Electromagnetic Theory II

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Chapter 7: Electrodynamics

- Maxwell's Equations
  - Electrodynamics before Maxwell
  - How Maxwell fixed Ampere's law
  - Magnetic Charge
  - Maxwell's equations in Matter
  - Boundary conditions

7.3.1 Electrodynamics before Maxwell

What we learned up to this point

$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$	Gauss's Law
$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$	Faraday's Law
$\vec{\nabla}\cdot\vec{B} = 0$	No name (no magnetic monopoles)
$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$	Ampere's Law
$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$	Continuity equation (Conservation of charge)
$\vec{\nabla}\cdot\vec{J} = 0$	For steady currents

$$\rightarrow 0 = \vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{E}\right) = -\frac{\partial\vec{\nabla}\cdot\vec{B}}{\partial t} = 0$$

Consistent

$$\rightarrow 0 = \vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{B}\right) = \mu_0\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} \neq 0$$

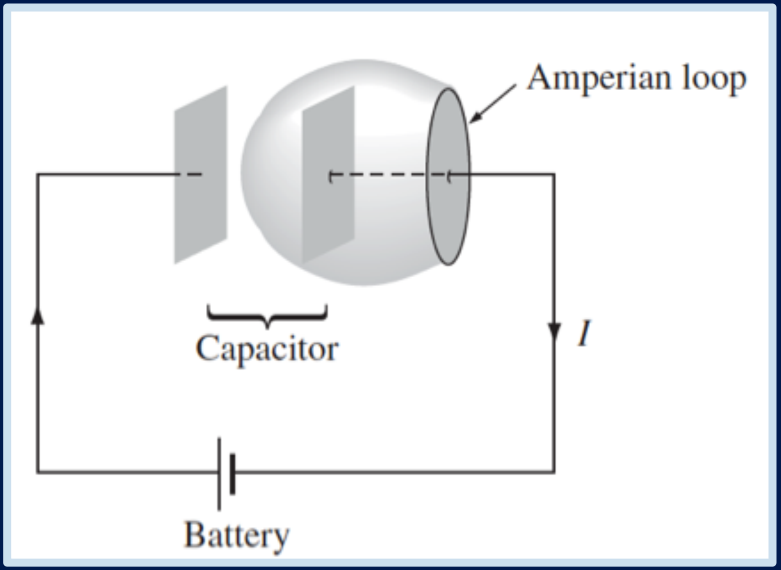
Inconsistent

Ampere's Law is not correct for non-steady currents

7.3.1 Electrodynamics before Maxwell

Lets see this inconsistency from charging a capacitor. Ampere's law gives

$$\int (\nabla\times\vec{B})\cdot d\vec{a} = \oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$



What to apply it to solve the problem, the Amperian loop or Amperian surface. For the flat surface , $I_{enc} = I$, while for the balloon-shaped surface, $I_{enc} = 0$ (No current passes through it)

The inconsistency arises only when charge is piling up somewhere (on the capacitor plates).. Maxwell fixed this inconsistency by purely theoretical arguments in 1861 which was confirmed experimentally with Hertz's experiments on electromagnetic waves. In 1888.

7.3.2 How Maxwell fixed Ampere's law

Starting from the continuity equation and Gauss's law,

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} = -\epsilon_0\frac{\partial\vec{\nabla}\cdot\vec{E}}{\partial t} = \vec{\nabla}\cdot\left(-\epsilon_0\frac{\partial\vec{E}}{\partial t}\right) \rightarrow \vec{\nabla}\cdot\left(\vec{J}+\epsilon_0\frac{\partial\vec{E}}{\partial t}\right) = 0$$

$$\vec{J}\ \textit{is replaced with}\ \vec{J}+\epsilon_0\frac{\partial\vec{E}}{\partial t}\ \textit{in Ampere's law}$$

$$\rightarrow 0 = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0\vec{\nabla}\cdot\left(\vec{J}+\epsilon_0\frac{\partial\vec{E}}{\partial t}\right) = 0$$

