

Problems and Solutions in a Graduate Course in Classical Electrodynamics (1)

Raza M. Syed

Department of Physics, Northeastern University, 360 Huntington Ave.,
Boston, MA 02115-5000.

ABSTRACT

The following is the very first set of the series in 'Problems and Solutions in a Graduate Course in Classical Electrodynamics'. In each of the sets of the problems we intend to follow a theme, which not only makes it unique but also deals with the investigation of a certain class of problems in a thorough and rigorous manner. Figures are not included.

Two Point Charge System

Two point charges q and λq and masses m and σm , respectively in vacuum are located at the points $A(a, 0)$ and $B(\mu a, 0)$ in the xy plane as shown in the figure. Assume throughout that $a > 0$ and $q > 0$.

PROBLEM 1

- I. A. If the sum of the charges is fixed, what value of λ will maximize the electric force between them? Is the force attractive or repulsive?
- B. If the midpoint of the line joining the charges is fixed, what value of μ will maximize the electric force between them?
- II. A. Show that the equation for the lines of force in the xy plane is given by

$$\frac{\lambda(x - \mu a)}{\sqrt{(x - \mu a)^2 + y^2}} + \frac{\lambda(x - a)}{\sqrt{(x - a)^2 + y^2}} = C,$$

where C is a constant.

- B. (i) Show that the equation for the line of force in part II.A. can be written as

$$\cos \theta_1 + \lambda \cos \theta_2 = C.$$

Interpret the angles θ_1 and θ_2 geometrically.

- (ii) Find the range of values of C for which
- (a) the equation in part II.(B.)(i) has a solution;
 - (b) the lines of force pass through charge q ;
 - (c) the lines of force pass through charge λq .
- C. What are the equations for the lines of force in the xz and yz planes?
- D. (i) Using the result of part II.A., show that the equation for the lines of force of an electric dipole is

$$\sqrt{(x^2 + y^2)^3} = -\frac{2a}{C}y^2.$$

- (ii) Hence, show that the equation describing the equipotential lines of an electric dipole is

$$\sqrt{(x^2 + y^2)^3} = Kx,$$

where K is a constant.

- III. Consider a plane, which is a perpendicular bisector of the line joining the charges.

- A. (i) For a fixed value of μ and λ , show that the electric field attains its maximum on a circle on the plane whose radius is

$$\frac{a}{2} \left| \frac{1 - \mu}{1 + \lambda} \right| \sqrt{3 - (\lambda - 2)^2}.$$

(ii) Find the magnitude of the electric field everywhere on the circle.

(iii) State the necessary restrictions on the parameters μ and λ .

- B. (i) With λ fixed, show that the electric field is maximized, if the parameter μ is chosen on a circle on the plane whose radius is

$$\frac{a |(1 - \mu)(1 - \lambda)|}{2 \sqrt{3 - (\lambda + 2)^2}}.$$

(ii) Find the magnitude of the electric field in this case.

(iii) State the necessary restrictions on the parameter λ .

- C. (i) With μ fixed, show that the electric field is maximized, if the parameter λ is determined on a circle on the plane whose radius is

$$\frac{a}{2} |(1 - \mu)| \sqrt{\frac{1 - \lambda}{1 + \lambda}}.$$

(ii) What is the corresponding magnitude of the maximum electric field.

- IV. A. Show that the electric flux passing through the circle described in parts III.(A.)(i), III.(B.)(i) and III.(C.)(i) is given by

$$2\pi q \sqrt{\frac{2}{3}} \left(\frac{1 - \lambda}{\sqrt{\lambda}} \right) \left[\pm \lambda - \sqrt{6\lambda} \pm 1 \right],$$

$$2\pi q \sqrt{\frac{2}{3}} \left(\frac{1 - \lambda}{\sqrt{-\lambda}} \right) \left[\pm \sqrt{3 - (\lambda + 2)^2} - 1 \right],$$

$$2\sqrt{2}\pi q (1 - \lambda) \left[\pm \sqrt{1 + \lambda} - \sqrt{2} \right],$$

respectively. The \pm sign corresponds to $\mu < 1$ and $\mu > 1$, respectively.

- B. Show that if,

$$\lambda = - \left[\frac{\frac{a}{\sqrt{a^2 + b^2}} - 1}{\frac{\mu a}{\sqrt{(\mu a)^2 + b^2}} - 1} \right],$$

then there is no net flux passing through the circle $y^2 + z^2 = b^2$ at $x = 0$.

V. A. Show that the electric field is zero (*neutral point*) at

$$\left(a \left[\frac{\mu \pm \sqrt{|\lambda|}}{1 \pm \sqrt{|\lambda|}} \right], 0, 0 \right).$$

B. Find the range of values of μ and λ for which each of the location of the neutral points is valid.

VI. A. Show that the equipotential surface for which the electric potential is zero is a sphere on $\overline{I_1 I_2}$ as diameter, where

$$\frac{AI_1}{BI_1} = \frac{1}{|\lambda|} = \frac{AI_2}{BI_2}.$$

That is, the points I_1 and I_2 divide \overline{AB} internally and externally in the ratio $1 : |\lambda|$

B. How would the answer to part VI.A. change, when two charges of opposite signs and equal magnitudes are considered?

C. State the necessary conditions on the parameters μ and λ in order for the the result in part VI.A. to be valid.

VII. A. Show that the asymptote (tangent at infinity) to any line of force must always pass through a fixed point G on \overline{AB} such that

$$\frac{AG}{BG} = |\lambda|.$$

B. What conclusion can you draw from the result of part VII.A.?

C. Show that the point G is the “center of gravity” of the charges.

VIII. Assume $\lambda > 0$ for this part of the problem.

A. Show that for a line of force which starts from charge q making an angle α with \overline{AB} satisfies

$$\cos^2 \left(\frac{1}{2} \theta_1 \right) + |\lambda| \cos^2 \left(\frac{1}{2} \theta_2 \right) = \cos^2 \left(\frac{1}{2} \alpha \right),$$

\forall values of θ_1 and θ_2 [see part II.(B.)(i)].

B. (i) Show that the asymptote to the line of force considered in part VIII.A. makes an angle

$$2 \cos^{-1} \left[\frac{1}{\sqrt{1+|\lambda|}} \cos \left(\frac{1}{2} \alpha \right) \right]$$

with \overline{AB} .

(ii) For what values of α will the result in part VIII.(B.)(i) hold true?

(iii) Show that the equation of this asymptote is

$$y = \pm \sqrt{\left(\frac{1 + |\lambda|}{\cos \alpha - |\lambda|}\right)^2 - 1} \left[x - \left(\frac{1 + \mu|\lambda|}{1 + |\lambda|}\right) a \right].$$

C. How would the result of parts VIII.A., VIII.(B.)(i) and VIII.(B.)(ii) change, if instead a line of force emanating from charge λq making an angle α with \overline{AB} , is considered?

IX. Assume $\lambda < 0$ for this part of the problem.

A. Show that for a line of force which starts from charge q making an angle α with \overline{AB} satisfies

$$\cos^2\left(\frac{1}{2}\theta_1\right) - |\lambda| \cos^2\left(\frac{1}{2}\theta_2\right) = \cos^2\left(\frac{1}{2}\alpha\right),$$

\forall values of θ_1 and θ_2 .

B. (i) If the line of force considered in part IX.A. is to end at charge λq , then show that it must make an angle

$$2 \cos^{-1} \left[\frac{1}{\sqrt{|\lambda|}} \sin \left(\frac{1}{2}\alpha \right) \right]$$

at B with \overline{AB} .

(ii) Find the range of values of α and λ for which the result in part IX.(B.)(i) is valid.

C. (i) If the line of force considered in part IX.A., is to go to infinity, then show that the asymptote to this line of force must make an angle

$$2 \cos^{-1} \left[\frac{1}{\sqrt{1 - |\lambda|}} \cos \left(\frac{1}{2}\alpha \right) \right]$$

with \overline{AB} .

(ii) Find the range of values of α and λ for which the result in part IX.(C.)(i) is valid.

(iii) Show that the equation of this asymptote is

$$y = \pm \sqrt{\left(\frac{1 - |\lambda|}{\cos \alpha + |\lambda|}\right)^2 - 1} \left[x - \left(\frac{1 - \mu|\lambda|}{1 - |\lambda|}\right) a \right].$$

D. (i) If the line of force considered in part IX.A., is an *extreme* line of force from charge q to charge λq (a line of force that separates the line of force going from charge q to λq from those going from charge q to infinity), then

$$\alpha = \cos^{-1} (1 - 2|\lambda|).$$

(ii) For what values of λ will the result in part IX.(D.)(i) hold true?

- E. (i) If the line of force considered in part IX.A., is to meet the plane that bisects \overline{AB} at right angles, in C, then show that the angle between \overline{AB} and \overline{AC} is

$$2 \sin^{-1} \left[\frac{1}{\sqrt{1 + |\lambda|}} \sin \left(\frac{1}{2} \alpha \right) \right].$$

- (ii) For what values of α will the result in part IX.(E.)(i) hold true.
 (iii) Show that this line of force which crosses the plane is at a distance from \overline{AB} not greater than

$$|\mu - 1| \frac{\sqrt{|\lambda|}}{1 - |\lambda|} a.$$

- F. (i) Show that for a line of force which terminates at charge λq making an angle α with \overline{AB} satisfies

$$|\lambda| \sin^2 \left(\frac{1}{2} \theta_2 \right) - \sin^2 \left(\frac{1}{2} \theta_1 \right) = |\lambda| \sin^2 \left(\frac{1}{2} \alpha \right),$$

\forall values of θ_1 and θ_2 .

