Electromagnetic Theory II

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جامعة بيرزيت

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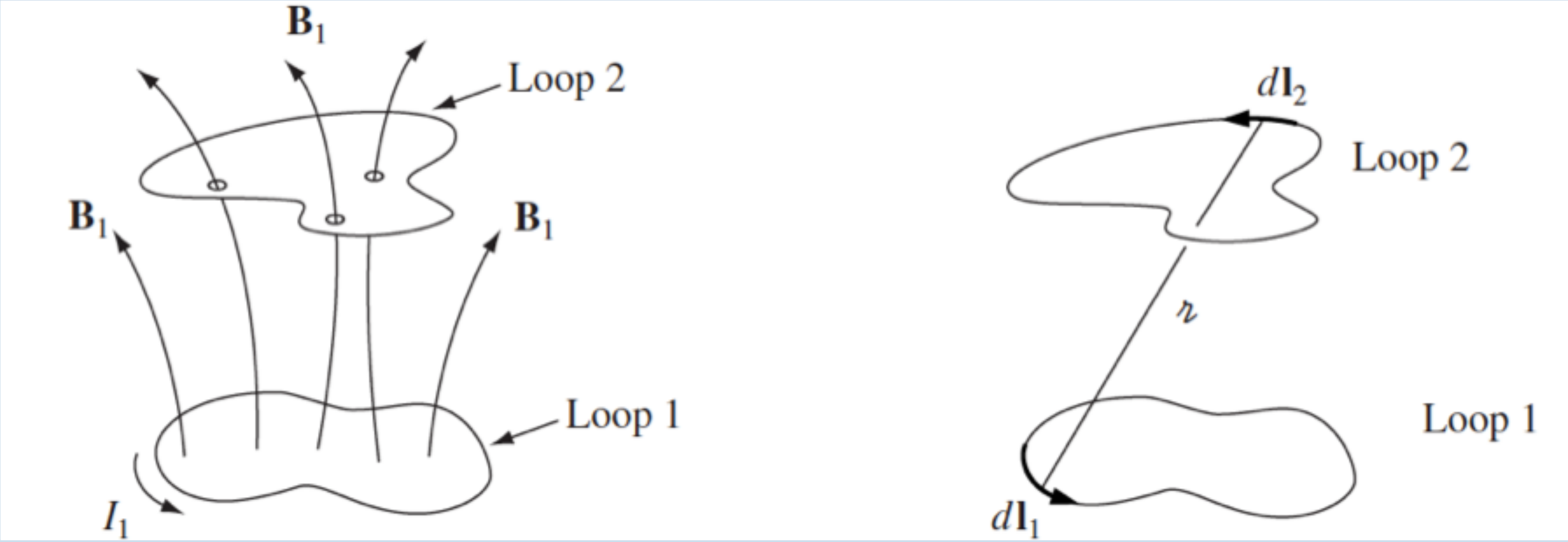
Chapter 7: Electrodynamics

- Electromagnetic Induction
  - Inductance
  - Energy in Magnetic field

7.2.3 Inductance

Consider two loops of wires at rest



$$\mathbf{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\mathbf{l}_1 \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2},$$

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2.$$

$$\Phi_2 = M_{21} I_1$$

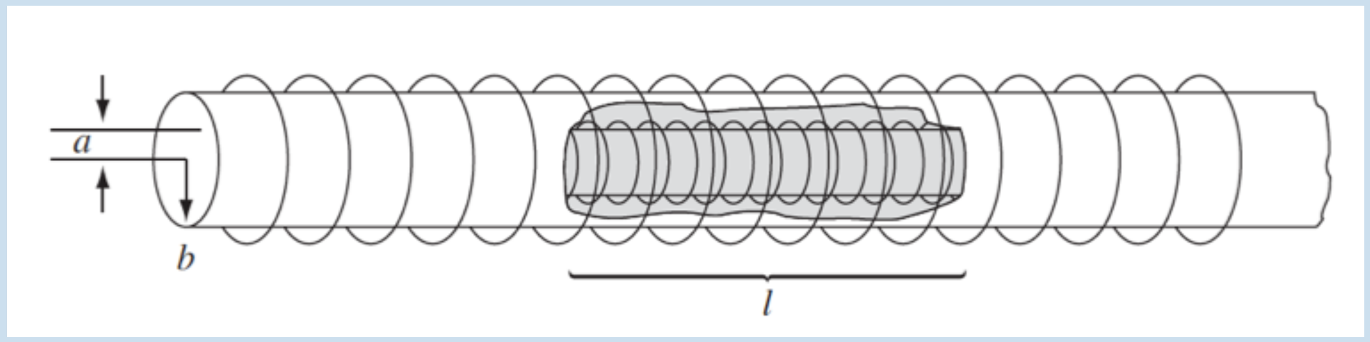
$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2 = \int (\nabla \times \mathbf{A}_1) \cdot d\mathbf{a}_2 = \oint \mathbf{A}_1 \cdot d\mathbf{l}_2. \quad \mathbf{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\mathbf{l}_1}{\mathscr{r}}$$