Ampere's Law is now consistent

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J} + \mu_0\vec{J}_d$$

Changing Electric Field induced Magnetic Field

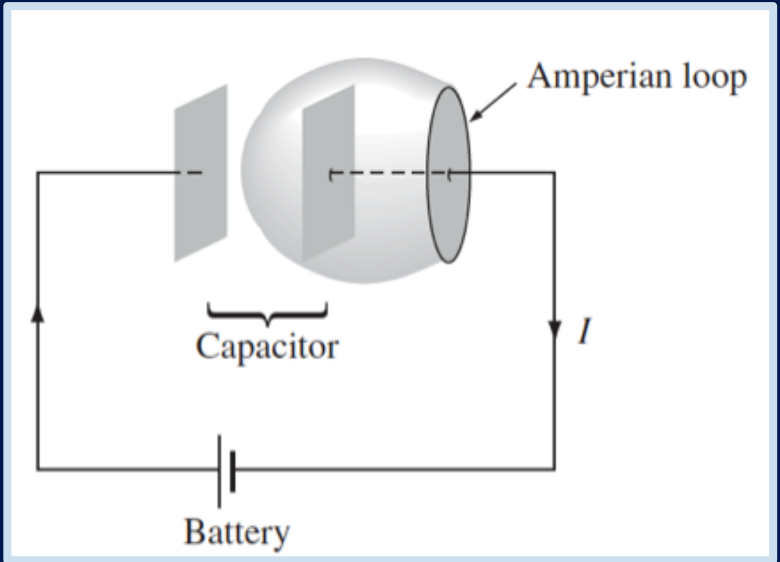
Displacement Current

7.3.2 How Maxwell fixed Ampere's law

Back to charging a capacitor. New Ampere's law gives

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a}$$

$$\vec{E} = \frac{Q}{\epsilon_0 A}\hat{z} \rightarrow \frac{\partial \vec{E}}{\partial t} = \frac{1}{\epsilon_0 A}\frac{dQ}{dt}\hat{z} = \frac{I}{\epsilon_0 A}\hat{z}$$



For the flat surface , $\vec{E} = 0 \; and \; I_{enc} = I \rightarrow$

For the balloon-shaped surface, $I_{enc} = 0$ and $\int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} = \frac{I}{\epsilon_0} \rightarrow \oint \vec{B} \cdot d\vec{l} = \mu_0 I$

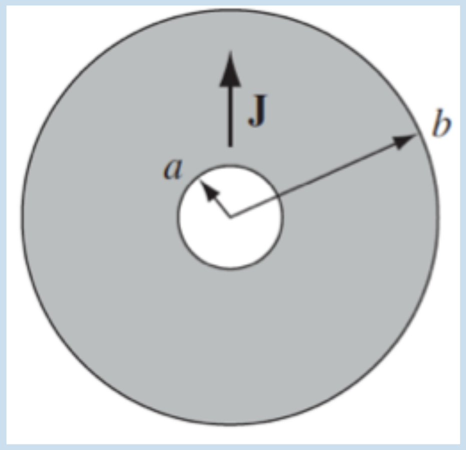
7.3.2 How Maxwell fixed Ampere's law

Example: Imagine two concentric metal spherical shells. The inner one (radius a) carries a charge $Q(t)$, and the outer one (radius b) an opposite charge $-Q(t)$. The space between them is filled with Ohmic material of conductivity σ, so a radial current flows:

$$\vec{J} = \sigma\vec{E} = \frac{\sigma Q}{4\pi\epsilon_0 r^2}\hat{r} \text{ and } I = -\dot{Q} = \int \vec{J}\cdot d\vec{a} = \sigma Q/\epsilon_0$$

This configuration is spherically symmetrical, so the magnetic field has to be zero (the only direction it could possibly point is radial)

$$\vec{\nabla}\cdot\vec{B} = 0 \rightarrow \int \vec{B}\cdot d\vec{a} = 4\pi r^2 B = 0 \rightarrow B = 0$$