- (ii) If the line of force considered in part IX.(F.)(i), is restricted to have been originated from charge q , then show that it must make an angle

$$2 \sin^{-1} \left[\sqrt{|\lambda|} \cos \left(\frac{1}{2} \alpha \right) \right],$$

at A with \overline{AB} . For what values of α and λ will this result hold true.

- (iii) If the line of force considered in IX.(F.)(i), is constrained to have been originated from infinity, then show that the asymptote to this line of force must make an angle

$$2 \sin^{-1} \left[\sqrt{\frac{|\lambda|}{|\lambda| - 1}} \sin \left(\frac{1}{2} \alpha \right) \right],$$

with \overline{AB} . Find the range of values for α and λ for which this result is valid.

- (iv) If the line of force considered in IX.(F.)(i), is an extreme line of force, then show that

$$\alpha = \cos^{-1} \left(\frac{2}{|\lambda|} - 1 \right).$$

Find the range of values for λ for which this result is valid.

- (v) If the line of force considered in part IX.(F.)(i), has crossed the plane that bisects \overline{AB} at right angles, in C, then show that the angle between \overline{AB} and \overline{AC} is

$$2 \cos^{-1} \left[\sqrt{\frac{|\lambda|}{1 + |\lambda|}} \cos \left(\frac{1}{2} \alpha \right) \right].$$

Find the range of values for α for which this result is valid. At what maximum distance from \overline{AB} can this line of force cross the plane.

X. A. Assume $\lambda > 0$ for this part of the problem.

(i) Show that the (limiting) line of force through the neutral point on \overline{AB} satisfies

$$\sin^2\left(\frac{1}{2}\theta_1\right) = |\lambda| \cos^2\left(\frac{1}{2}\theta_2\right),$$

\forall values of θ_1 and θ_2 .

(ii) Show that this limiting line of force has the asymptote

$$y = \pm \frac{2\sqrt{|\lambda|}}{1-|\lambda|} \left[x - \left(\frac{1+\mu|\lambda|}{1+|\lambda|} \right) a \right].$$

B. Assume $\lambda < 0$ for this part of the problem.

(i) Show that the (limiting) line of force through the neutral point on \overline{AB} satisfies

$$\begin{aligned} \sin^2\left(\frac{1}{2}\theta_1\right) &= |\lambda| \sin^2\left(\frac{1}{2}\theta_2\right) & \text{if } -1 < \lambda < 0, \\ \cos^2\left(\frac{1}{2}\theta_1\right) &= |\lambda| \cos^2\left(\frac{1}{2}\theta_2\right) & \text{if } \lambda < -1, \end{aligned}$$

\forall values of θ_1 and θ_2 .

(ii) (a) In the case when $-1 < \lambda < 0$, find the angle w.r.t. \overline{AB} at A, the line of force through neutral point, leaves charge q .

(b) In the case when $\lambda < -1$, find the angle w.r.t. \overline{AB} at B, the line of force through neutral point, ends at charge λq .

(iii) What conclusion can you draw from the results of part **X.(B.)(ii)**

XI. Assume $\lambda < 0$ for this part of the problem.

A. In particular, consider the situation when $-1 < \lambda < 0$. Further let P be a point on the line of force through the neutral point, N. Show that, if bisectors of the $\angle PAN$ and $\angle PBN$ meet at point Q, the locus of Q is the circle on \overline{MN} as diameter, where point M lies on \overline{AB} and

$$\frac{AM}{BM} = \frac{1}{\sqrt{|\lambda|}}.$$

Compare this ratio with $\frac{AN}{BN}$.

B. Show that the locus of the points at which the lines of force are parallel to \overline{AB} is a sphere of radius

$$\sqrt[3]{|\lambda|} \left| \frac{\mu - 1}{1 - \sqrt[3]{|\lambda|^2}} \right| a.$$

- XII.** A. Show that the ratio of the component of electric field perpendicular to \overline{AB} to the component of electric field parallel to \overline{AB} at a point on the limiting line of force close to neutral point is approximately

$$\begin{aligned} -\frac{2}{3} \left(\frac{1}{\pi - \theta_1 - \theta_2} \right) & \quad \text{if } \lambda > 0, \\ -\frac{2}{3} \left(\frac{1}{\theta_1 + \theta_2} \right) & \quad \text{if } -1 < \lambda < 0, \\ -\frac{2}{3} \left(\frac{1}{\theta_1 + \theta_2 - 2\pi} \right) & \quad \text{if } \lambda < -1. \end{aligned}$$

- B. At what angle does the line of force through N crosses \overline{AB} .

- XIII.** A. Show that the form of equipotential surfaces in the neighborhood of the neutral point are

- hyperboloid of one sheet
- or
- hyperboloid of two sheets
- or
- right circular cone

- B. Show that the equipotential near the neutral point makes a constant angle

$$\tan^{-1}(\sqrt{2})$$

with \overline{AB}

- C. Show that the lines of force near the neutral point are the curves

$$y^2 \left[x - \left(\frac{\mu \pm \sqrt{|\lambda|}}{1 \pm \sqrt{|\lambda|}} \right) a \right] = \text{Constant}$$

PROBLEM 2

You may assume throughout this problem that $\mu > 1$.

- I.** A. If the charges in vacuum are now released from rest from their initial positions,
(i) show that at time t , their relative position $x(t) \equiv x_2(t) - x_1(t)$ is such that

$$t = \sqrt{\frac{ma}{2q^2} \frac{\sigma(\mu - 1)}{|\lambda|(\sigma + 1)}} \times \begin{cases} \frac{a}{2}(\mu - 1) \left[\pi - \sin^{-1} \sqrt{\frac{2x}{a(\mu - 1)}} \right] - \sqrt{x[a(\mu - 1) - x]} & \text{if } \lambda < 0 \\ \frac{1}{2}a(\mu - 1) \cosh^{-1} \left[\frac{2x}{a(\mu - 1)} - 1 \right] + \sqrt{x[x - a(\mu - 1)]} & \text{if } \lambda > 0 \end{cases}$$

where x_2 and x_1 are the positions of the charge λq and q at time t , respectively.

(ii) For the case of charges with unlike signs, find the collision time.

B. Now assume that charges are immersed in a medium where the Coulomb force ($F_{1,2}$) is proportional to the velocity ($v_{1,2}$) rather than acceleration of the particle: $F_2 = \beta v_2$ and $F_1 = v_1$, where β is a proportionality constant. The charges are now released from rest from their initial positions.

(i) Show that the relative position x and time t , in this case are related by

$$t = \frac{1}{3q^2} \frac{\beta}{|\lambda|(1+\beta)} \times \begin{cases} \left[\frac{a^3}{8}(\mu-1)^3 - x^3 \right] & \text{if } \lambda < 0 \\ \left[x^3 - \frac{a^3}{8}(\mu-1)^3 \right] & \text{if } \lambda > 0 \end{cases}$$

(ii) For the case of charges with unlike signs, find the collision time.

II. Assume that the charges are connected by means of a light, nonconducting inextensible string.

A. The charge q is fixed while the charge λq ($\lambda > 0$) is free to rotate about q . Show that the minimum velocity with which the charge λq is projected so that it completes one revolution is given by

$$\sqrt{ga(\mu-1)(\sigma+2+2\cos\alpha) - \frac{|\lambda|q^2}{ma(\mu-1)}},$$

where α is the angle the string makes with the downward vertical, just before the charge λq is projected.

B. Take $\lambda < 0$. The entire charge and the string assembly is released and a uniform electric field, $-E_0\hat{x}$ is turned on. Show that

$$\mu = 1 + \frac{1}{a} \sqrt{\frac{q}{E_0} \frac{|\lambda|(1+\sigma)}{\sigma+|\lambda|}},$$

in order for the string to remain taut at all times during the subsequent motion of the assembly.

III. Assume that the charges q and λq ($\lambda < 0$) are held together at the ends of a massless rigid non-conducting rod. The whole arrangement is immersed in a region of a uniform electric field $-E_0\hat{x}$, with the rod constrained to only rotate about an axis through its center and perpendicular to its length. If the rod is rotated through a small angle from its equilibrium position, show that it performs simple harmonic motion with time period

$$2\pi \sqrt{\frac{ma}{2qE_0} \frac{(\mu-1)(\sigma+1)}{|\lambda+1|}}$$

SOLUTIONS

PROBLEM 1

- I. A. We are given that $q + \lambda q \equiv \text{constant } (Q)$. This implies $q = \frac{Q}{1 + \lambda}$. The Coulomb force, F_1 between charges q and λq is given by

$$\begin{aligned} F_1 &= \frac{(q)(\lambda q)}{(a - \mu a)^2} = \left[\frac{Q}{a(1 - \mu)} \right]^2 \frac{\lambda}{(1 + \lambda)^2} \\ \frac{\partial F_1}{\partial \lambda} &= \left[\frac{Q}{a(1 - \mu)} \right]^2 \frac{(1 + \lambda)^2 - 2\lambda(1 + \lambda)}{(1 + \lambda)^4} \\ &= \left[\frac{Q}{a(1 - \mu)} \right]^2 \frac{1 - \lambda}{(1 + \lambda)^3} = 0 \\ &\Rightarrow \boxed{\lambda = 1} \end{aligned}$$

Further, note that :

$$\begin{aligned} \frac{\partial^2 F_1}{\partial \lambda^2} &= \left[\frac{Q}{a(1 - \mu)} \right]^2 \frac{-(1 + \lambda)^3 - 3(1\lambda)(1 + \lambda)^2}{(1 + \lambda)^6} \\ &= 2 \left[\frac{Q}{a(1 - \mu)} \right]^2 \frac{\lambda - 2}{(1 + \lambda)^4} < 0 \text{ for } \lambda = 1 \end{aligned}$$

Thus for $\lambda = 1$, $q = \frac{Q}{2}$. This means the total fixed charge will be divided equally among the the two charges to give the maximum force of *repulsion*:

$$\boxed{F_1|_{\max} = \left[\frac{q}{a(1 - \mu)} \right]^2}$$

- B. If the midpoint of the line joining the two charges is fixed, then $\frac{a + \mu a}{2} = \text{constant } (d)$.

