Chapter 5. Magnetostatics

5.1. The Magnetic Field

Consider two parallel straight wires in which current is flowing. The wires are neutral and therefore there is no net electric force between the wires. Nevertheless, if the current in both wires is flowing in the same direction, the wires are found to attract each other. If the current in one of the wires is reversed, the wires are found to repel each other. The force responsible for the attraction and repulsion is called the **magnetic force**. The magnetic force acting on a moving charge q is defined in terms of the **magnetic field**:

$$\overline{F}_{magnetic} = q(\overline{v} \times \overline{B})$$

The vector product is required since observations show that the force acting on a moving charge is perpendicular to the direction of the moving charge. In a region where there is an electric field and a magnetic field the total force on the moving force is equal to

$$\overline{F}_{total} = \overline{F}_{electric} + \overline{F}_{magnetic} = q\overline{E} + q(\overline{v} \times \overline{B})$$

This equation is called the **Lorentz force law** and provides us with the total electromagnetic force acting on q. An important difference between the electric field and the magnetic field is that the electric field does work on a charged particle (it produces acceleration or deceleration) while the magnetic field does not do any work on the moving charge. This is a direct consequence of the Lorentz force law:

$$dW_{magnetic} = \overline{F}_{magnetic} \bullet d\overline{l} = q\left[(\overline{v} \times \overline{B}) \bullet \overline{v}\right]dt = 0$$

We conclude that the magnetic force can alter the direction in which a particle moves, but can not change its velocity.

Example: Problem 5.1

A particle of charge q enters the region of uniform magnetic field $\overline{B}$ (pointing into the page). The field deflects the particle a distance d above the original line of flight, as shown in Figure 5.1. Is the charge positive or negative? In terms of a, d, B, and q, find the momentum of the particle.

In order to produce the observed deflection, the force on q at the entrance of the field region must be directed upwards (see Figure 5.1). Since direction of motion of the particle and the direction of the magnetic field are known, the Lorentz force law can be used to determine the direction of the magnetic force acting on a positive charge and on a negative charge. The vector product between $\bar{v}$ and $\bar{B}$ points upwards in Figure 5.1 (use the right-hand rule). This shows that the charge of the particle is positive.

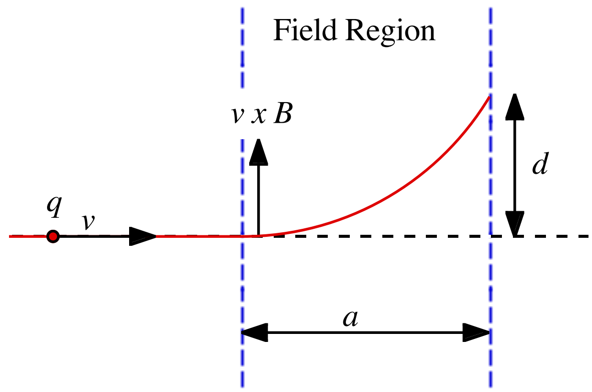

Figure 1. Problem 5.1.

The magnitude of the force acting on the moving charge is equal to

$$F_{magnetic} = qvB$$

As a result of the magnetic force, the charged particle will follow a spherical trajectory. The radius of the trajectory is determined by the requirement that the magnetic force provides the centripetal force:

$$F_{cent} = \frac{mv^2}{r} = F_{magnetic} = qvB$$

In this equation r is the radius of the circle that describes the circular part of the trajectory of charge q. The equation can be used to calculate r:

$$r = \frac{mv}{qB} = \frac{p}{qB}$$

where p is the momentum of the particle. Figure 5.2 shows the following relation between r, d and a:

$$(r - d)^2 + a^2 = r^2$$

This equation can be used to express r in terms of d and a:

$$r = \frac{d^2 + a^2}{2d}$$

The momentum of the charge q is therefore equal to

$$p = qBr = \frac{d^2 + a^2}{2d} qB$$

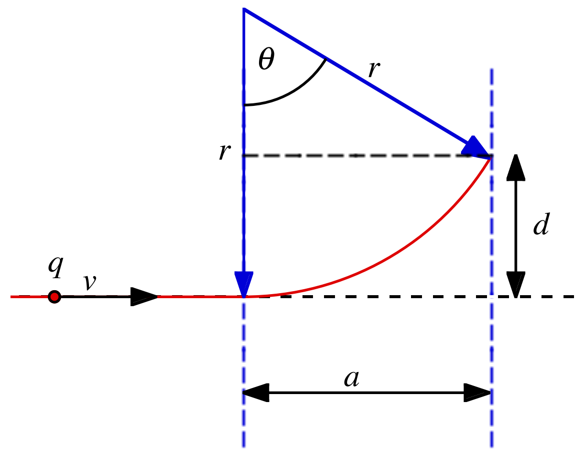

Figure 2. Problem 5.2.

