Electromagnetic Theory I

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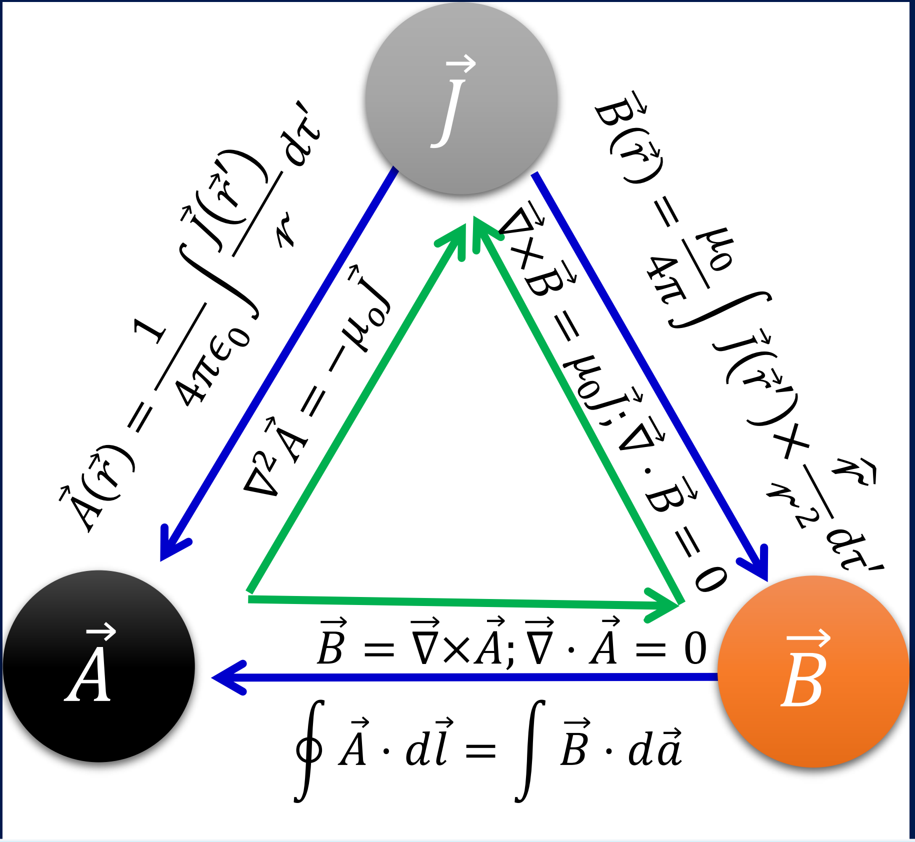
Chapter 5: Magnetostatics

- The Lorentz Force Law
- The Biot-Savart Law
- The Divergence and Curl of $\vec{B}$
- Magnetic Vector Potential ($\vec{A}$)