Therefore, $a = \frac{2d}{1 + \mu}$. The Coulomb force, F_2 in this case is

$$\begin{aligned} F_2 &= \frac{(q)(\lambda q)}{(a - \mu a)^2} = \frac{\lambda q^2}{4d^2} \left(\frac{1 - \mu}{1 + \mu} \right)^2 \\ \frac{\partial F_2}{\partial \mu} &= \left(\frac{\lambda q^2}{4d^2} \right) \frac{(1 - \mu)^2 2(1 + \mu) - (1 + \mu)^2 (2\mu - 2)}{(1 - \mu)^4} \\ &= \frac{\lambda q^2}{d^2} \frac{1 + \mu}{(1 - \mu)^3} = 0 \end{aligned}$$

$$\begin{aligned}
& \Rightarrow \boxed{\mu = -1} \\
\text{Further, } \frac{\partial^2 F_2}{\partial \lambda^2} &= \left(\frac{\lambda q^2}{4d^2} \right) \frac{(1-\mu)^3 + 3(1-\mu)^2(1+\mu)}{(1-\mu)^6} \\
&= \left(\frac{2\lambda q^2}{d^2} \right) \frac{\mu+2}{(\mu-1)^4} \\
\frac{\partial^2 F_2}{\partial \lambda^2} \Big|_{\mu=-1} &= \frac{\lambda q^2}{8d^2} \begin{cases} < 0 & \text{if } \lambda < 0 \\ > 0 & \text{if } \lambda > 0 \end{cases}
\end{aligned}$$

Thus for $\mu = -1$, $d = 0$. That is, the charges are placed symmetrically about the origin, giving rise to a maximum *attractive* (*repulsive*) force for $\lambda < 0$ ($\lambda > 0$):

$$\boxed{F_2|_{\max} = -|\lambda| \left(\frac{q}{2a} \right)^2}$$

II. A. The electric field, \mathbf{E} for the system is given by

$$\mathbf{E} = \frac{q(\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3} + \frac{\lambda q(\mathbf{r} - \mathbf{r}_2)}{|\mathbf{r} - \mathbf{r}_2|^3}.$$

The vectors $\mathbf{r}_1 = a\hat{\mathbf{x}}$, $\mathbf{r}_2 = \mu a\hat{\mathbf{x}}$ and $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ are shown in Figure 1.

$$\begin{aligned}
|\mathbf{r} - \mathbf{r}_i| &= [(\mathbf{r} - \mathbf{r}_i) \cdot (\mathbf{r} - \mathbf{r}_i)]^{\frac{1}{2}} = \mathbf{r}^2 - 2\mathbf{r} \cdot \mathbf{r}_i + \mathbf{r}_i^2 \\
\text{Thus, } |\mathbf{r} - \mathbf{r}_1|^3 &= [(x-a)^2 + y^2 + z^2]^{\frac{3}{2}} \\
\text{and } |\mathbf{r} - \mathbf{r}_2|^3 &= [(x-\mu a)^2 + y^2 + z^2]^{\frac{3}{2}}.
\end{aligned}$$

$$\begin{aligned}
\text{Writing, } \mathbf{E} &= E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}} \\
\text{where } E_x &= \frac{q(x-a)}{[(x-a)^2 + y^2 + z^2]^{\frac{3}{2}}} + \frac{\lambda q(x-\mu a)}{[(x-\mu a)^2 + y^2 + z^2]^{\frac{3}{2}}}; \\
E_y &= \frac{qy}{[(x-a)^2 + y^2 + z^2]^{\frac{3}{2}}} + \frac{\lambda qy}{[(x-\mu a)^2 + y^2 + z^2]^{\frac{3}{2}}}; \\
E_z &= \frac{qz}{[(x-a)^2 + y^2 + z^2]^{\frac{3}{2}}} + \frac{\lambda qz}{[(x-\mu a)^2 + y^2 + z^2]^{\frac{3}{2}}}. \tag{1}
\end{aligned}$$

The lines of force represent the direction of electric field in space: $d\mathbf{l} = k\mathbf{E}$ (k is a constant). In rectangular coordinates this equation takes the form,

$$\begin{aligned}
dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}} &= k(E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}) \\
\Rightarrow \frac{dx}{E_x} &= \frac{dy}{E_y} = \frac{dz}{E_z}.
\end{aligned}$$

Therefore, the differential equation for the lines of force in the xy plane ($z = 0$) takes the form

$$\frac{dy}{dx} = \frac{E_y}{E_x}$$

$$\frac{dy}{dx} = \frac{y \left\{ [(x - \mu a)^2 + y^2]^{\frac{3}{2}} + \lambda [(x - a)^2 + y^2]^{\frac{3}{2}} \right\}}{(x - a) [(x - \mu a)^2 + y^2]^{\frac{3}{2}} + \lambda (x - \mu a) [(x - a)^2 + y^2]^{\frac{3}{2}}} \quad (2)$$

$$= \frac{(1 + u^2)^{\frac{3}{2}} + \lambda(1 + v^2)^{\frac{3}{2}}}{v(1 + u^2)^{\frac{3}{2}} + \lambda u(1 + v^2)^{\frac{3}{2}}}, \quad (3)$$

where in the last equation, we have used the following sets of transformations:

$$u = \frac{x - \mu a}{y}, \quad v = \frac{x - a}{y}$$

$$\text{Eliminating, } x : \quad y = \frac{a(\mu - 1)}{v - u} \Rightarrow dy = \frac{a(\mu - 1)(du - dv)}{(v - u)^2}$$

$$\text{Eliminating, } y : \quad x = \frac{a(v\mu - u)}{v - u} \Rightarrow dx = \frac{a(\mu - 1)(vdu - u dv)}{(v - u)^2}$$

$$\text{Thus Eq.(3),} \quad \frac{du - dv}{vdu - u dv} = \frac{(1 + u^2)^{\frac{3}{2}} + \lambda(1 + v^2)^{\frac{3}{2}}}{v(1 + u^2)^{\frac{3}{2}} + \lambda u(1 + v^2)^{\frac{3}{2}}}$$

$$\text{simplifying,} \quad \lambda \frac{du}{(1 + u^2)^{\frac{3}{2}}} = - \frac{dv}{(1 + v^2)^{\frac{3}{2}}}$$

$$\text{integrating,} \quad \lambda \int \frac{du}{(1 + u^2)^{\frac{3}{2}}} = - \int \frac{dv}{(1 + v^2)^{\frac{3}{2}}} + \text{constant } (C)$$

$$\lambda \frac{u}{(1 + u^2)^{\frac{1}{2}}} = C - \frac{v}{(1 + v^2)^{\frac{1}{2}}} \quad [\text{see appendix}].$$

$$\text{And finally,} \quad \boxed{\frac{(x - a)}{[(x - a)^2 + y^2]^{\frac{1}{2}}} + \frac{\lambda(x - \mu a)}{[(x - \mu a)^2 + y^2]^{\frac{1}{2}}} = C}. \quad (4)$$

B. (i) Define θ_1 and θ_2 to be the angles of elevation (see Figure 2) as seen by the charges q and λq , respectively:

$$\boxed{\cos \theta_1 = \frac{(x - a)}{[(x - a)^2 + y^2]^{\frac{1}{2}}}, \quad \boxed{\cos \theta_2 = \frac{(x - \mu a)}{[(x - \mu a)^2 + y^2]^{\frac{1}{2}}}.$$

Then Eq.(4) can be rewritten as

$$\cos \theta_1 + \lambda \cos \theta_2 = C \quad (5)$$

(ii) (a) Note that

$$\begin{aligned}
 & -1 \leq \cos \theta_1 \leq 1 \\
 & -|\lambda| \leq \lambda \cos \theta_2 \leq |\lambda| \\
 \text{adding:} \quad & -1 - |\lambda| \leq \cos \theta_1 + \lambda \cos \theta_2 \leq |1 + \lambda| \\
 \text{Eq.(5)} \Rightarrow \quad & \boxed{-1 - |\lambda| \leq C \leq |1 + \lambda|}
 \end{aligned}$$

(b) We write Eq.(5) as

$$\cos \theta_1 + \lambda \frac{(x - \mu a)}{[(x - \mu a)^2 + y^2]^{\frac{1}{2}}} = C$$

and take the limit as $(x, y) \rightarrow (a, 0)$. Therefore,

$$\begin{aligned}
 \lim_{(x,y) \rightarrow (a,0)} \cos \theta_1 + \lambda \lim_{(x,y) \rightarrow (a,0)} \frac{(x - \mu a)}{[(x - \mu a)^2 + y^2]^{\frac{1}{2}}} &= \lim_{(x,y) \rightarrow (a,0)} C \\
 \lim_{(x,y) \rightarrow (a,0)} \cos \theta_1 &= C - \lambda \operatorname{sgn}(1 - \mu)
 \end{aligned}$$

where we have defined

$$\operatorname{sgn}(1 - \mu) \equiv \frac{1 - \mu}{|1 - \mu|} = \begin{cases} 1; & \mu < 1 \\ -1; & \mu > 1 \end{cases}$$

and since $-1 \leq \cos \theta_1 \leq 1$, we get

$$\boxed{-1 + \lambda \operatorname{sgn}(1 - \mu) \leq C \leq 1 + \lambda \operatorname{sgn}(1 - \mu)}.$$