The electric current in a wire is due to the motion of the electrons in the wire. The direction of current is defined to be the direction in which the positive charges move. Therefore, in a conductor the current is directed opposite to the direction of the electrons. The magnitude of the current is defined as the total charge per unit time passing a given point of the wire ($I = dq/dt$). If the current flows in a region with a non-zero magnetic field then each electron will experience a magnetic force. Consider a tiny segment of the wire of length dl. Assume that the electron density is -λ C/m and that each electron is moving with a velocity v. The magnetic force exerted by the magnetic field on a single electron is equal to

$$d\overline{F}_{1e} = -e(\overline{v} \times \overline{B})$$

A segment of the wire of length dl contains $\lambda\, dl/e$ electrons. Therefore the magnetic force acting in this segment is equal to

$$d\overline{F}_{magnetic} = \frac{\lambda dl}{e} d\overline{F}_{1e} = -\lambda dl(\overline{v} \times \overline{B}) = \lambda v(d\overline{l} \times \overline{B}) = I(d\overline{l} \times \overline{B})$$

Here we have used the definition of the current I in terms of dq and dt:

$$I = \frac{dq}{dt} = \frac{dq}{dl}\frac{dl}{dt} = \lambda v$$

In this derivation we have defined the direction of $d\bar{l}$ to be equal to the direction of the current (and therefore opposite to the direction of the velocity of the electrons). The total force on the wire is therefore equal to

$$\bar{F}_{magnetic} = \int_{wire} d\bar{F}_{magnetic} = I \int_{wire} \left(d\bar{l} \times \bar{B}\right)$$

Here I have assumed that the current is constant throughout the wire. If the current is flowing over a surface, it is usually described by a **surface current density** $\bar{K}$, which is the current per unit length-perpendicular-to-flow. The force on a surface current is equal to

$$\bar{F}_{magnetic} = \int_{surface} \left(\bar{K} \times \bar{B}\right) da$$

If the current flows through a volume, is it is usually described in terms of a **volume current density** $\bar{J}$. The magnetic force on a volume current is equal to

$$\bar{F}_{magnetic} = \int_{volume} \left(\bar{J} \times \bar{B}\right) d\tau$$

The surface integral of the current density $\bar{J}$ across the surface of a volume V is equal to the total charge leaving the volume per unit time (charge conservation):

$$\oint_{Surface} \bar{J} \bullet d\bar{a} = -\frac{d}{dt} \int_{Volume} \rho d\tau$$

Using the divergence theorem we can rewrite this expression as

$$\oint_{Surface} \bar{J} \bullet d\bar{a} = \int_{Volume} \left[\bar{\nabla} \bullet \bar{J}\right] d\tau = -\frac{d}{dt} \int_{Volume} \rho d\tau$$

Since this must hold for any volume V we must require that

$$\bar{\nabla} \bullet \bar{J} = -\frac{d\rho}{dt}$$

This equation is known as the **continuity equation**.

5.2. The Biot-Savart Law

In this Section we will discuss the magnetic field produced by a **steady curren**t. A steady current is a flow of charge that has been going on forever, and will be going on forever. These currents produce magnetic fields that are constant in time. The magnetic field produced by a steady line current is given by the **Biot-Savart Law**:

$$\bar{B}(P) = \frac{\mu_0}{4\pi} \int_{Line} \frac{\bar{I} \times \Delta\hat{r}}{\Delta r^2} dl = \frac{\mu_0 I}{4\pi} \int_{Line} \frac{d\bar{l} \times \Delta\hat{r}}{\Delta r^2}$$

where $d\bar{l}$ is an element of the wire, $\hat{r}$ is the vector connecting the element of the wire and P, and μ_0 is the permeability constant which is equal to

$$\mu_0 = 4\pi\, 10^{-7}\ N / A^2$$

The unit of the magnetic field is the **Tesla** (T). For surface and volume currents the Biot-Savart law can be rewritten as

$$\bar{B}(P) = \frac{\mu_0}{4\pi} \int_{Surface} \frac{\bar{K} \times \Delta\hat{r}}{\Delta r^2} da$$

and

$$\bar{B}(P) = \frac{\mu_0}{4\pi} \int_{Volume} \frac{\bar{J} \times \Delta\hat{r}}{\Delta r^2} d\tau$$

Example: Problem 5.9

Find the magnetic field at point P for each of the steady current configurations shown in Figure 5.3.

a) The total magnetic field at P is the vector sum of the magnetic fields produced by the four segments of the current loop. Along the two straight sections of the loop, $\hat{r}$ and $d\bar{l}$ are parallel or opposite, and thus $d\bar{l} \times \hat{r} = 0$. Therefore, the magnetic field produced by these two straight segments is equal to zero. Along the two circular segments $\hat{r}$ and $d\bar{l}$ are perpendicular. Using the right-hand rule it is easy to show that

$$\bar{B}_b(P) = \frac{\mu_0 I}{4\pi} \int_{Line} \frac{d\bar{l} \times \hat{b}}{b^2} = -\frac{\mu_0 I}{4\pi} \frac{\frac{1}{2}\pi b}{b^2} \hat{k} = -\frac{\mu_0 I}{8b} \hat{k}$$

and

$$\overline{B}_a(P) = \frac{\mu_0 I}{4\pi} \int_{Line} \frac{d\bar{l} \times \hat{a}}{a^2} = \frac{\mu_0 I}{4\pi} \frac{\frac{1}{2}\pi a}{a^2} \hat{k} = \frac{\mu_0 I}{8a} \hat{k}$$

where $\hat{z}$ is pointing out of the paper. The total magnetic field at P is therefore equal to

$$\overline{B}_{total}(P) = \frac{\mu_0 I}{8} \left(\frac{1}{a} - \frac{1}{b} \right) \hat{k}$$