This is not a static configuration: Q, E, and J are all functions of time; Ampere and Biot-Savart do not apply.

$$\vec{J}_d = \epsilon_0\frac{\partial\vec{E}}{\partial t} = \frac{\dot{Q}}{4\pi r^2}\hat{r} = -\frac{\sigma Q}{4\pi\epsilon_0 r^2}\hat{r} = -\vec{J}$$

exactly cancels the conduction current

7.3.3 Maxwell Equations

All classical electrodynamics can be studied using

$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\epsilon_0}$	Gauss's Law	Electric fields can be produced either by charges or by changing magnetic fields.
$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$	Faraday's Law	
$\vec{\nabla}\cdot\vec{B}=0$	No name (no magnetic monopoles)	
$\vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$	Ampere's Law with Maxwell's correction. Magnetic fields can be produced either by current or by changing electric fields.	
$\vec{F}=q(\vec{v}\times\vec{B}+\vec{E})$	Lorentz force	

- Maxwell's equations describe how (static or dynamic) charges produce electromagnetic fields, while the force law describes how fields affect charges.
- No need for continuity equation as it can be derived from Maxwell's equations

7.3.4 Magnetic charges

There is no symmetry between E and B in Maxwell equation. If there is a density of magnetic charge (ρ_m) and a current of magnetic charge ($\vec{J}_m$),

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho_e}{\epsilon_0}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \rightarrow \quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} - \mu_0\vec{J}_m$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \rightarrow \quad \vec{\nabla}\cdot\vec{B} = \mu_0\rho_m$$

$$\vec{\nabla}\times\vec{B} = \mu_0\overrightarrow{J_e} + \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$$

Note what happens when we replace $\vec{E}$ by $\vec{B}$ and $\vec{B}$ by $-\epsilon_0\mu_0\vec{E}$

- Maxwell's equations suggest magnetic charge to exist. No one has ever found any magnetic charge. Apparently God just didn't make any magnetic charge.
- Dirac showed that the existence of magnetic charge would explain why electric charge is quantized.

7.3.5 Maxwell equations in matter

The total charge density,

$$\rho = \rho_f + \rho_b = \rho_f - \vec{\nabla}\cdot\vec{P} \rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} = \frac{1}{\epsilon_0}\left(\rho_f - \vec{\nabla}\cdot\vec{P}\right) \rightarrow \vec{\nabla}\cdot\left(\epsilon_0\vec{E} + \vec{P}\right) = \rho_f = \vec{\nabla}\cdot\vec{D}$$

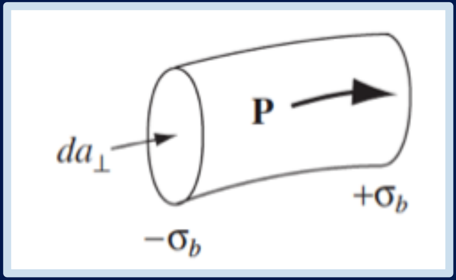
The total current density,

Polarization current density

$$\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_p = \overrightarrow{J_f} + \vec{\nabla}\times\vec{M} + \frac{\partial\vec{P}}{\partial t}$$

$$\rightarrow \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} = \mu_0\left(\vec{J}_f + \vec{\nabla}\times\vec{M} + \frac{\partial\vec{P}}{\partial t}\right) + \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$$

$$\rightarrow \vec{\nabla}\times\left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right) = \vec{J}_f + \frac{\partial}{\partial t}\left(\epsilon_0\vec{E} + \vec{P}\right) = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}$$