(c) We now investigate the behavior of $\cos \theta_2$ as $(x, y) \rightarrow (\mu a, 0)$ and find that

$$\begin{aligned}
 \lambda \lim_{(x,y) \rightarrow (\mu a, 0)} \cos \theta_2 + \frac{\mu - 1}{|\mu - 1|} &= C \\
 \lambda \lim_{(x,y) \rightarrow (\mu a, 0)} \cos \theta_2 &= \frac{C + \operatorname{sgn}(1 - \mu)}{\lambda}.
 \end{aligned}$$

Using the fact $-1 \leq \cos \theta_2 \leq 1$, we get

$$\boxed{-1 \leq \frac{C + \operatorname{sgn}(1 - \mu)}{\lambda} \leq 1}.$$

C. Lines of force in the yz plane can be obtained by simply setting $y = 0$ in Eq.(1) to get,

$$\frac{dz}{dx} = \frac{E_z}{E_x} = \frac{y \left\{ [(x - \mu a)^2 + z^2]^{\frac{3}{2}} + \lambda [(x - a)^2 + z^2]^{\frac{3}{2}} \right\}}{(x - a) [(x - \mu a)^2 + z^2]^{\frac{3}{2}} + \lambda (x - \mu a) [(x - a)^2 + z^2]^{\frac{3}{2}}}.$$

This equation is the same as Eq.(2), with the replacement: $y \rightarrow z$. Hence, the solution to this differential equation follows from Eq.(4)

$$\boxed{\frac{(x - a)}{[(x - a)^2 + z^2]^{\frac{1}{2}}} + \frac{\lambda(x - \mu a)}{[(x - \mu a)^2 + z^2]^{\frac{1}{2}}} = C'}$$

where C' is a new constant.

Similarly, the equation in the yz plane can be obtained by letting $x = 0$ in Eq.(1). In this case, the differential equation take a very simple form:

$$\frac{dy}{dz} = \frac{E_y}{E_z} = \frac{y}{z}.$$

Therefore,

$$\int \frac{dy}{y} = \int \frac{dz}{z}$$

$$\ln y = \ln z + \ln C''$$

$$\boxed{y = C'' z} \quad (\text{straight lines}). \quad (6)$$

D. (i) For an electric dipole, we set $\lambda = -1$ and $\mu = -1$ in Eq.(4):

$$\frac{(x + a)}{[x^2 + a^2 + 2ax + y^2]^{\frac{1}{2}}} - \frac{\lambda(x - a)}{[x^2 + a^2 - 2ax + y^2]^{\frac{1}{2}}} = -C.$$

Defining, $r^2 = x^2 + y^2$ expanding this equation in powers of a/r , we get

$$\frac{x + a}{r \left[1 + \left(\frac{a}{r}\right)^2 + \frac{2ax}{r} \right]^{\frac{1}{2}}} - \frac{x - a}{r \left[1 + \left(\frac{a}{r}\right)^2 - \frac{2ax}{r} \right]^{\frac{1}{2}}} = -C.$$

Neglecting terms like $(a/r)^2$ and higher ($a \rightarrow 0$) and using the binomial expansion $(1 \pm \epsilon)^{1/2} \approx 1 \pm 1/2\epsilon$ for $\epsilon \ll 1$, we get

$$\frac{1}{r} \left\{ \frac{x + a}{1 + \frac{ax}{r^2}} - \frac{x - a}{1 - \frac{ax}{r^2}} \right\} = -C$$

$$\frac{2a}{r} \frac{1 - \left(\frac{x}{r}\right)^2}{1 - \left(\frac{a}{r}\right)^2 \left(\frac{x}{r}\right)^2} = -C$$

$$\frac{2a}{\sqrt{x^2 + y^2}} \left[1 - \frac{x^2}{x^2 + y^2}\right] = -C$$

$$\boxed{(x^2 + y^2)^{\frac{3}{2}} = \left(\frac{-2a}{C}\right) y^2.} \quad (7)$$

Eq.(7) can be write in polar coordinates, if we make the transformation: $x = r \cos \theta$ and $y = r \sin \theta$. Then the equation for the lines of force take the more convenient form

$$r = \left(\frac{-2a}{C}\right) \sin^2 \theta.$$

(ii) Taking the natural logs of both sides of Eq.(7), we get

$$\frac{3}{2} \ln(x^2 + y^2) = \ln\left(\frac{-2a}{C}\right) + 2 \ln y.$$

Differentiating this last equation w.r.t. x , we get

$$\frac{3}{x^2 + y^2} \left(x + y \frac{dy}{dx}\right) = \frac{2}{y} \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{3xy}{2x^2 - y^2}. \quad (8)$$

Eq.(8) represents the slope of the lines of force at the point (x, y) . Therefore, the slope of the corresponding equipotential line would be

$$\frac{dy}{dx} = \frac{-1}{\frac{3xy}{2x^2 - y^2}}$$

simplifying,
$$\frac{dy}{dx} = \frac{1}{3} \left(\frac{y}{x}\right) - \frac{2}{3} \left(\frac{x}{y}\right).$$

To separate the variables we make the transformation:

$$w = \frac{y}{x}$$

$$\Rightarrow \frac{dw}{dx} = \frac{x \frac{dy}{dx} - y}{x^2}$$

$$\frac{dy}{dx} = x \frac{dw}{dx} + w.$$

Therefore, the differential equation for the equipotential lines in the wx plane are given by

$$\begin{aligned}
& \frac{1}{3}w - \frac{2}{3}\frac{1}{w} = x\frac{dw}{dx} + w \\
\text{integrating, } & 3 \int \frac{w}{w^2+1} = -2 \int \frac{dx}{x} \\
& \frac{3}{2} \int \frac{dp}{p} = -2 \int \frac{dx}{x} \quad [p \equiv w^2 + 1] \\
& \frac{3}{2} \ln(w^2 + 1) = -2 \ln x + \ln K \\
& \left(\frac{y^2}{x^2} + 1\right)^{\frac{3}{2}} = \frac{K}{x^2} \\
& \boxed{(x^2 + y^2)^{\frac{3}{2}} = Kx}. \tag{9}
\end{aligned}$$

This last equation can be rewritten in polar coordinates as

$$r^2 = K \cos \theta.$$

III. A. (i) Equation for the orthogonal bisecting plane is $x = \frac{(1 + \mu)a}{2}$. With this value for x in Eq.(5), we get

$$\begin{aligned}
E_x &= \frac{4qa(1 - \mu)(\lambda - 1)}{[a^2(1 - \mu)^2 + 4(y^2 + z^2)]^{\frac{3}{2}}} \\
E_y &= \frac{8qy(\lambda + 1)}{[a^2(1 - \mu)^2 + 4(y^2 + z^2)]^{\frac{3}{2}}} \\
E_z &= \frac{8qz(\lambda + 1)}{[a^2(1 - \mu)^2 + 4(y^2 + z^2)]^{\frac{3}{2}}}. \tag{10}
\end{aligned}$$

Now since, $|\mathbf{E}| = (E_x^2 + E_y^2 + E_z^2)^{\frac{1}{2}}$. Therefore,

$$\begin{aligned}
\frac{\partial|\mathbf{E}|}{\partial y} &= \frac{E_x \frac{\partial E_x}{\partial y} + E_y \frac{\partial E_y}{\partial y} + E_z \frac{\partial E_z}{\partial y}}{(E_x^2 + E_y^2 + E_z^2)^{\frac{1}{2}}} \\
\frac{\partial|\mathbf{E}|}{\partial z} &= \frac{E_x \frac{\partial E_x}{\partial z} + E_y \frac{\partial E_y}{\partial z} + E_z \frac{\partial E_z}{\partial z}}{(E_x^2 + E_y^2 + E_z^2)^{\frac{1}{2}}}.
\end{aligned}$$

For maximum value of the electric field,

$$\frac{\partial|\mathbf{E}|}{\partial y} = 0, \quad \frac{\partial|\mathbf{E}|}{\partial z} = 0,$$

and therefore

$$E_x \frac{\partial E_x}{\partial y} + E_y \frac{\partial E_y}{\partial y} + E_z \frac{\partial E_z}{\partial y} = 0 \quad (11)$$

$$E_x \frac{\partial E_x}{\partial z} + E_y \frac{\partial E_y}{\partial z} + E_z \frac{\partial E_z}{\partial z} = 0. \quad (12)$$

The expressions for the derivatives of the component of electric fields are given by,

$$\begin{aligned} \frac{\partial E_x}{\partial y} &= \frac{48qa(1-\mu)(1-\lambda)y}{[a^2(1-\mu)^2 + 4(y^2 + z^2)]^{\frac{5}{2}}} \\ \frac{\partial E_y}{\partial y} &= \frac{8q(1+\lambda)[a^2(1-\mu)^2 + 4(z^2 - 2y^2)]}{[a^2(1-\mu)^2 + 4(y^2 + z^2)]^{\frac{5}{2}}} \\ \frac{\partial E_z}{\partial y} &= \frac{-96q(1+\lambda)yz}{[a^2(1-\mu)^2 + 4(y^2 + z^2)]^{\frac{5}{2}}}. \end{aligned} \quad (13)$$

The other derivatives can be simply obtained by the transformation, $y \rightarrow z$ in Eq.(13). Use of Eq.(13) in Eq.(11) gives

$$\frac{64qy \{-3a^2(\lambda-1)^2(1-\mu)^2 + (\lambda+1)^2[a^2(1-\mu)^2 + 4(z^2 - 2y^2)] - 12(\lambda+1)^2z^2\}}{[a^2(1-\mu)^2 + 4(y^2 + z^2)]^4} = 0,$$

which can be simplified to

$$\begin{aligned} a^2(1-\mu)^2 - 8(y^2 + z^2) &= 3a^2(1-\mu)^2 \frac{(\lambda-1)^2}{(\lambda+1)^2} \\ y^2 + z^2 &= \left[\frac{a(1-\mu)}{2(1+\lambda)} \right]^2 [3 - (\lambda-2)^2] \equiv r_1^2. \end{aligned} \quad (14)$$

Hence, the electric field is maximized on a circle, centered about the origin and lying on the $x = \frac{a(1+\mu)}{2}$ plane. The radius of the circle being

$$\boxed{\frac{a}{2} \left| \frac{1-\mu}{1+\lambda} \right| \sqrt{3 - (\lambda-2)^2}}. \quad (15)$$

Use of Eq.(12) gives an identical result.

