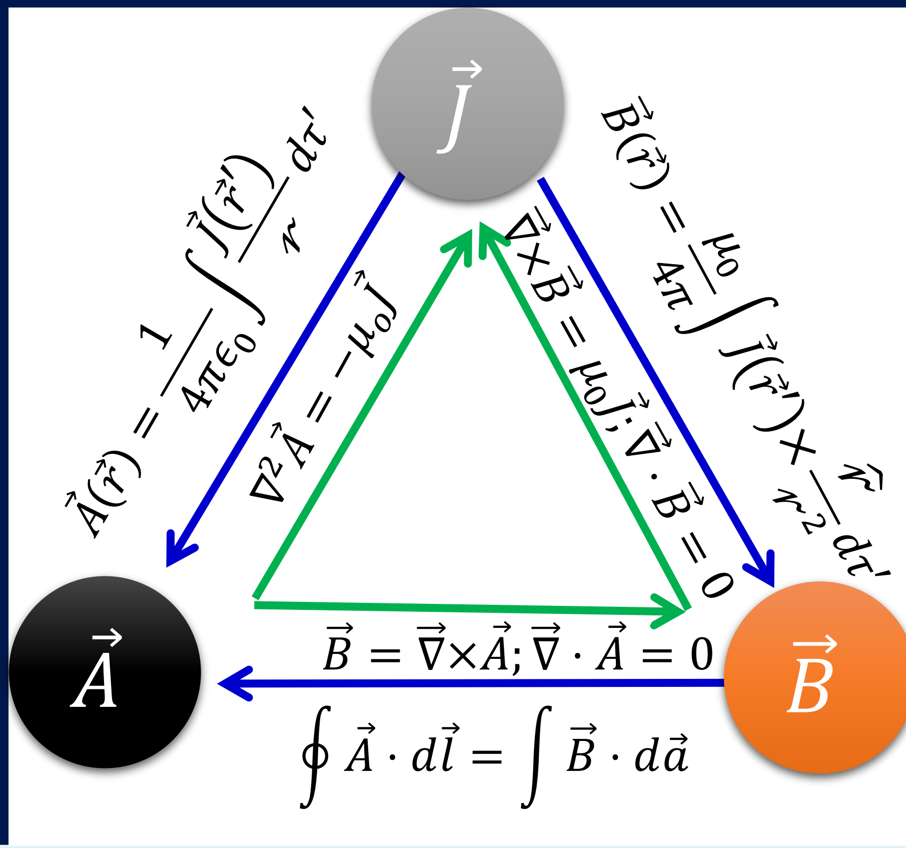
5.4 The Magnetic Vector Potential

$$\vec{A} = \begin{cases} \dfrac{s\mu_0 nI}{2}\hat{\phi} & inside \\ \dfrac{R^2\mu_0 nI}{2s}\hat{\phi} & outside \end{cases}$$