7.3.5 Maxwell equations in matter

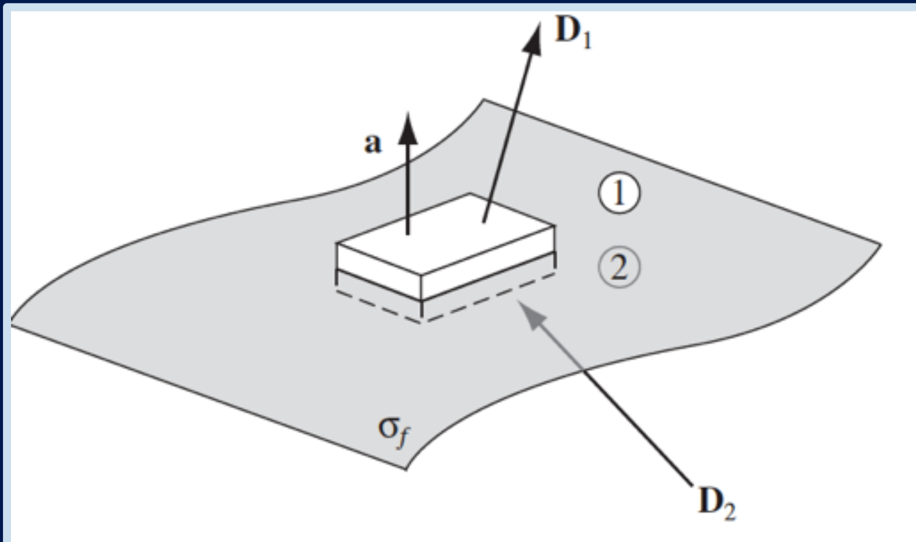
$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\epsilon_0} \quad \rightarrow \quad \vec{\nabla}\cdot\vec{D}=\rho_f$$

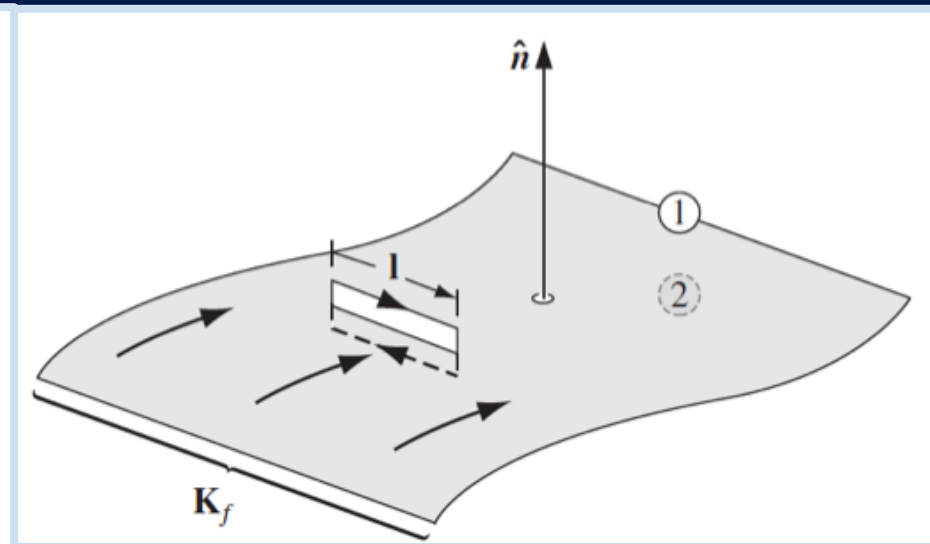
$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \quad \rightarrow \quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B}=0 \quad \rightarrow \quad \vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \quad \rightarrow \quad \vec{\nabla}\times\vec{H}=\vec{J_f}+\frac{\partial\vec{D}}{\partial t}$$

7.3.6 Boundary Conditions





$$D_1^{\perp} - D_2^{\perp} = \sigma_f$$

$$B_2^{\perp} - B_1^{\perp} = 0$$

(i) $\oint_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$

(ii) $\oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a} = 0$

over any closed surface $\mathcal{S}$.

(iii) $\oint_{\mathcal{P}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a}$

(iv) $\oint_{\mathcal{P}} \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}} + \frac{d}{dt} \int_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{a}$

for any surface $\mathcal{S}$ bounded by the closed loop $\mathcal{P}$.

$$E_2^{\parallel} - E_1^{\parallel} = 0$$

$$\vec{H}_1^{\parallel} - \vec{H}_2^{\parallel} = K_f \times \hat{n}$$