- (ii) It is first useful to compute the quantity, $a^2(1-\mu)^2 + 4(y^2 + z^2)$. Using the expression for $y^2 + z^2$ from Eq.(14), we get for $a^2(1-\mu)^2 + 4(y^2 + z^2) = 6\lambda a^2 \left(\frac{1-\mu}{1+\lambda} \right)^2$. Using this result in Eqs.(10), we get

$$\begin{aligned} E_x^2|_{\max} &= \frac{2q^2(\lambda-1)^2(\lambda+1)^6}{27a^4(1-\mu)^4\lambda^3} \\ (E_y^2 + E_z^2)|_{\max} &= \frac{2q^2(-\lambda^2 + 4\lambda - 1)(\lambda+1)^6}{27a^4(1-\mu)^4\lambda^3}. \end{aligned}$$

Therefore,

$$|\mathbf{E}|_{\max} = \frac{2}{3\sqrt{3}} \frac{q}{a^2(1-\mu)^2} \left| \frac{(\lambda+1)^3}{\lambda} \right|.$$

(iii) Looking at the expressions for the radius and $|\mathbf{E}|$, we conclude that

$$\begin{aligned} \lambda \neq 0, \quad \lambda \neq -1 \quad \& \quad 3 - (\lambda - 2)^2 > 0 \\ \Rightarrow \quad \lambda \in \left(2 - \sqrt{3}, 2 + \sqrt{3} \right). \end{aligned}$$

B. (i) From the condition $\frac{\partial |\mathbf{E}|}{\partial \mu} = 0$, one can derive an equation similar to Eqs.(11) and (12):

$$E_x \frac{\partial E_x}{\partial \mu} + E_y \frac{\partial E_y}{\partial \mu} + E_z \frac{\partial E_z}{\partial \mu} = 0. \quad (16)$$

Taking the derivatives of E_x , E_y and E_z w.r.t μ in Eqs.(10), we have

$$\begin{aligned} \frac{\partial E_x}{\partial \mu} &= \frac{8qa(\lambda-1)[a^2(1-\mu)^2 - 2(y^2+z^2)]}{[a^2(1-\mu)^2 + 4(y^2+z^2)]^{\frac{5}{2}}} \\ \frac{\partial E_y}{\partial \mu} &= \frac{24qa^2(1+\lambda)(1-\mu)y}{[a^2(1-\mu)^2 + 4(y^2+z^2)]^{\frac{5}{2}}} \\ \frac{\partial E_z}{\partial \mu} &= \frac{24qa^2(1+\lambda)(1-\mu)z}{[a^2(1-\mu)^2 + 4(y^2+z^2)]^{\frac{5}{2}}}. \end{aligned} \quad (17)$$

Use of Eqs.(10) and (17) in Eq.(16) gives,

$$\frac{32qa^2(1-\mu)\{(\lambda-1)^2[a^2(1-\mu)^2 - 2(y^2+z^2)] + 6(\lambda+1)^2(y^2+z^2)\}}{[a^2(1-\mu)^2 + 4(y^2+z^2)]^4} = 0.$$

After simple algebraic manipulation, we arrive at

$$y^2 + z^2 = \left[\frac{a(1-\mu)(1-\lambda)}{2} \right]^2 \frac{1}{3 - (\lambda+2)^2} \equiv r_2^2. \quad (18)$$

where,

$$r_2 = \frac{a}{2} \frac{|(1-\mu)(1-\lambda)|}{\sqrt{3 - (\lambda+2)^2}}. \quad (19)$$

From Eq.(18), one can compute μ for a particular value of y and z :

$$\mu = 1 \pm \frac{2}{a|1-\lambda|} \sqrt{(y^2+z^2)[3 - (\lambda+2)^2]}.$$

(ii) As before we first compute $a^2(1 - \mu)^2 + 4(y^2 + z^2)$ using the expression for $y^2 + z^2$ from Eq.(18) to get $a^2(1 - \mu)^2 + 4(y^2 + z^2) = 6\lambda a^2 \frac{(1 - \mu)^2}{(\lambda + 2)^2 - 3}$. Using this result in Eqs.(10), we get

$$\begin{aligned} E_x^2|_{\max} &= \frac{2q^2[(\lambda + 2)^2 - 3]^3(\lambda - 1)^2}{27a^4(1 - \mu)^4} \\ (E_y^2 + E_z^2)|_{\max} &= \frac{-2q^2[3 - (\lambda + 2)]^2(\lambda - 1)^2(\lambda + 1)^2}{27a^4(1 - \mu)^4\lambda^3}. \end{aligned}$$

Finally,
$$|\mathbf{E}|_{\max} = \frac{2}{3\sqrt{3}} \frac{q}{a^2(1 - \mu)^2} \left| \frac{[3 - (\lambda + 2)]^2(\lambda - 1)}{\lambda} \right|.$$

(iii) From the results of the previous two parts, we infer that

$$\begin{aligned} \lambda \neq 0, \quad \lambda \neq 1 \quad \& \quad 3 - (\lambda + 2)^2 > 0 \\ \Rightarrow \quad \lambda \in \left(-2 - \sqrt{3}, -2 + \sqrt{3} \right). \end{aligned}$$

C. (i) In this case the extremum condition, $\frac{\partial |\mathbf{E}|}{\partial \lambda} = 0$, leads to

$$E_x \frac{\partial E_x}{\partial \lambda} + E_y \frac{\partial E_y}{\partial \lambda} + E_z \frac{\partial E_z}{\partial \lambda} = 0. \quad (20)$$

Taking the derivatives of E_x , E_y and E_z w.r.t. λ in Eqs.(10), we have

$$\begin{aligned} \frac{\partial E_x}{\partial \lambda} &= \frac{4qa(1 - \mu)}{[a^2(1 - \mu)^2 + 4(y^2 + z^2)]^{\frac{3}{2}}} \\ \frac{\partial E_y}{\partial \lambda} &= \frac{8qy}{[a^2(1 - \mu)^2 + 4(y^2 + z^2)]^{\frac{3}{2}}} \\ \frac{\partial E_z}{\partial \lambda} &= \frac{8qz}{[a^2(1 - \mu)^2 + 4(y^2 + z^2)]^{\frac{3}{2}}}. \end{aligned} \quad (21)$$

Use of Eqs.(10) and (21) in Eq.(20) gives,

$$\frac{16q^2 [a^2(1 - \mu)^2(\lambda - 1) + 4(\lambda + 1)(y^2 + z^2)]}{[a^2(1 - \mu)^2 + 4(y^2 + z^2)]^3} = 0.$$

Simplifying this expression gives,

$$y^2 + z^2 \equiv r_3^2 \quad (22)$$

$$\boxed{r_3 = \frac{a}{2} |(1 - \mu)| \sqrt{\frac{1 - \lambda}{1 + \lambda}}}. \quad (23)$$

From Eqs.(22) and (23), one can compute λ for a particular value of y and z :

$$\lambda = \frac{a^2(1 - \mu)^2 - 4(y^2 + z^2)}{a^2(1 - \mu)^2 + 4(y^2 + z^2)}$$

(ii) Using Eqs.(22) and (23) in Eq.(10), we get

$$\begin{aligned} E_x^2|_{\max} &= \frac{2q^2(\lambda - 1)^2(\lambda + 1)^3}{a^4(1 - \mu)^4} \\ (E_y^2 + E_z^2)|_{\max} &= \frac{2q^2(\lambda + 1)^5}{a^4(1 - \mu)^4}. \end{aligned}$$

Now since, $|\mathbf{E}|_{\max} = \sqrt{E_x^2|_{\max} + E_y^2|_{\max} + E_z^2|_{\max}}$

$$\boxed{|\mathbf{E}|_{\max} = \frac{2q}{a^2(1 - \mu)^2} \sqrt{(\lambda^2 + 1)(\lambda + 1)^3}} \quad (24)$$

IV. A. The electric flux, N passing through the circles (see Figure 3)

$$y^2 + z^2 = R^2 \equiv \begin{cases} r_1^2 &= \left[\frac{a(1 - \mu)}{2(1 + \lambda)} \right]^2 [3 - (\lambda - 2)^2] \\ r_2^2 &= \left[\frac{a(1 - \mu)(1 - \lambda)}{2} \right]^2 \frac{1}{3 - (\lambda + 2)^2} \\ r_3^2 &= \left[\frac{a(1 - \mu)}{2} \right]^2 \frac{1 - \lambda}{1 + \lambda} \end{cases} \quad (25)$$

at $x = \frac{a(1 + \mu)}{2}$, can be readily computed from

$$\begin{aligned} N &= \iint \mathbf{D} \cdot d\mathbf{s}, \\ \text{where } \mathbf{D} &= \mathbf{E} \quad \text{and} \quad d\mathbf{s} = \hat{\mathbf{x}} \, dydz \\ \text{Therefore } N &= \iint E_x|_{x = \frac{a(1 + \mu)}{2}} \, dydz. \end{aligned}$$