$$\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \left(\oint \frac{d\mathbf{l}_1}{\mathscr{r}} \right) \cdot d\mathbf{l}_2$$

$$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\mathscr{r}}$$ **Mutual Inductance of the two loops**

7.2.3 Inductance

Example: A short solenoid (length l and radius a, with n_1 turns per unit length) lies on the axis of a very long solenoid (radius b , n_2 turns per unit length). Current I flows in the short solenoid. What is the flux through the long solenoid?



Solving the problem directly is just too difficult, Instead we consider the opposite case of having the current flowing in the long solenoid

$$\Phi = BAn_2 l = \pi a^2 \mu_o n_1 n_2 I l = MI, where\ M = \mu_0 \pi a^2 n_1 n_2 l$$

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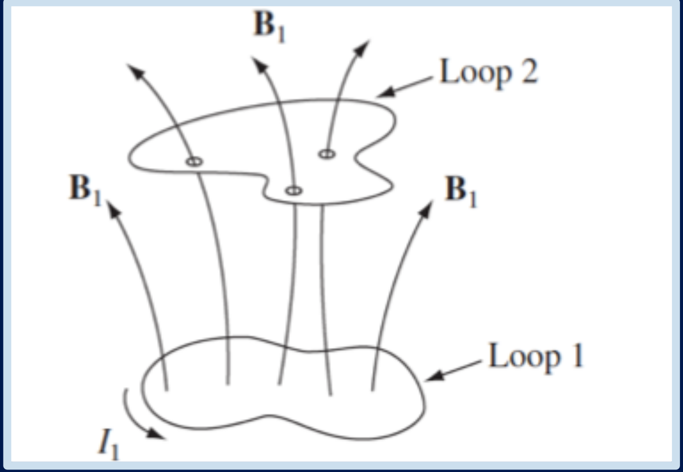
Now if the current is varying slowly in one of the loops (quasistatic)



$$\Phi = LI.$$ **Self Inductance of a loop**

$$\mathcal{E} = -L\frac{dI}{dt}$$ **Back emf**

A changing current not only induces an emf in any nearby loops, It also induces an emf in the source loop itself.

If the current in loop 1 changes, the flux through loop 2 will vary accordingly. The changing flux will induce an emf in loop 2

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt}.$$

When the current in a wire is changed, back emf opposes this change. It is like that inductor plays the same role as mass plays in mechanical systems: The greater L is, the harder it is to change the current, just as the larger the mass, the harder it is to change an object's velocity.

7.2.3 Inductance

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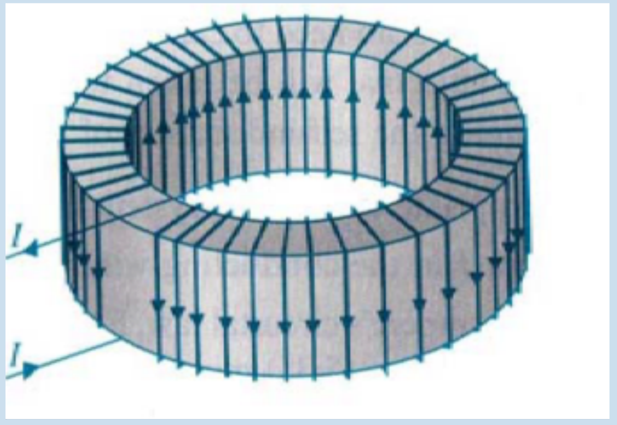
Example: A Find the self-inductance of a toroidal coil with rectangular cross section (inner radius a, outer radius b, height h), that carries a total of N turns.