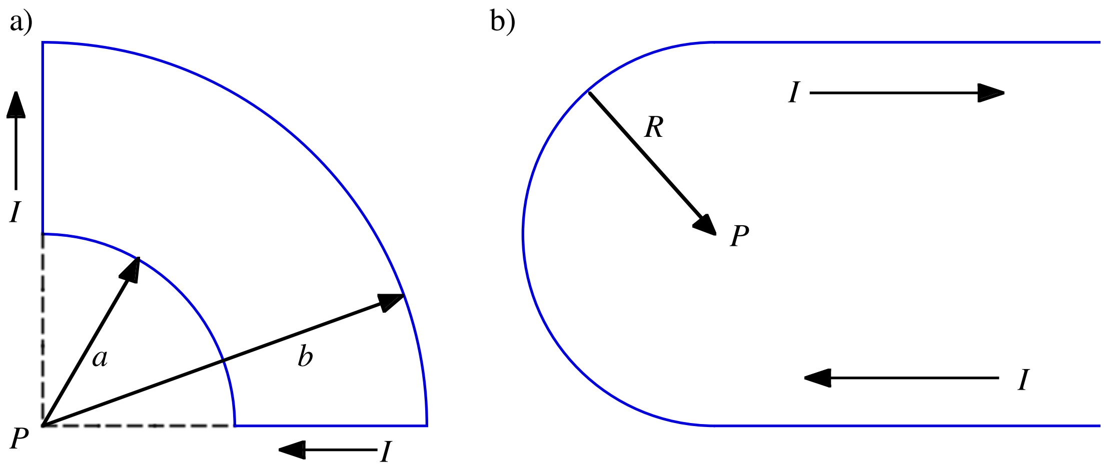

Figure 5.3. Problem 5.9.

b) The magnetic field at P produced by the circular segment of the current loop is equal to

$$\overline{B}_R(P) = \frac{\mu_0 I}{4\pi} \int_{Line} \frac{d\bar{l} \times \hat{R}}{R^2} = -\frac{\mu_0 I}{4\pi} \frac{\pi R}{R^2} \hat{k} = -\frac{\mu_0 I}{4R} \hat{k}$$

where $\hat{z}$ is pointing out of the paper. The magnetic field produced at P by each of the two linear segments will also be directed along the negative z axis. The magnitude of the magnetic field produced by each linear segment is just half of the field produced by an infinitely long straight wire (see Example 5 in Griffiths):

$$\overline{B}_{linear}(P) = -2\frac{\mu_0 I}{4\pi R} \hat{k} = -\frac{\mu_0 I}{2\pi R} \hat{k}$$

The total field at P is therefore equal to

$$\overline{B}_{total}(P) = -\frac{\mu_0 I}{4R} \hat{k} - \frac{\mu_0 I}{2\pi R} \hat{k} = -\frac{\mu_0 I}{R} \left(\frac{1}{4} + \frac{1}{2\pi} \right) \hat{k}$$

Example: Problem 5.12

Suppose you have two infinite straight-line charges λ, a distance d apart, moving along at a constant v (see Figure 5.4). How fast would v have to be in order for the magnetic attraction to balance the electrical repulsion?

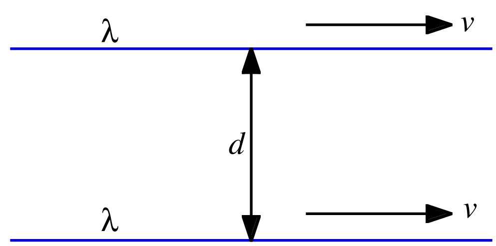

Figure 5.4. Problem 5.12.

When a line charge moves it looks like a current of magnitude $I = \lambda v$. The two parallel currents attract each other, and the attractive force per unit length is

$$f_{magnetic} = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d} = \frac{\mu_0}{2\pi}\frac{\lambda^2 v^2}{d}$$

and is attractive. The electric generated by one of the wires can be found using Gauss' law and is equal to

$$E(r) = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r}$$

The electric force per unit length acting on the other wire is equal to

$$f_{electric} = \lambda\, E(d) = \frac{1}{2\pi\varepsilon_0}\frac{\lambda^2}{d}$$

and is repulsive (like charges). The electric and magnetic forces are balanced when

$$\frac{1}{2\pi\varepsilon_0}\frac{\lambda^2}{d} = \frac{\mu_0}{2\pi}\frac{\lambda^2 v^2}{d}$$

or

$$\mu_0 v^2 = \frac{1}{\varepsilon_0}$$

This requires that

$$v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3\ 10^8 \text{ m / s}$$

This requires that the speed v is equal to the speed of light, and this can therefore never be achieved. Therefore, at all velocities the electric force will dominate.

5.3. The Divergence and Curl of *B*.

Using the Biot-Savart law for a volume current $\bar{J}$ we can calculate the divergence and curl of $\bar{B}$:

$$\bar{\nabla} \bullet \bar{B} = 0$$

and

$$\bar{\nabla} \times \bar{B} = \mu_0 \bar{J}$$

This last equation is called **Ampere's law in differential form**. This equation can be rewritten, using Stokes' law, as

$$\int_{Surface} \left[\bar{\nabla} \times \bar{B} \right] \bullet d\bar{a} = \oint_{Line} \bar{B} \bullet d\bar{l} = \mu_0 \int_{Surface} \bar{J} \bullet d\bar{a} = \mu_0 I_{encl}$$

This equation is called **Ampere's law in integral form.** The direction of evaluation of the line integral and the direction of the surface element vector $d\bar{a}$ must be consistent with the right-hand rule.

Ampere's law is always true, but is only a useful tool to evaluate the magnetic field if the symmetry of the system enables you to pull $\bar{B}$ outside the line integral. The configurations that can be handled by Ampere's law are:

1. Infinite straight lines
2. Infinite planes
3. Infinite solenoids
4. Toroids

Example: Problem 5.14

A thick slab extending from $z = -a$ to $z = a$ carries a uniform volume current $\bar{J} = J\,\hat{i}$. Find the magnetic field both inside and outside the slab.