Using Eq.(10), we obtain

$$\begin{aligned}
N &= 2[4qa(1-\mu)(\lambda-1)] \int_{y=0}^{y=R} dy \int_{z=-\sqrt{R^2-y^2}}^{z=+\sqrt{R^2-y^2}} \frac{dz}{[a^2(1-\mu)^2+4(y^2+z^2)]^{\frac{3}{2}}} \\
&= 16qa(1-\mu)(\lambda-1) \int_{y=0}^{y=R} \frac{dy}{a^2(1-\mu)^2+4y^2} \left\{ \frac{z}{[a^2(1-\mu)^2+4(y^2+z^2)]^{\frac{1}{2}}} \right\}_{z=-\sqrt{R^2-y^2}}^{z=+\sqrt{R^2-y^2}} \\
&= \frac{8qa(1-\mu)(\lambda-1)}{[a^2(1-\mu)^2+4R^2]^{\frac{1}{2}}} \int_{y=0}^{y=R} \frac{\sqrt{R^2-y^2}}{\left[\frac{a(1-\mu)}{2}\right]^2+y^2} dy \\
&= \frac{8qa(1-\mu)(\lambda-1)}{[a^2(1-\mu)^2+4R^2]^{\frac{1}{2}}} \left\{ -\sin^{-1}\left(\frac{y}{R}\right) - \frac{\sqrt{R^2+\frac{a^2(1-\mu)^2}{4}}}{\frac{a(1-\mu)}{2}} \sin^{-1}\left[\frac{\frac{a(1-\mu)}{2}}{R} \sqrt{\frac{R^2-y^2}{\frac{a^2(1-\mu)^2}{4}+y^2}}\right] \right\}_{y=0}^{y=R} \\
&= 4\pi q(\lambda-1) \left\{ 1 - \frac{a(1-\mu)}{[a^2(1-\mu)^2+4R^2]^{\frac{1}{2}}} \right\}, \tag{26}
\end{aligned}$$

where in the second and fourth steps above, we have made use of Eqs.(#) and (#), respectively, from the appendix. One can now fully compute N in Eq.(26) for $R = r_1, r_2$ and r_3 given by Eq.(25). Thus

$$\begin{aligned}
\frac{a(1-\mu)}{[a^2(1-\mu)^2+4r_1^2]^{\frac{1}{2}}} &= \operatorname{sgn}(1-\mu) \frac{|1+\lambda|}{\sqrt{6\lambda}} \\
\frac{a(1-\mu)}{[a^2(1-\mu)^2+4r_2^2]^{\frac{1}{2}}} &= \operatorname{sgn}(1-\mu) \sqrt{\frac{3-(\lambda+2)^2}{-6\lambda}} \\
\frac{a(1-\mu)}{[a^2(1-\mu)^2+4r_3^2]^{\frac{1}{2}}} &= \operatorname{sgn}(1-\mu) \sqrt{\frac{1+\lambda}{2}}.
\end{aligned}$$

Finally,

$$\boxed{N_1 = 2\sqrt{\frac{2}{3}}\pi q \left(\frac{1-\lambda}{\sqrt{\lambda}}\right) \left[\operatorname{sgn}(1-\mu) (1+\lambda) - \sqrt{6\lambda}\right]}$$

$$\boxed{N_2 = 2\sqrt{\frac{2}{3}}\pi q \left(\frac{1-\lambda}{\sqrt{-\lambda}}\right) \left[\operatorname{sgn}(1-\mu) \sqrt{3-(\lambda+2)^2} - 1\right]}$$

$$\boxed{N_3 = 2\sqrt{2}\pi q (1-\lambda) \left[\operatorname{sgn}(1-\mu) \sqrt{1+\lambda} - \sqrt{2}\right]}.$$

B. For this part of the problem, we first calculate the flux passing through the circle $y^2 + z^2 = b^2$ at $x = 0$ (see Figure 4). Using Eq.(1), we have

$$\begin{aligned}
N &= \int \int E_x|_{x=0} dydz \\
&= -2qa \int_{y=0}^{y=b} dy \int_{z=-\sqrt{b^2-y^2}}^{z=+\sqrt{b^2-y^2}} dz \left\{ \frac{1}{[a^2 + y^2 + z^2]^{\frac{3}{2}}} + \frac{\lambda\mu}{[(\mu a)^2 + y^2 + z^2]^{\frac{3}{2}}} \right\} \\
&= -2qa \int_{y=0}^{y=b} dy \left\{ \frac{z}{[a^2 + y^2][a^2 + y^2 + z^2]^{\frac{1}{2}}} + \frac{\lambda\mu z}{[(\mu a)^2 + y^2][(\mu a)^2 + y^2 + z^2]^{\frac{1}{2}}} \right\} \Bigg|_{z=-\sqrt{b^2-y^2}}^{z=+\sqrt{b^2-y^2}} \\
&= -4qa \left\{ \frac{1}{\sqrt{a^2 + b^2}} \int_{y=0}^{y=b} dy \frac{\sqrt{b^2 - y^2}}{a^2 + y^2} + \frac{\lambda\mu}{\sqrt{(\mu a)^2 + b^2}} \int_{y=0}^{y=b} dy \frac{\sqrt{b^2 - y^2}}{(\mu a)^2 + y^2} \right\} \\
&= -4qa \left\{ \frac{1}{\sqrt{a^2 + b^2}} \left[-\sin^{-1} \left(\frac{y}{b} \right) - \frac{\sqrt{a^2 + b^2}}{a} \sin^{-1} \left(\frac{a}{b} \sqrt{\frac{b^2 - y^2}{a^2 + y^2}} \right) \right]_{y=0}^{y=b} \right. \\
&\quad \left. + \frac{\lambda\mu}{\sqrt{(\mu a)^2 + b^2}} \left[-\sin^{-1} \left(\frac{y}{b} \right) - \frac{\sqrt{(\mu a)^2 + b^2}}{\mu a} \sin^{-1} \left(\frac{\mu a}{b} \sqrt{\frac{b^2 - y^2}{(\mu a)^2 + y^2}} \right) \right]_{y=0}^{y=b} \right\}
\end{aligned}$$

where again, in the second and fourth steps above, we have made use of Eqs.(#) and (#), respectively, from the appendix. Finally, evaluating the last expression between the limits, yields

$$N = 2\pi q \left\{ \frac{a}{\sqrt{a^2 + b^2}} - 1 + \lambda \left[\frac{\mu a}{\sqrt{(\mu a)^2 + b^2}} - 1 \right] \right\}$$

Now if, $N = 0$, then

$$\lambda = - \frac{\left[\frac{a}{\sqrt{a^2 + b^2}} - 1 \right]}{\left[\frac{\mu a}{\sqrt{(\mu a)^2 + b^2}} - 1 \right]}$$

V. A. The neutral point or the equilibrium position can be located by setting $|\mathbf{E}| = 0$. This implies $E_x = 0$, $E_y = 0$, $E_z = 0$. Use of Eqs.(1) gives

$$E_y = 0 \Rightarrow qy \left[\frac{1}{[(x-a)^2 + y^2 + z^2]^{\frac{3}{2}}} + \frac{\lambda}{[(x-\mu a)^2 + y^2 + z^2]^{\frac{3}{2}}} \right] = 0 \Rightarrow y = 0$$

$$E_z = 0 \Rightarrow qz \left[\frac{1}{[(x-a)^2 + y^2 + z^2]^{\frac{3}{2}}} + \frac{\lambda}{[(x-\mu a)^2 + y^2 + z^2]^{\frac{3}{2}}} \right] = 0 \Rightarrow z = 0.$$

Then,

$$E_x|_{x=0,y=0} \Rightarrow \frac{q(x-a)}{[(x-a)^2 + y^2 + z^2]^{\frac{3}{2}}} + \frac{\lambda q(x-\mu a)}{[(x-\mu a)^2 + y^2 + z^2]^{\frac{3}{2}}} \Big|_{x=0,y=0} = 0.$$

This last equation gives,

$$\frac{x-a}{|x-a|^3} + \frac{\lambda(x-\mu a)}{|x-\mu a|^3} = 0. \quad (27)$$

For $(x-a > 0, x-\mu a > 0)$

or

$(x-a < 0, x-\mu a < 0)$

For $(x-a < 0, x-\mu a > 0)$

or

$(x-a > 0, x-\mu a < 0)$

$$\text{Eq.(27)} \Rightarrow \frac{1}{(x-a)^2} + \frac{\lambda}{(x-\mu a)^2} = 0$$

$$\Rightarrow \left(\frac{x-\mu a}{x-a} \right)^2 = -\lambda$$

$$\Rightarrow \lambda < 0, \text{ set } \lambda = -|\lambda|$$

$$\text{Eq.(27)} \Rightarrow \frac{1}{(x-a)^2} - \frac{\lambda}{(x-\mu a)^2} = 0$$

$$\Rightarrow \left(\frac{x-\mu a}{x-a} \right)^2 = \lambda$$

$$\Rightarrow \lambda > 0, \text{ set } \lambda = |\lambda|$$

In either case, one gets $(\lambda < 0 \text{ or } \lambda > 0)$

$$\left(\frac{x-\mu a}{x-a} \right)^2 = |\lambda|.$$

Solving for x gives,

$$\boxed{x = a \left[\frac{\mu \pm \sqrt{|\lambda|}}{1 \pm \sqrt{|\lambda|}} \right]} \quad (28)$$

B.