For, $a < s < b$ $\qquad \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$

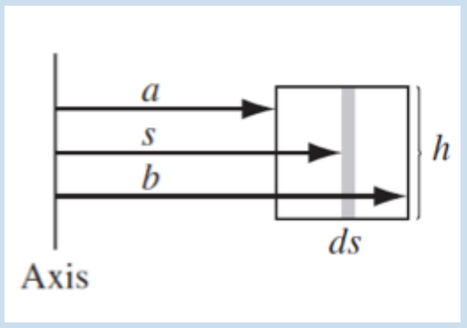


$$2\pi s B = \mu_0 I N \rightarrow \vec{B} = \frac{\mu_0 I N}{2\pi s}\hat{\Phi}$$

$$\Phi_1 = \int \vec{B} \cdot d\vec{a} = \int_0^h \int_a^b \mu_0 I \frac{N}{2\pi s}\hat{\Phi} \cdot dsdz\hat{\phi}$$

$$\Phi_1 = \mu_0 \frac{INh}{2\pi} \ln\frac{b}{a}$$

$$\Phi_N = \mu_0 \frac{IN^2 h}{2\pi} \ln\frac{b}{a} = IL \rightarrow L = \mu_0 \frac{N^2 h}{2\pi} \ln\frac{b}{a}$$



7.2.3 Inductance

Example: A Find the self-inductance of a long solenoid of radius R and have n turns per unit length

$$\vec{B} = \mu_0 I n \hat{z}$$

$$\Phi_1 = \int \vec{B} \cdot d\vec{a} = BA = \mu_0 I n \pi R^2$$

$$\Phi_N = \mu_0 I n \pi R^2 (nl) = \mu_0 n^2 \pi R^2 l I = IL \rightarrow L = \mu_0 n^2 \pi R^2 l$$

$$\mathcal{L} = \mu_0 n^2 \pi R^2 \ (Inductance\ per\ unit\ length)$$

7.2.3 Inductance

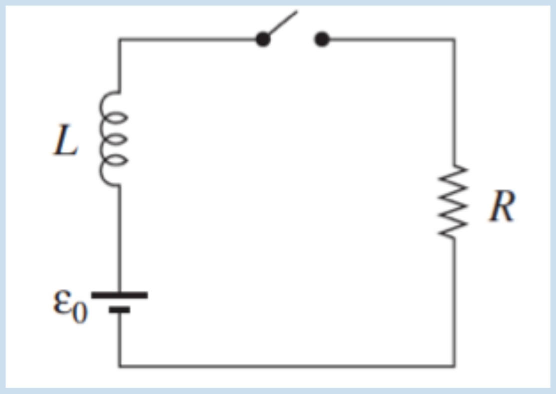
Example: When you plug in a toaster or iron. Induction opposes the sudden increase in current, prescribing instead a smooth and continuous buildup. Suppose that a battery (a constant emf ε_0) is connected to a circuit of resistance R and inductance L. What current flows?

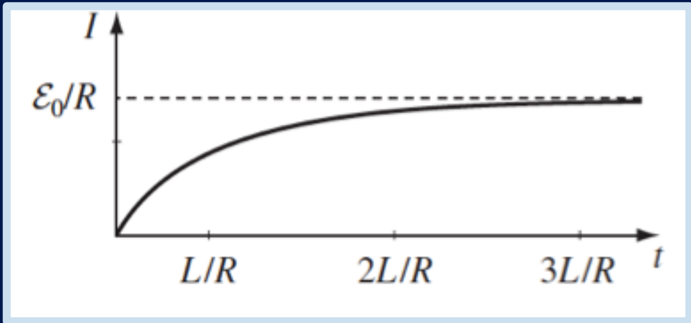
$$\varepsilon_0 - \varepsilon_{back} = IR \rightarrow \varepsilon_0 - L\frac{dI}{dt} = IR$$

$$\rightarrow L\frac{dI}{dt} = \varepsilon_0 - IR \rightarrow \frac{-dI}{\varepsilon_0 - IR} = -\frac{1}{L}dt$$

$$\rightarrow \int_0^I \frac{-RdI}{\varepsilon_0 - IR} = -\frac{R}{L}t \rightarrow \ln\frac{\varepsilon_0 - RI}{\varepsilon_0} = -\frac{R}{L}t$$

$$\ln\frac{\varepsilon_0 - RI}{\varepsilon_0} = -\frac{R}{L}t \rightarrow I = \frac{\varepsilon_0}{R}(1 - e^{-R/Lt})$$





7.2.4 Energy in Magnetic Fields

It takes an energy to let a current flowing in a circuit as you need to overcome the "back emf" from L. You get this energy back when the current is turned off. But while the circuit is not off this energy remains latent in the circuit as energy stored in the magnetic field.