5.4 The Magnetic Vector Potential

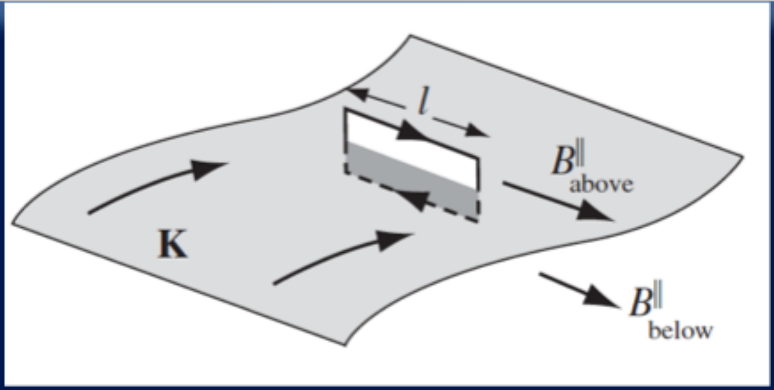
Summary of Magnetostatics

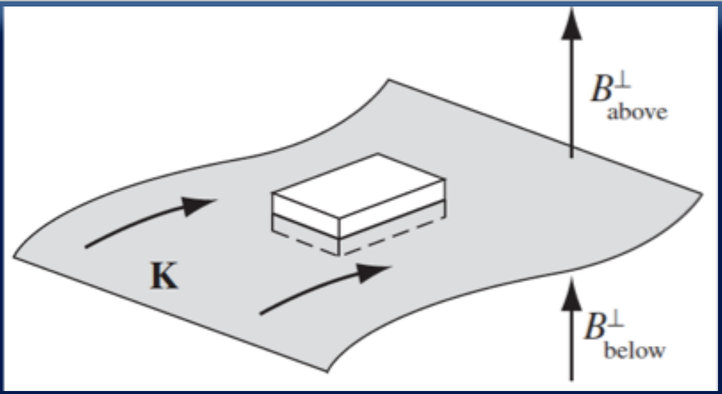
Just from two experimental observations: Biot-Savart Law and Superposition



5.4 The Magnetic Vector Potential

Magnetostatics Boundary Conditions





$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I = \mu_0 \kappa l$$

$$B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 \kappa$$

The *tangential component of* $\vec{B}$ is discontinuous

$$\vec{\nabla} \cdot \vec{B} = 0$$

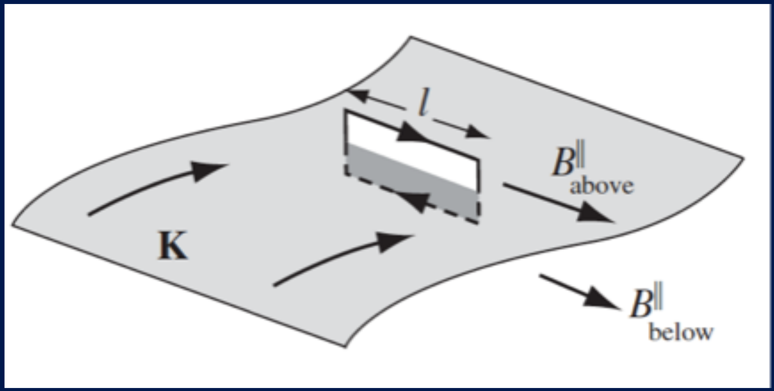
$$\oint \vec{B} \cdot d\vec{a} = 0$$

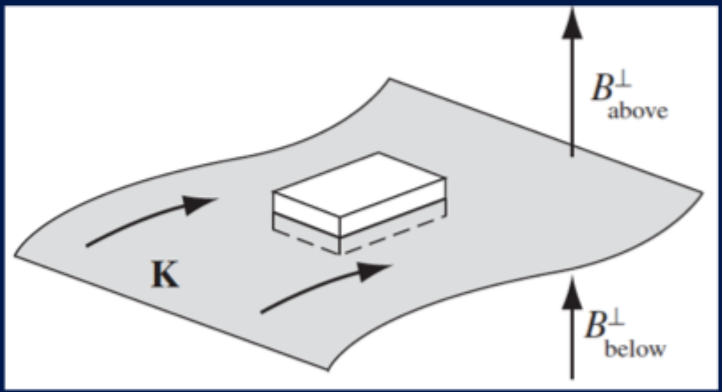
$$B^{\perp}_{above} - B^{\perp}_{below} = 0$$

The *normal component of* $\vec{B}$ is continuous

$$\vec{B}_{above} - \vec{B}_{below} = \mu_0 \vec{\kappa} \times \hat{n}$$

5.4 The Magnetic Vector Potential





$$\oint \vec{A}\cdot d\vec{l} = \int (\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \int \vec{B}\cdot d\vec{a} = 0$$