CASE 1: $\mu > 1$

- $\lambda < 0$

$$\Rightarrow x < a \text{ or } x > \mu a$$

$$\text{For } x = a \left[\frac{\mu - \sqrt{|\lambda|}}{1 - \sqrt{|\lambda|}} \right]$$

$$x < a \Rightarrow \mu > 1 \text{ and } \lambda < -1$$

$$x > \mu a \Rightarrow \mu > 1 \text{ and } -1 < \lambda < 0$$

$$\text{For } x = a \left[\frac{\mu + \sqrt{|\lambda|}}{1 + \sqrt{|\lambda|}} \right]$$

$$x < a \Rightarrow \text{unacceptable solutions}$$

$$x > \mu a \Rightarrow \text{unacceptable solutions}$$

- $\lambda > 0$

$$\Rightarrow a < x < \mu a$$

For $x = a \left[\frac{\mu - \sqrt{|\lambda|}}{1 - \sqrt{|\lambda|}} \right]$
 $a < x < \mu a \Rightarrow$ unacceptable solutions

For $x = a \left[\frac{\mu + \sqrt{|\lambda|}}{1 + \sqrt{|\lambda|}} \right]$
 $a < x < \mu a \Rightarrow \mu > 1$

CASE 2: $0 < \mu < 1$

- $\lambda < 0$

$$\Rightarrow x > a \text{ or } x < \mu a$$

For $x = a \left[\frac{\mu - \sqrt{|\lambda|}}{1 - \sqrt{|\lambda|}} \right]$
 $x > a \Rightarrow 0 < \mu < 1$ and $\lambda < -1$
 $x < \mu a \Rightarrow 0 < \mu < 1$ and $-1 < \lambda < 0$

For $x = a \left[\frac{\mu + \sqrt{|\lambda|}}{1 + \sqrt{|\lambda|}} \right]$
 $x > a \Rightarrow$ unacceptable solutions
 $x < \mu a \Rightarrow$ unacceptable solutions

- $\lambda > 0$

$$\Rightarrow \mu a < x < a$$

For $x = a \left[\frac{\mu - \sqrt{|\lambda|}}{1 - \sqrt{|\lambda|}} \right]$
 $\mu a < x < a \Rightarrow$ unacceptable solutions

For $x = a \left[\frac{\mu + \sqrt{|\lambda|}}{1 + \sqrt{|\lambda|}} \right]$
 $\mu a < x < a \Rightarrow 0 < \mu < 1$

CASE 3: $\mu < 0$

- $\lambda < 0$

$$\Rightarrow x < -|\mu|a \text{ or } x > a$$

For $x = a \left[\frac{-|\mu| - \sqrt{|\lambda|}}{1 - \sqrt{|\lambda|}} \right]$
 $x < -|\mu|a \Rightarrow \mu < 1$ and $-1 < \lambda < 0$
 $x > a \Rightarrow \mu < 1$ and $\lambda < -1$

For $x = a \left[\frac{-|\mu| + \sqrt{|\lambda|}}{1 + \sqrt{|\lambda|}} \right]$
 $x < -|\mu|a \Rightarrow$ unacceptable solutions
 $x > a \Rightarrow$ unacceptable solutions

- $\lambda > 0$

$$\Rightarrow -|\mu|a < x < a$$

$$\text{For } x = a \left[\frac{-|\mu| - \sqrt{|\lambda|}}{1 - \sqrt{|\lambda|}} \right]$$

$$-|\mu|a < x < a \Rightarrow \text{unacceptable solutions}$$

$$\text{For } x = a \left[\frac{-|\mu| + \sqrt{|\lambda|}}{1 + \sqrt{|\lambda|}} \right]$$

$$-|\mu|a < x < a \Rightarrow \mu < 0$$

CASE 4: $\forall \mu$

- $\lambda = -1$
No solutions

- $\lambda = +1$

$$x = \frac{a}{2}(1 + \mu), \text{ which follows directly from } x = a \left[\frac{\mu + \sqrt{|\lambda|}}{1 + \sqrt{|\lambda|}} \right]$$

Therefore,

$$x = a \left[\frac{\mu - \sqrt{|\lambda|}}{1 - \sqrt{|\lambda|}} \right] \text{ is valid for } \mu \in (-\infty, 1) \cup (1, \infty) \text{ and } \lambda \in (-\infty, -1) \cup (-1, 0)$$

$$x = a \left[\frac{\mu + \sqrt{|\lambda|}}{1 + \sqrt{|\lambda|}} \right] \text{ is valid for } \mu \in (-\infty, 1) \cup (1, \infty) \text{ and } \lambda \in (0, \infty)$$

VI. A. The equation for the electric potential, $V = 0$, the two charge system takes the form:

$$V = \frac{q}{[(x-a)^2 + y^2 + z^2]} + \frac{\lambda q}{[(x-\mu a)^2 + y^2 + z^2]} = 0$$

$$\Rightarrow \left[\frac{(x-\mu a)^2 + y^2 + z^2}{(x-a)^2 + y^2 + z^2} \right]^{\frac{1}{2}} = -\lambda \quad (29)$$

This last equation will hold true if $\lambda < 0$. Squaring and simplifying we get,

$$x^2(\lambda^2 - 1) + 2a(\mu - \lambda^2)x + y^2(\lambda^2 - 1) + z^2(\lambda^2 - 1) = a^2(\mu^2 - \lambda^2)$$

$$x^2 + 2a \left(\frac{\mu - \lambda^2}{\lambda^2 - 1} \right) x + \left[a \left(\frac{\mu - \lambda^2}{\lambda^2 - 1} \right) \right]^2 + y^2 + z^2 = \frac{a^2(\mu^2 - \lambda^2)}{\lambda^2 - 1} + \left[a \left(\frac{\mu - \lambda^2}{\lambda^2 - 1} \right) \right]^2$$

$$\left[x + a \left(\frac{\mu - \lambda^2}{\lambda^2 - 1} \right) \right]^2 + y^2 + z^2 = \left[\frac{a\lambda(\mu - 1)}{\lambda^2 - 1} \right]^2 \equiv r^2$$

Thus the equipotential surface is a sphere with center, O' and radius, r , where

$$O' : \left(a \left[\frac{\lambda^2 - \mu}{\lambda^2 - 1} \right], 0, 0 \right) \quad \text{and} \quad r = \left| \frac{a\lambda(\mu - 1)}{\lambda^2 - 1} \right|$$

If we denote the origin by O , then (see Figure 5)

$$\begin{aligned} Ol_1 &= |OO' - r| & Ol_2 &= |OO' + r| \\ &= \left| a \left(\frac{\lambda^2 - \mu}{\lambda^2 - 1} \right) - \frac{a\lambda(\mu - 1)}{\lambda^2 - 1} \right| & &= \left| a \left(\frac{\lambda^2 - \mu}{\lambda^2 - 1} \right) + \frac{a\lambda(\mu - 1)}{\lambda^2 - 1} \right| \\ &= a \left| \frac{\lambda - \mu}{\lambda - 1} \right| & &= a \left| \frac{\lambda + \mu}{\lambda + 1} \right| \end{aligned}$$

One can now use Ol_1 and Ol_2 , calculated above, to find Al_1 , Bl_1 , Al_2 and Bl_2 as follows (see Figure 6):

$$\begin{aligned} Al_1 &= |Ol_1 - OA| & Bl_1 &= |OB - Ol_1| \\ &= \left| a \left(\frac{\lambda - \mu}{\lambda - 1} \right) - a \right| & &= \left| \mu a - a \left(\frac{\lambda - \mu}{\lambda - 1} \right) \right| \\ &= a \left| \frac{1 - \mu}{\lambda - 1} \right| & &= a \left| \frac{\lambda(\mu - 1)}{\lambda - 1} \right| \end{aligned}$$

Therefore, it follows from this last result

$$\boxed{\frac{Al_1}{Bl_1} = \frac{1}{|\lambda|}}$$

Further,

$$\begin{aligned} Al_2 &= |Ol_2 - OA| & Bl_2 &= |OB - Ol_2| \\ &= \left| a \left(\frac{\lambda + \mu}{\lambda + 1} \right) - a \right| & &= \left| a \left(\frac{\lambda + \mu}{\lambda + 1} - \mu a \right) \right| \\ &= a \left| \frac{\mu - 1}{\lambda + 1} \right| & &= a \left| \frac{\lambda(1 - \mu)}{\lambda + 1} \right| \end{aligned}$$

Hence,

$$\boxed{\frac{Al_2}{Bl_2} = \frac{1}{|\lambda|}}$$

B. Setting $\lambda = -1$ in Eq.(29), we get

$$\begin{aligned} 2a(\mu - 1)x &= a^2(\mu^2 - 1) \\ x &= \frac{a}{2}(\mu - 1) \end{aligned} \tag{30}$$

Thus for equal and opposite charges, *the equipotential is a plane, bisecting the line of charges orthogonally.*

C. Since $\lambda < 0$ and further since the radius of the sphere is $\left| \frac{a\lambda(\mu - 1)}{\lambda^2 - 1} \right|$, λ cannot be equal to -1 . Therefore

$$\boxed{\lambda \in (-\infty, -1) \cup (-1, 0)}$$

VII. A.

CASE 1: $\lambda > 0$ ($\lambda = |\lambda|$)