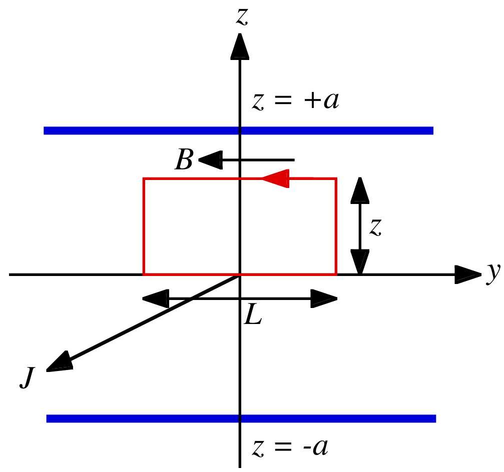

Figure 5.5. Problem 5.14

Because of the symmetry of the problem the magnetic field will be directed parallel to the y axis. The magnetic field in the region above the xy plane ($z > 0$) will be the mirror image of the field in the region below the xy plane ($z < 0$). The magnetic field in the xy plane ($z = 0$) will be equal to zero. Consider the Amperian loop shown in Figure 5.5. The current is flowing out of the paper, and we choice the direction of $d\bar{a}$ to be parallel to the direction of $\bar{J}$. Therefore,

$$\int_{Surface} \bar{J} \bullet d\bar{a} = JzL \qquad 0 < z < a$$

$$\int_{Surface} \bar{J} \bullet d\bar{a} = JaL \qquad z > a$$

The direction of evaluation of the line integral of $\bar{B}$ must be consistent with our choice of the direction of $d\bar{a}$ (right-hand rule). This requires that the line integral of $\bar{B}$ must be evaluated in a counter-clockwise direction. The line integral of $\bar{B}$ is equal to

$$\oint_{Line} \bar{B} \bullet d\bar{l} = BL$$

Applying Ampere's law we obtain for $\bar{B}$:

$$B = \frac{\mu_0}{L} \int_{Surface} \bar{J} \bullet d\bar{a} = \mu_0 Jz \qquad 0 < z < a$$

$$B = \frac{\mu_0}{L} \int_{Surface} \bar{J} \bullet d\bar{a} = \mu_0 Ja \qquad z > a$$

Thus

$$B(z) = -\mu_0 J a \hat{j} \qquad a < z$$

$$B(z) = -\mu_0 J z \hat{j} \qquad -a < z < a$$

$$B(z) = \mu_0 J a \hat{j} \qquad z < -a$$

5.4. The Vector Potential

The magnetic field generated by a static current distribution is uniquely defined by the so-called **Maxwell equations for magnetostatics**:

$$\overline{\nabla} \bullet \overline{B} = 0$$

$$\overline{\nabla} \times \overline{B} = \mu_0 \overline{J}$$

Similarly, the electric field generated by a static charge distribution is uniquely defined by the so-called **Maxwell equations for electrostatics**:

$$\overline{\nabla} \bullet \overline{E} = \frac{\rho}{\varepsilon_0}$$

$$\overline{\nabla} \times \overline{E} = 0$$

The fact that the divergence of $\overline{B}$ is equal to zero suggests that there are no point charges for $\overline{B}$. Magnetic field lines therefore do not begin or end anywhere (in contrast to electric field lines that start on positive point charges and end on negative point charges). Since a magnetic field is created by moving charges, a magnetic field can never be present without an electric field being present. In contrast, only an electric field will exist if the charges do not move.

Maxwell's equations for magnetostatics show that if the current density is known, both the divergence and the curl of the magnetic field are known. The Helmholtz theorem indicates that in that case there is a **vector potential** $\overline{A}$ such that

$$\overline{B} = \overline{\nabla} \times \overline{A}$$

However, the vector potential is not uniquely defined. We can add to it the gradient of any scalar function *f* without changing its curl:

$$\overline{\nabla} \times \left(\overline{A} + \overline{\nabla} f\right) = \overline{\nabla} \times \overline{A} + \overline{\nabla} \times \overline{\nabla} f = \overline{\nabla} \times \overline{A}$$

The divergence of $\overline{A} + \overline{\nabla} f$ is equal to

$$\overline{\nabla}\bullet\left(\overline{A}+\overline{\nabla}f\right)=\overline{\nabla}\bullet\overline{A}+\overline{\nabla}\bullet\overline{\nabla}f=\overline{\nabla}\bullet\overline{A}+\overline{\nabla}^2 f$$

It turns out that we can always find a scalar function f such that the vector potential $\overline{A}$ is divergence-less. The main reason for imposing the requirement that $\overline{\nabla}\bullet\overline{A}=0$ is that it simplifies many equations involving the vector potential. For example, Ampere's law rewritten in terms of $\overline{A}$ is

$$\overline{\nabla}\times\overline{B}=\overline{\nabla}\times\left(\overline{\nabla}\times\overline{A}\right)=\overline{\nabla}\left(\overline{\nabla}\bullet\overline{A}\right)-\overline{\nabla}^2\overline{A}=-\overline{\nabla}^2\overline{A}=\mu_0\overline{J}$$

or

$$\overline{\nabla}^2\overline{A}=-\mu_0\overline{J}$$

This equation is similar to Poisson's equation for a charge distribution ρ:

$$\overline{\nabla}^2 V=-\frac{\rho}{\varepsilon_0}$$

Therefore, the vector potential $\overline{A}$ can be calculated from the current $\overline{J}$ in a manner similar to how we obtained V from ρ. Thus

$$\overline{A}=\frac{\mu_0}{4\pi}\int_{Volume}\frac{\overline{J}}{\Delta r}d\tau \qquad \text{for a volume current}$$

$$\overline{A}=\frac{\mu_0}{4\pi}\int_{Surface}\frac{\overline{K}}{\Delta r}da \qquad \text{for a surface current}$$

$$\overline{A}=\frac{\mu_0}{4\pi}\int_{Line}\frac{\overline{I}}{\Delta r}dl=\frac{\mu_0 I}{4\pi}\int_{Line}\frac{d\overline{l}}{\Delta r} \qquad \text{for a line current}$$

Note: these solutions require that the currents go to zero at infinity (similar to the requirement that ρ goes to zero at infinity).