$$A^{\parallel}_{above} - A^{\parallel}_{below} = 0$$

$$\vec{\nabla}\cdot\vec{A} = 0$$

$$\oint \vec{A}\cdot d\vec{a} = 0$$

$$A^{\perp}_{above} - A^{\perp}_{below} = 0$$

$$\vec{A}_{above} = \vec{A}_{below}$$

5.4 The Magnetic Vector Potential

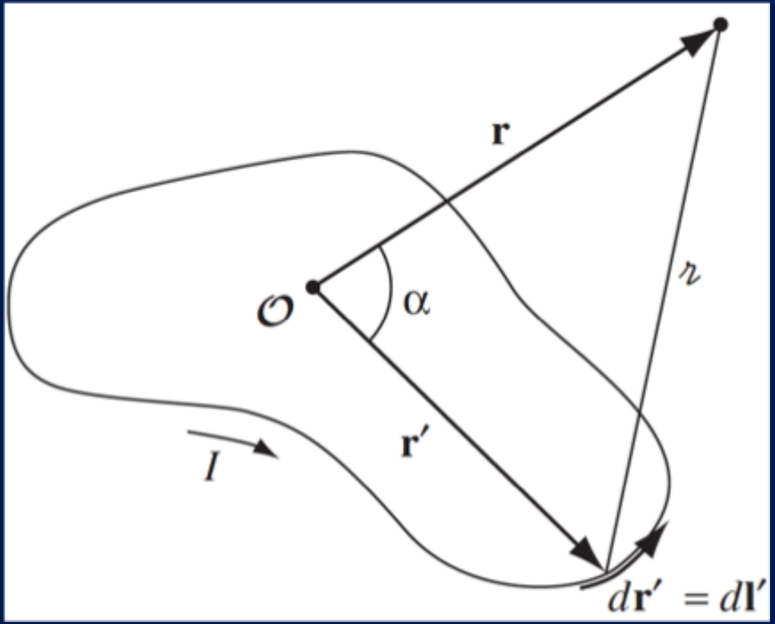
$$\vec{B}_{above} - \vec{B}_{below} = \mu_0 \vec{\kappa} \times \hat{n} \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{\kappa}$$

5.4 The Magnetic Vector Potential

Multipole Expansion of Vector Potential

To approximate formula for the vector potential of a localized current distribution, valid at distant points, a multipole expansion is required



$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\oint \frac{\vec{I}}{r} dl'$$

$$\frac{1}{r} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\gamma)$$

5.4 The Magnetic Vector Potential

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\oint \frac{\vec{I}}{\mathscr{r}} dl' = \frac{\mu_0 I}{4\pi}\oint \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\gamma)\, d\vec{l}'$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\vec{l}' + \frac{1}{r^2}\oint r' P_1(\cos\gamma)d\vec{l}' + \frac{1}{r^3}\oint r'^2 P_2(\cos\gamma)d\vec{l}' + \cdots\right]$$

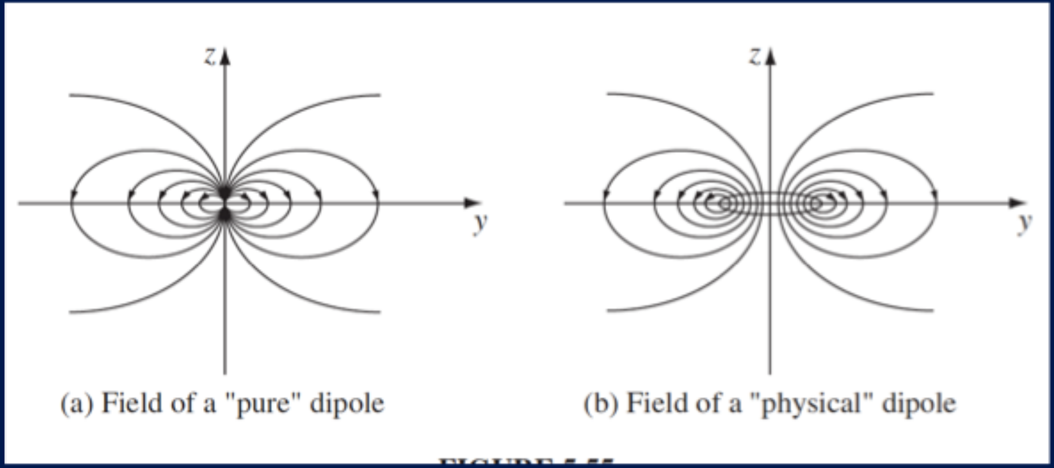
(Magnetic Monopole Term) is Always zero!