5.4 The Magnetic Vector Potential

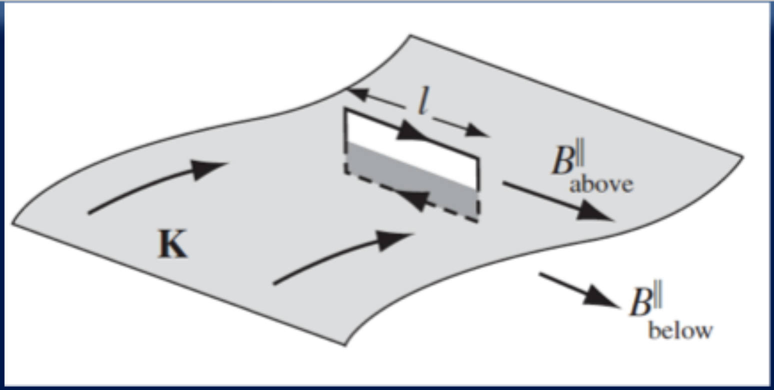
Summary of Magnetostatics

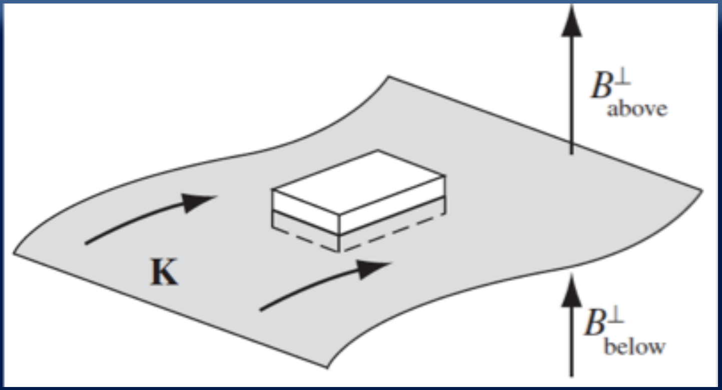
Just from two experimental observations: Biot-Savart Law and Superposition



5.4 The Magnetic Vector Potential

Magnetostatics Boundary Conditions





$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I = \mu_0 \kappa l$$

$$B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 \kappa$$

The *tangential component of* $\vec{B}$ is discontinuous

$$\vec{\nabla} \cdot \vec{B} = 0$$

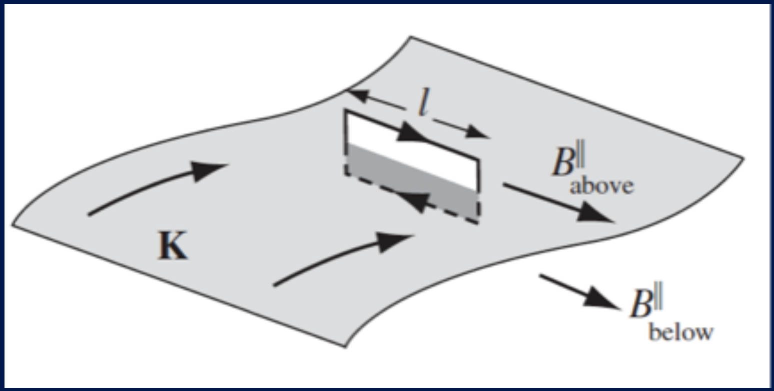
$$\oint \vec{B} \cdot d\vec{a} = 0$$

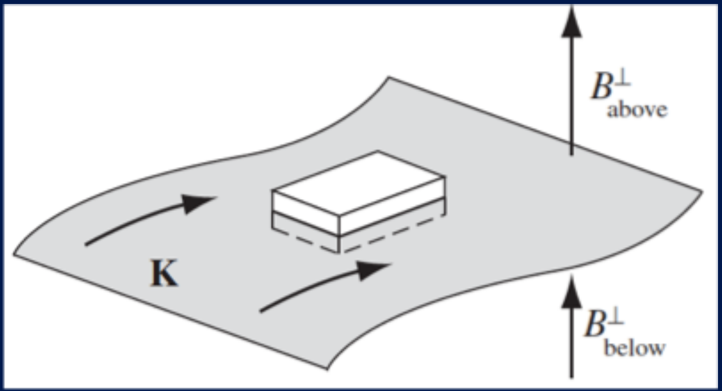
$$B^{\perp}_{above} - B^{\perp}_{below} = 0$$

The *normal component of* $\vec{B}$ is continuous

$$\vec{B}_{above} - \vec{B}_{below} = \mu_0 \vec{\kappa} \times \hat{n}$$

5.4 The Magnetic Vector Potential





$$\oint \vec{A}\cdot d\vec{l} = \int (\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \int \vec{B}\cdot d\vec{a} = 0$$

$$A^{\parallel}_{above} - A^{\parallel}_{below} = 0$$

$$\vec{\nabla}\cdot\vec{A} = 0$$

$$\oint \vec{A}\cdot d\vec{a} = 0$$

$$A^{\perp}_{above} - A^{\perp}_{below} = 0$$

$$\vec{A}_{above} = \vec{A}_{below}$$

5.4 The Magnetic Vector Potential

$$\vec{B}_{above} - \vec{B}_{below} = \mu_0 \vec{\kappa} \times \hat{n}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{\kappa}$$