Example: Problem 5.22

Find the magnetic vector potential of a finite segment of straight wire carrying a current I. Check that your answer is consistent with eq. (5.35) of Griffiths.

The current at infinity is zero in this problem, and therefore we can use the expression for $\overline{A}$ in terms of the line integral of the current I. Consider the wire located along the z axis between z_1 and z_2 (see Figure 5.6) and use cylindrical coordinates. The vector potential at a point P is independent of ϕ (cylindrical symmetry) and equal to

$$\overline{A} = \frac{\mu_0}{4\pi}\int_{Line}\frac{d\overline{l}}{\Delta r} = \frac{\mu_0 I}{4\pi}\int_{z_1}^{z_2}\frac{dz'}{\sqrt{r^2+z'^2}}\hat{k} = \frac{\mu_0 I}{4\pi}\ln\left[\frac{z_2+\sqrt{r^2+z_2{}^2}}{z_1+\sqrt{r^2+z_1{}^2}}\right]\hat{k}$$

Here we have assumed that the origin of the coordinate system is chosen such that P has $z = 0$. The magnetic field at P can be obtained from the vector potential and is equal to

$$\overline{B} \;=\; \overline{\nabla}\times\overline{A} = -\frac{\partial A_z}{\partial r}\hat{\phi} = -\frac{\mu_0 I}{4\pi}\left[\frac{r}{\sqrt{r^2+z_2{}^2}}\frac{1}{z_2+\sqrt{r^2+z_2{}^2}} - \frac{r}{\sqrt{r^2+z_1{}^2}}\frac{1}{z_1+\sqrt{r^2+z_1{}^2}}\right]\hat{\phi} =$$

$$= \frac{\mu_0 I}{4\pi\, r}\left[\frac{z_2}{\sqrt{r^2+z_2{}^2}} - \frac{z_1}{\sqrt{r^2+z_1{}^2}}\right]\hat{\phi} = \frac{\mu_0 I}{4\pi\, r}\left[\sin\theta_2 - \sin\theta_1\right]\hat{\phi}$$

where θ_1 and θ_2 are defined in Figure 5.6. This result is identical to the result of Example 5 in Griffiths.

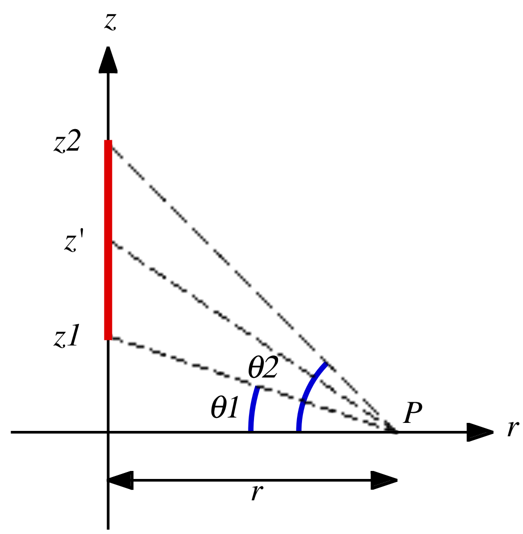

Figure 5.6. Problem 5.25.

Example: Problem 5.24

If $\overline{B}$ is *uniform*, show that $\overline{A} = -(\overline{r}\times\overline{B})/2$, where $\overline{r}$ is the vector from the origin to the point in question. That is check that $\overline{\nabla}\times\overline{A} = \overline{B}$ and $\overline{\nabla}\bullet\overline{A} = 0$.

The curl of $\overline{A} = -(\overline{r}\times\overline{B})/2$ is equal to

$$\overline{\nabla}\times\overline{A} = -\frac{1}{2}\overline{\nabla}\times(\overline{r}\times\overline{B}) = -\frac{1}{2}\left[(\overline{B}\bullet\overline{\nabla})\overline{r} - (\overline{r}\bullet\overline{\nabla})\overline{B} + \overline{r}(\overline{\nabla}\bullet\overline{B}) - \overline{B}(\overline{\nabla}\bullet\overline{r})\right]$$

Since $\overline{B}$ is uniform it is independent of r, θ, and ϕ and therefore the second and third term on the right-hand side of this equation are zero. The first term, expressed in Cartesian coordinates, is equal to

$$\left(\overline{B} \bullet \overline{\nabla}\right)\overline{r} = \left(B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z}\right)\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = B_x\hat{i} + B_y\hat{j} + B_z\hat{k} = \overline{B}$$

The fourth term, expressed in Cartesian coordinates, is equal to

$$\overline{B}\left(\overline{\nabla} \bullet \overline{r}\right) = \overline{B}\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z\right) = 3\overline{B}$$

Therefore, the curl of $\overline{A}$ is equal to

$$\overline{\nabla} \times \overline{A} = -\frac{1}{2}\left(\overline{B} - 3\overline{B}\right) = \overline{B}$$

The divergence of $\overline{A}$ is equal to

$$\overline{\nabla} \bullet \overline{A} = -\frac{1}{2}\overline{\nabla} \bullet \left(\overline{r} \times \overline{B}\right) = -\frac{1}{2}\left[\overline{B} \bullet \left(\overline{\nabla} \times \overline{r}\right) - \overline{r} \bullet \left(\overline{\nabla} \times \overline{B}\right)\right] = 0$$

Example: Problem 5.26

Find the vector potential above and below the plane surface current of Example 5.8 in Griffiths.