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\left[\frac{1}{r^2}\oint r' P_1(\cos\gamma)d\vec{l}' + \frac{1}{r^3}\oint r'^2 P_2(\cos\gamma)d\vec{l}' + \cdots\right]$$

dipole *quadrupole*

5.4 The Magnetic Vector Potential

In general, the dipole term is dominant



(a) Field of a "pure" dipole (b) Field of a "physical" dipole

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r' P_1(\cos\gamma) d\vec{l}' = \frac{\mu_0 I}{4\pi r^2}\oint r'\cos\gamma\, d\vec{l}' = \frac{\mu_0 I}{4\pi r^2}\oint (\vec{r}'\cdot\hat{r}) d\vec{l}'$$

$$= -\frac{\mu_0 I}{4\pi r^2}\hat{r}\times\int d\vec{a} = -\frac{\mu_0}{4\pi r^2}\hat{r}\times I\int d\vec{a} = \frac{\mu_0}{4\pi r^2}\vec{m}\times\hat{r} = \frac{\mu_0(\vec{m}\times\vec{r})}{4\pi r^3}$$

$$= \frac{\mu_0 m}{4\pi r^2}\sin\theta\,\hat{\phi}$$ **If $\vec{m}$ is along z**

5.4 The Magnetic Vector Potential

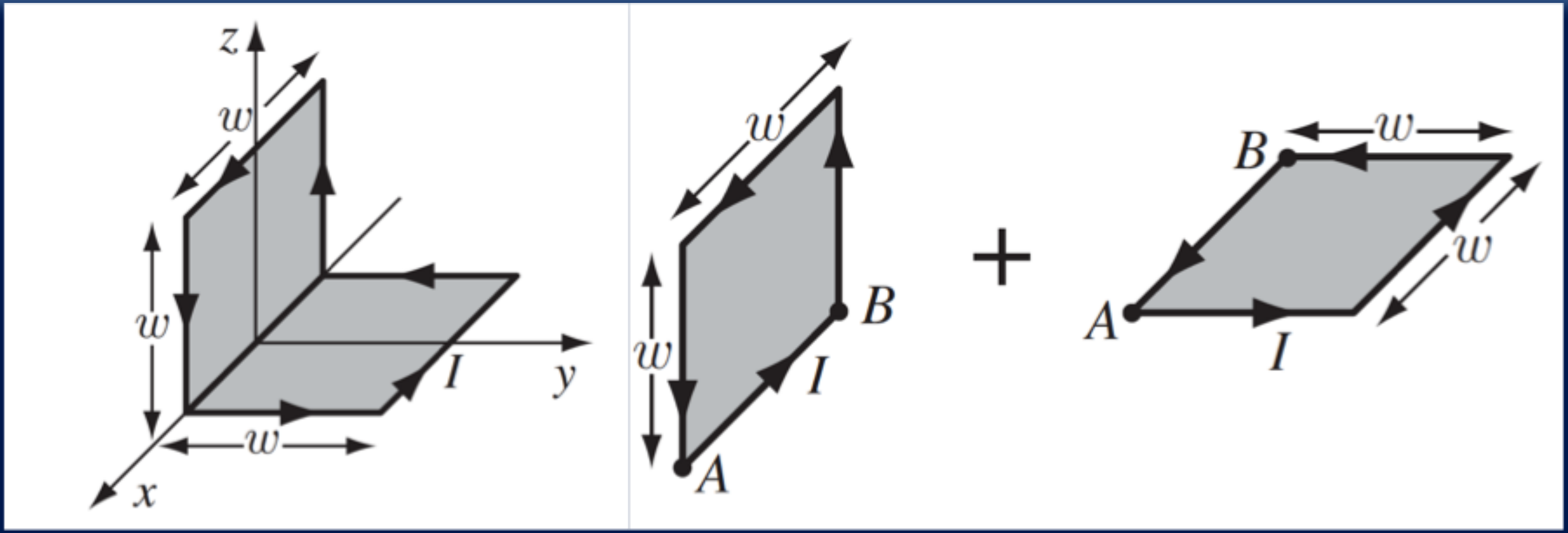
$$\vec{m} = I\int d\vec{a} = I\vec{a} = \frac{1}{2}\int \vec{r}\times\vec{J}\,d\tau' \text{ is the magnetic dipole moment}$$