The work done on a unit charge, against the back emf, in one trip around the circuit is $-\varepsilon$ (the minus sign is for the work done against the emf)

The work done per unit time is

$$\frac{dW}{dt} = -\varepsilon I = LI\frac{dI}{dt} \rightarrow W = \frac{1}{2}I^2L$$

The work does not depend on how long it takes current to reach its steady state value, It only depends on L that in turns depends on the geometry of the loop and the final current I.

7.2.4 Energy in Magnetic Fields

$$W = \frac{1}{2}I^2L = \frac{1}{2}I\Phi = \frac{1}{2}I\int \vec{B}\cdot d\vec{a} = \frac{1}{2}I\int (\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \frac{1}{2}I\oint \vec{A}\cdot d\vec{l}$$

$$= \frac{1}{2}\oint \vec{A}\cdot\vec{I}dl = \frac{1}{2}\int \vec{A}\cdot\vec{J}d\tau$$

$$W = \frac{1}{2}\int \vec{A}\cdot\vec{J}d\tau = \frac{1}{2\mu_0}\int \vec{A}\cdot(\vec{\nabla}\times\vec{B})d\tau$$

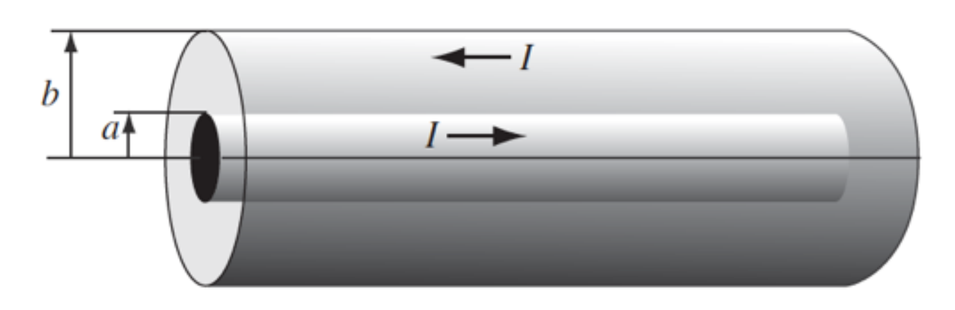
$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

$$W = \frac{1}{2\mu_0}\int \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{\nabla}\cdot(\vec{A}\times\vec{B})d\tau = \frac{1}{2\mu_0}\int \vec{B}\cdot\vec{B}d\tau = \frac{1}{2\mu_0}\int B^2 d\tau$$

7.2.4 Energy in Magnetic Fields

Example: A long coaxial cable carries current I (the current flows down the surface of the inner cylinder, radius a, and back along the outer cylinder, radius b). Find the magnetic energy stored in a section of length l.

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi} \quad for\, a < s < b$$



$$W = \frac{1}{2\mu_0}\int B^2 d\tau = \frac{1}{2\mu_0}\int_0^l \int_0^{2\pi} \int_a^b \left(\frac{\mu_0 I}{2\pi s}\right)^2 s\,ds\,d\phi\,dz = \frac{\mu_0 I^2 l}{4\pi}\ln\frac{b}{a} = \frac{1}{2}I^2 L$$

$$\rightarrow L = \frac{\mu_0 l}{2\pi}\ln\frac{b}{a}$$

7.2.4 Energy in Magnetic Fields

Example: Find the magnetic energy stored in a section of length l of a solenoid of radius R, number of turns n and current passing through it I.

$\vec{B} = \mu_0 n I \ \ inside$

Method 1: $W = \frac{1}{2\mu_0}\int B^2 d\tau = \frac{1}{2\mu_0}\int_0^l \int_0^{2\pi} \int_0^R (\mu_0 n I)^2 s\,ds\,d\phi\,dz$

$= \frac{1}{2}\pi\mu_0 I^2 n^2 l R^2$

Method 2: $W = \frac{1}{2} I^2 L = \frac{1}{2} I^2 (\mu_0 n^2 \pi R^2 l) = \frac{1}{2}\pi\mu_0 I^2 n^2 l R^2$

Method 3: $W_1 = I \oint \vec{A}\cdot d\vec{l} = \frac{1}{2} I \oint \frac{\mu_0 I n R}{2}\hat{\phi}\cdot R d\phi \hat{\phi} = \frac{\mu_0 I^2 n R^2 \pi}{2} \rightarrow W$

$= n l W_1 = \frac{1}{2}\pi\mu_0 I^2 n^2 l R^2$