In Example 5.8 of Griffiths a uniform surface current is flowing in the *xy* plane, directed parallel to the *x* axis:

$$\overline{K} = K\,\hat{i}$$

However, since the surface current extends to infinity, we can not use the surface integral of $\overline{K} / \Delta r$ to calculate $\overline{A}$ and an alternative method must be used to obtain $\overline{A}$. Since Example 8 showed that $\overline{B}$ is uniform above the plane of the surface current and $\overline{B}$ is uniform below the plane of the surface current, we can use the result of Problem 5.27 to calculate $\overline{A}$:

$$\overline{A} = -\frac{1}{2}\left(\overline{r} \times \overline{B}\right)$$

In the region above the *xy* plane ($z > 0$) the magnetic field is equal to

$$\overline{B} = -\frac{\mu_0}{2} K \hat{j}$$

Therefore,

$$\overline{A} = -\frac{1}{2}\left(\overline{r} \times \overline{B}\right) = -\frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 0 & -\frac{\mu_0}{2}K & 0 \end{vmatrix} = -\frac{\mu_0}{4} K z \,\hat{i} + \frac{\mu_0}{4} K x \,\hat{k}$$

In the region below the *xy* plane ($z < 0$) the magnetic field is equal to

$$\overline{B} = \frac{\mu_0}{2} K \hat{j}$$

Therefore,

$$\overline{A} = -\frac{1}{2}\left(\overline{r} \times \overline{B}\right) = -\frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 0 & \frac{\mu_0}{2}K & 0 \end{vmatrix} = \frac{\mu_0}{4} K z \,\hat{i} - \frac{\mu_0}{4} K x \,\hat{k}$$

We can verify that our solution for $\overline{A}$ is correct by calculating the curl of $\overline{A}$ (which must be equal to the magnetic field). For $z > 0$:

$$\overline{\nabla} \times \overline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\mu_0}{4} K z & 0 & \frac{\mu_0}{4} K z \end{vmatrix} = -\frac{\mu_0}{2} K \hat{j} = \overline{B}$$

The vector potential $\overline{A}$ is however not uniquely defined. For example, $\overline{A} = -(\mu_0 / 2) K z \,\hat{i}$ and $\overline{A} = (\mu_0 / 2) K x \,\hat{k}$ are also possible solutions that generate the same magnetic field. These solutions also satisfy the requirement that $\overline{\nabla} \bullet \overline{A} = 0$.

5.5. The Three Fundamental Quantities of Magnetostatics

Our discussion of the magnetic fields produced by steady currents has shown that there are three fundamental quantities of magnetostatics:

1. The current density $\overline{J}$
2. The magnetic field $\overline{B}$
3. The vector potential $\overline{A}$

These three quantities are related and if one of them is known, the other two can be calculated. The following table summarizes the relations between $\overline{J}$, $\overline{B}$, and $\overline{A}$:

Known ↓	$\overline{J} =$	$\overline{B} =$	$\overline{A} =$
$\overline{J}$		$\frac{\mu_0}{4\pi}\int \frac{\overline{J}\times\Delta\hat{r}}{\Delta r^2}d\tau$	$\frac{\mu_0}{4\pi}\int \frac{\overline{J}}{\Delta r}d\tau$
$\overline{B}$	$\frac{1}{\mu_0}\left(\overline{\nabla}\times\overline{B}\right)$		$\frac{1}{4\pi}\int \frac{B\times\Delta\hat{r}}{\Delta r^2}d\tau$
$\overline{A}$	$-\frac{1}{\mu_0}\overline{\nabla}^2\overline{A}$	$\overline{\nabla}\times\overline{A}$	

5.6. The Boundary Conditions of *B*

In Chapter 2 we studied the boundary conditions of the electric field and concluded that the electric field suffers a discontinuity at a surface charge. Similarly, the magnetic field suffers a discontinuity at a surface current.

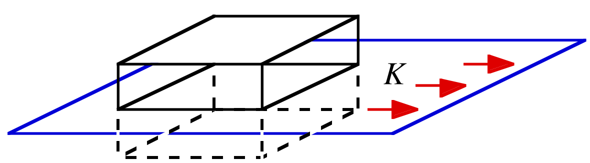

Figure 5.7. Boundary conditions for $\overline{B}$.

Consider the surface current $\overline{K}$ (see Figure 5.7). The surface integral of $\overline{B}$ over a wafer thin pillbox is equal to

$$\int_{Surface} \overline{B}\bullet d\overline{a} = B_{\perp,above}A - B_{\perp,below}A$$

where A is the area of the top and bottom of the pill box. The surface integral of $\overline{B}$ can be rewritten using the divergence theorem:

$$\int_{Surface} \overline{B} \bullet d\overline{a} = \int_{Volume} \left(\overline{\nabla} \bullet \overline{B}\right) d\tau = 0$$

since $\overline{\nabla} \bullet \overline{B} = 0$ for any magnetic field $\overline{B}$. Therefore, the perpendicular component of the magnetic field is continuous at a surface current:

$$B_{\perp,above} = B_{\perp,below}$$

The line integral of $\overline{B}$ around the loop shown in Figure 5.8 (in the limit $\varepsilon \to 0$) is equal to

$$\oint_{Loop} \overline{B} \bullet d\overline{l} = B_{\parallel,above} L - B_{\parallel,below} L$$