Note that magnetic dipole moment is independent of the choice of origin.

$$\vec{B}_{dip}(\vec{r}) = \vec{\nabla}\times\vec{A}_{dip} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

5.4 The Magnetic Vector Potential

Example: Find the magnetic dipole moment of the "bookend-shaped" loop shown in Figure. All sides have length w, and it carries a current I .



$$\vec{m} = \vec{m}_1 + \vec{m}_2 = Iw^2\hat{y} + Iw^2\hat{z}$$

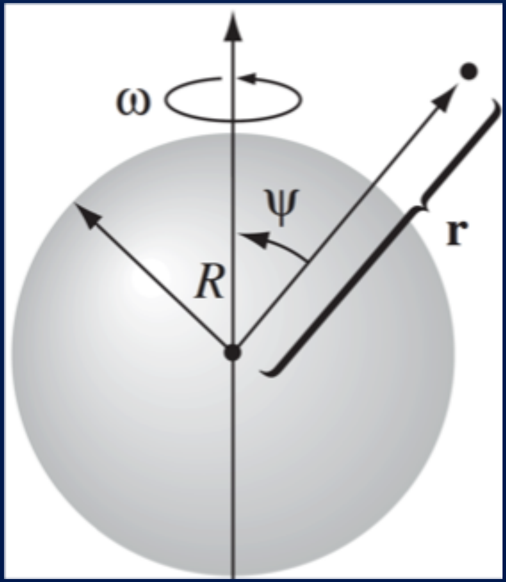
$$\vec{m} = Iw^2(\hat{y} + \hat{z})$$

5.4 The Magnetic Vector Potential

Example: A spherical shell, of radius R, carrying a uniform surface charge σ, is set spinning at angular velocity ω. Find the magnetic moment of the rotating sphere.

$$\vec{J} = \vec{\kappa}\delta(r-R) = \sigma\vec{v}\delta(r-R)$$

$$= R\sigma\,\omega\sin\theta\,\delta(r-R)\hat{\phi} =$$



$$\vec{m} = \frac{1}{2}\int \vec{r}\times\vec{J}\,d\tau = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} R^4\sigma\,\omega\sin^2\theta\,\delta(r-R)(\hat{r}\times\hat{\phi})\,drd\theta d\phi$$

$$= -\frac{1}{2}R^4\sigma\omega\int_0^{2\pi}\int_0^{\pi}\sin^2\theta\;\hat{\theta}\;d\theta d\phi$$

$$= -\frac{1}{2}R^4\sigma\omega\int_0^{2\pi}\int_0^{\pi}\sin^2\theta\;[\cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}]\,d\theta d\phi$$

$$= \pi R^4\sigma\omega\hat{z}\int_0^{\pi}\sin^3\theta\;d\theta = \frac{4}{3}\pi R^4\sigma\omega\hat{z}$$

5.4 The Magnetic Vector Potential

Excersise: An ellipsoidal shell with a charge density

$$\rho = \frac{Q}{2\pi a^2 b}\delta\left(\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{b^2}-1\right),$$

Rotates with an angular frequency $\vec{\omega} = \omega\hat{\mathbf{z}}$. Find the magnetic field at the origin