According to Ampere's law the line integral of $\overline{B}$ around this loop is equal to

$$\oint_{Loop} \overline{B} \bullet d\overline{l} = \mu_0 I_{encl} = \mu_0 K L$$

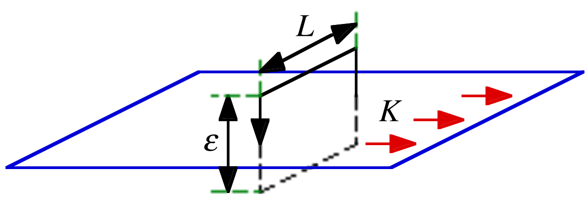

Figure 5.8. Boundary conditions for $\overline{B}$.

Therefore, the boundary condition for the component of $\overline{B}$, parallel to the surface and perpendicular to the current, is equal to

$$B_{\parallel,above} - B_{\parallel,below} = \mu_0 K$$

The boundary conditions for $\overline{B}$ can be combined into one equation:

$$\overline{B}_{above} - \overline{B}_{below} = \mu_0 \left(\overline{K} \times \hat{n}\right)$$

where $\hat{n}$ is a unit vector perpendicular to the surface and the surface current and pointing "upward". The vector potential $\overline{A}$ is continuous at a surface current, but its normal derivative is not:

$$\frac{\partial \overline{A}_{above}}{\partial n} - \frac{\partial \overline{A}_{below}}{\partial n} = -\mu_0 \overline{K}$$

5.7. The Multipole Expansion of the Magnetic Field

To calculate the vector potential of a localized current distribution at large distances we can use the multipole expansion. Consider a current loop with current I. The vector potential of this current loop can be written as

$$\overline{A} = \frac{\mu_0 I}{4\pi} \oint_{Line} \frac{d\bar{l}}{\Delta r} = \frac{\mu_0 I}{4\pi} \left\{ \sum_{n=0}^{\infty} \left[\frac{1}{r^{n+1}} \oint_{Line} r'^n P_n(\cos\theta) d\bar{l} \right] \right\}$$

At large distance only the first couple of terms of the multipole expansion need to be considered:

$$\overline{A} \cong \frac{\mu_0 I}{4\pi} \left\{ \frac{1}{r} \oint_{Line} d\bar{l} + \frac{1}{r^2} \oint_{Line} r' \cos\theta \, d\bar{l} + ... \right\}$$

The first term is called the **monopole term** and is equal to zero (since the line integral of $d\bar{l}$ is equal to zero for any closed loop). The second term, called the **dipole term**, is usually the dominant term. The vector potential generated by the dipole terms is equal to

$$\overline{A}_{dipole} = \frac{\mu_0 I}{4\pi} \frac{1}{r^2} \oint_{Line} r' \cos\theta \, d\bar{l} = \frac{\mu_0 I}{4\pi} \frac{1}{r^2} \oint_{Line} \left(\bar{r}' \bullet \hat{r} \right) d\bar{l}$$

This equation can be rewritten as

$$\overline{A}_{dipole} = \frac{\mu_0 I}{4\pi} \frac{1}{r^2} \left\{ -\frac{1}{2} \hat{r} \times \oint_{Line} \bar{r}' \times d\bar{l} \right\} = \frac{\mu_0}{4\pi} \frac{\overline{m} \times \hat{r}}{r^2}$$

where $\overline{m}$ is called the **magnetic dipole moment** of the current loop. It is defined as

$$\overline{m} = \frac{1}{2} I \oint_{Line} \bar{r}' \times d\bar{l}$$

If the current loop is a plane loop (current located on the surface of a plane) then $\left(\bar{r}' \times d\bar{l} \right) / 2$ is the area of the triangle shown in Figure 5.9. Therefore,

$$\frac{1}{2} \oint_{Line} \bar{r}' \times d\bar{l} = a$$

where a is the area enclosed by the current loop. In this case, the dipole moment of the current loop is equal to

$$\overline{m} = I\,\overline{a}$$

where the direction of $\overline{a}$ must be consistent with the direction of the current in the loop (right-hand rule).

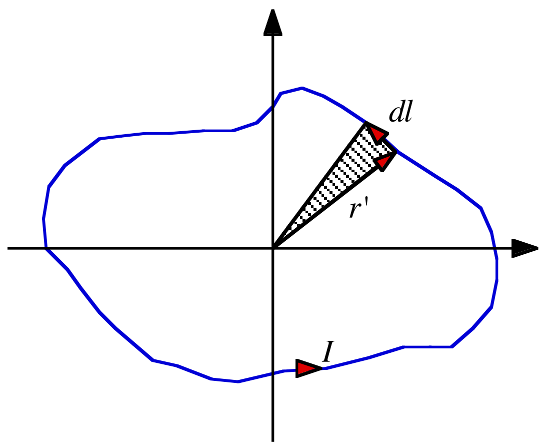

Figure 5.9. Calculation of $\overline{m}$.

Assuming that the magnetic dipole is located at the origin of our coordinate system and that $\overline{m}$ is pointing along the positive z axis, we obtain for $\overline{A}$:

$$\overline{A}_{dipole} = \frac{\mu_0}{4\pi}\frac{\overline{m}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}$$

The corresponding magnetic field is equal to

$$\overline{B}_{dipole} = \overline{\nabla}\times\overline{A}_{dipole} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\right)\hat{\theta} =$$

$$= \frac{\mu_0}{4\pi}\frac{m}{r^3}\left\{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right\}$$

The shape of the field generated by a magnetic dipole is identical to the shape of the field generated by an electric dipole.

Example: Problem 5.33

Show that the magnetic field of a dipole can be written in the following coordinate free form:

$$\overline{B} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left\{3\left(\overline{m}\bullet\hat{r}\right)\hat{r} - \overline{m}\right\}$$

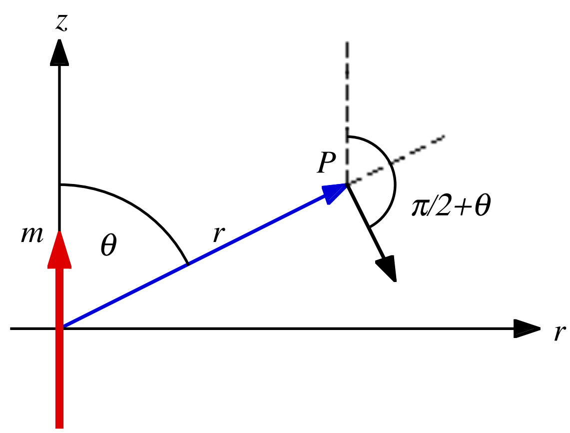

Figure 5.10. Problem 5.33.

Consider the configuration shown in Figure 5.10. The scalar product between $\hat{r}$ and $\overline{m}$ is equal to

$$\overline{m} \bullet \hat{r} = m\cos\theta$$

The scalar product between $\hat{\theta}$ and $\overline{m}$ is equal to

$$\overline{m} \bullet \hat{\theta} = m\cos\left(\frac{1}{2}\pi + \theta\right) = -m\sin\theta$$

Therefore,

$$\begin{aligned}\overline{B} &= \frac{\mu_0}{4\pi}\frac{1}{r^3}\left\{2m\cos\theta\,\hat{r} + m\sin\theta\,\hat{\theta}\right\} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left\{2(\overline{m}\bullet\hat{r})\hat{r} - \left(\overline{m}\bullet\hat{\theta}\right)\hat{\theta}\right\} = \\ &= \frac{\mu_0}{4\pi}\frac{1}{r^3}\left\{3(\overline{m}\bullet\hat{r})\hat{r} - (\overline{m}\bullet\hat{r})\hat{r} - \left(\overline{m}\bullet\hat{\theta}\right)\hat{\theta}\right\} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left\{3(\overline{m}\bullet\hat{r})\hat{r} - \overline{m}\right\}\end{aligned}$$

Example: Problem 5.34

A circular loop of wire, with radius *R*, lies in the *xy* plane, centered at the origin, and carries a current *I* running counterclockwise as viewed from the positive *z* axis.

a) What is its magnetic dipole moment?

b) What is its (approximate) magnetic field at points far from the origin?

c) Show that, for points on the *z* axis, your answer is consistent with the exact field as calculated in Example 6 of Griffiths.

a) Since the current loop is a plane loop, its dipole moment is easy to calculate. It is equal to

$$\overline{m} = I\overline{a} = \pi R^2 I\,\hat{k}$$

b) The magnetic field at large distances is approximately equal to

$$\overline{B} = \frac{\mu_0}{4\pi}\frac{\pi R^2 I}{r^3}\left\{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right\}$$

c) For points on the positive z axis $\theta = 0°$. Therefore, for $z>0$

$$\overline{B} = \frac{\mu_0}{4\pi}\frac{\pi R^2 I}{r^3}2\hat{k} = \frac{\mu_0}{2}\frac{R^2 I}{r^3}\hat{k}$$

Fore points on the negative z axis $\theta = 180°$. Therefore, for $z<0$

$$\overline{B} = \frac{\mu_0}{4\pi}\frac{\pi R^2 I}{r^3}\left(-2\hat{k}\right) = -\frac{\mu_0}{2}\frac{R^2 I}{r^3}\hat{k}$$

The exact solution for $\overline{B}$ on the positive z axis is

$$\overline{B} = \frac{\mu_0 I}{2}\frac{R^2}{\left(R^2 + z^2\right)^{3/2}}\hat{k}$$

For z » R the field is approximately equal to

$$\overline{B} \cong \frac{\mu_0}{2}\frac{R^2 I}{z^{32}}\hat{k}$$

which is consistent with the dipole field of the current loop.

Example: Problem 5.35

A phonograph record of radius R, carrying a uniform surface charge σ, is rotating at constant angular velocity ω. Find its magnetic dipole moment.

The rotational period of the disk is equal to

$$T = \frac{2\pi}{\omega}$$

Consider the disk to consist of a large number of thin rings. Consider a single ring of inner radius r and with dr. The charge on such a ring is equal to

$$dq = \sigma\left(\pi\left(r + dr\right)^2 - \pi r^2\right) \cong 2\pi\sigma\, r dr$$

Since the charge is rotating, the moving charge corresponds to a current *dI*:

$$dI = \frac{dq}{dt} = \frac{2\pi\sigma\, rdr}{\dfrac{2\pi}{\omega}} = \sigma\omega\, rdr$$

The dipole moment of this ring is therefore equal to

$$d\overline{m} = \left(\pi\, r^{2}\right) dI\, \hat{k} = \pi\sigma\omega\, r^{3} dr\, \hat{k}$$

The total dipole moment of the disk is equal to

$$\overline{m} = \int d\overline{m} = \pi\sigma\omega \int_{0}^{R} r^{3} dr\, \hat{k} = \frac{\pi}{4}\sigma\omega R^{4}\, \hat{k}$$