Since both charges have same sign (positive), all the lines of force emanating from either charges must go off to infinity. Let P_1 be a point on the lines of force originating from A (see Figure 7). Assume that the tangent at P_1 passes through \overline{AB} in C_1 ¹. By Law of sines one has

$$\begin{aligned} \frac{\sin \beta_1}{AC_1} &= \frac{\sin \delta_1}{AP_1} \\ \frac{\sin \gamma_1}{BC_1} &= \frac{\sin(\pi - \delta_1)}{BP_1} \end{aligned}$$

Dividing first equation by the second, we get

$$\frac{\sin \beta_1}{\sin \gamma_1} = \left(\frac{BP_1}{AP_1} \right) \left(\frac{AC_1}{BC_1} \right). \quad (31)$$

Also, at P_1 , the resultant electric field is in the tangential direction, hence the normal component of the resultant electric field must vanish at P_1 (see Figure 8):

$$\begin{aligned} \frac{|\lambda|q}{BP_1^2} \sin \gamma_1 &= \frac{q}{AP_1^2} \sin \beta_1 \\ \Rightarrow \frac{\sin \beta_1}{\sin \gamma_1} &= |\lambda| \left(\frac{AP_1}{BP_1} \right)^2 \end{aligned} \quad (32)$$

Eqs.(31) and (32) implies

$$\frac{AC_1}{BC_1} = |\lambda| \left(\frac{AP_1}{BP_1} \right)^3 \quad (33)$$

As $P_1 \rightarrow \infty$ along the line of force, $\frac{AP_1}{BP_1} \rightarrow 1$ and $C_1 \rightarrow$ some fixed point G_1 (say) in \overline{AB} .

With this limit, Eq.(33) becomes

$$\boxed{\frac{AG_1}{BG_1} = |\lambda|}. \quad (34)$$

For this case, lines of force are sketched in Figure 9.

¹Considering point P_1 on a line of force, originating from B instead of A would still lead to a point C_1 in between A and B.

CASE 2: $\lambda < 0$ ($\lambda = -|\lambda|$)

Here two situations occur (see Figures 10a and 10b)

- $0 < |\lambda| < 1$

In this case consider a point P_2 on the line of force starting from A and going off to infinity (see Figure 11a). Then from $\triangle P_2C_2A$ and $\triangle P_2C_2B$, we have

$$\begin{aligned} \frac{\sin \beta_2}{AC_2} &= \frac{\sin(\pi - \delta_2)}{AP_2} \\ \frac{\sin(\beta_2 + \gamma_2)}{BC_2} &= \frac{\sin(\pi - \delta_2)}{BP_2} \\ \Rightarrow \frac{\sin \beta_2}{\sin(\beta_2 + \gamma_2)} &= \left(\frac{BP_2}{AP_2}\right) \left(\frac{AC_2}{BC_2}\right) \end{aligned} \quad (35)$$

Further, we know that the normal component of the electric field at P_2 must vanish (see Figure 12 a):

$$\frac{q}{AP_2^2} \sin \beta_2 - \frac{|\lambda|q}{BP_2^2} \sin(\beta_2 + \gamma_2) = 0 \quad (36)$$

Then Eqs.(35) and (36) gives

$$\frac{AC_2}{BC_2} = |\lambda| \left(\frac{AP_2}{BP_2}\right)^3 \quad (37)$$

As before, if we let $P_2 \rightarrow \infty$ along the line of force, then $\frac{AP_2}{BP_2} \rightarrow 1$ and $C_2 \rightarrow$ some fixed point G_2 (say) in \overline{AB} , leading Eq.(37) to

$$\boxed{\frac{AG_2}{BG_2} = |\lambda|}. \quad (38)$$

The lines of force are sketched in Figure 10a.

- $|\lambda| > 1$

In this case we need to consider a point P_3 on the line of force coming from infinity and terminating at B (see Figure 11b). From $\triangle P_3C_3A$ and $\triangle P_3C_3B$, we have

$$\begin{aligned} \frac{\sin(\beta_3 + \gamma_3)}{AC_3} &= \frac{\sin(\pi - \delta_3)}{AP_3} \\ \frac{\sin \beta_3}{BC_3} &= \frac{\sin(\pi - \delta_3)}{BP_3} \\ \Rightarrow \frac{\sin(\beta_3 + \gamma_3)}{\sin \beta_3} &= \left(\frac{BP_3}{AP_3}\right) \left(\frac{AC_3}{BC_3}\right) \end{aligned} \quad (39)$$

Again the normal component of the electric field at P_3 must vanish (see Figure 12 b):

$$\frac{q}{AP_3^2} \sin(\beta_3 + \gamma_3) - \frac{|\lambda|q}{BP_3^2} \sin \beta_3 = 0 \quad (40)$$

These last two equations again lead to the familiar result

$$\frac{AC_3}{BC_3} = |\lambda| \left(\frac{AP_3}{BP_3} \right)^3 \quad (41)$$

Let P_3 approach infinity along the line of force, then $\frac{AP_3}{BP_3}$ would approach 1 and C_2 tends to some fixed point G_3 (say) in \overline{AB} , leading Eq.(41) to

$$\boxed{\frac{AG_3}{BG_3} = |\lambda|}. \quad (42)$$

The lines of force are sketched in Figure 10b.

- B. From Eqs.(34), (38) and (42), the ratio $\frac{AG}{BG}$ ($= |\lambda|$) is independent of any angle, it follows that *tangents to all the lines of force at infinity (asymptotes) must pass through the fixed point G.*
- C. Referring to Figure 13, one can define a point G (in analogy with the center of mass of a system) as follows:

$$\begin{aligned} OG &= \frac{q(OA) \pm |\lambda|q(OB)}{q \pm \lambda q} \\ &= \frac{OA \pm |\lambda|(OB)}{1 \pm |\lambda|} \end{aligned} \quad (43)$$

where \pm refers to $\lambda = \pm|\lambda|$ for $\lambda > 0$ and $\lambda < 0$, respectively. Further, from Eq.(43), one can calculate AG and BG

$$\begin{aligned} AG &= |OG - OA| \\ &= \left| \frac{OA \pm |\lambda|(OB) - OA \mp |\lambda|(OA)}{1 \pm |\lambda|} \right| \\ &= \frac{|\lambda|}{1 \pm |\lambda|} AB \quad (\text{since } AB = |OB - OA|) \end{aligned} \quad (44)$$

$$\begin{aligned} BG &= |OB - OG| \\ &= \left| \frac{OB \pm |\lambda|(OB) - OA \mp |\lambda|(OB)}{1 \pm |\lambda|} \right| \\ &= \frac{1}{1 \pm |\lambda|} AB \end{aligned} \quad (45)$$

Thus from Eqs.(44) and (45) , it follows immediately

$$\boxed{\frac{AG}{BG} = |\lambda|}$$

For future purposes we record OG, AG and BG purely in terms of the given parameters, λ and μ . Since, $OA = a$ and $OB = \mu a$, this implies $AB = |\mu - 1|a$ and thus, it follows from Eqs. (43), (44) and (45)

$$OG = \left(\frac{1 \pm \mu|\lambda|}{1 \pm |\lambda|} \right) a \quad (46)$$

$$AG = \left| \frac{\lambda(\mu - 1)}{1 \pm |\lambda|} \right| a \quad (47)$$

$$BG = \left| \frac{\mu - 1}{1 \pm |\lambda|} \right| a \quad (48)$$

VIII. A. First recall from part II.(B.)(i), that the equation for the lines of force can be written as

$$\cos \theta_1 + |\lambda| \cos \theta_2 = C, \quad (49)$$

where we have set $\lambda = |\lambda|$, since $\lambda > 0$. Let Q be a point on a particular line of force emanating from A at an angle α to \overline{AB} as shown in Figure 14. When $Q \rightarrow A$ on the particular line of force considered, then $\theta_1 \rightarrow \alpha$ and $\theta_2 \rightarrow \pi$. Eq.(49) gives $C = \cos \alpha - |\lambda|$. This Value of C in Eq.(49) gives

$$\cos \theta_1 + |\lambda| \cos \theta_2 = \cos \alpha - |\lambda|.$$

On using the half angle formula, the last equation takes the form

$$\begin{aligned} 2 \cos^2 \left(\frac{1}{2} \theta_1 \right) - 1 + |\lambda| \left[2 \cos^2 \left(\frac{1}{2} \theta_2 \right) - 1 \right] &= 2 \cos^2 \left(\frac{1}{2} \alpha \right) - 1 - |\lambda| \\ \Rightarrow \boxed{\cos^2 \left(\frac{1}{2} \theta_1 \right) + |\lambda| \cos^2 \left(\frac{1}{2} \theta_2 \right) = \cos^2 \left(\frac{1}{2} \alpha \right)} &\quad (50) \end{aligned}$$

B. (i) When Q has receded to infinity on the particular line of force, then $\theta_1 = \theta_2 \equiv \theta$ (see Figure 16). This is because $AQ \parallel BQ$ at infinity. Therefore, Eq.(50) gives

$$\begin{aligned} \cos^2 \left(\frac{1}{2} \theta \right) + |\lambda| \cos^2 \left(\frac{1}{2} \theta \right) &= \cos^2 \left(\frac{1}{2} \alpha \right) \\ \Rightarrow \boxed{\theta = 2 \cos^{-1} \left[\frac{1}{\sqrt{1 + |\lambda|}} \cos \left(\frac{1}{2} \alpha \right) \right]} &\quad (51) \end{aligned}$$

(ii) To determine the range of values of α , it is convenient to rewrite Eq.(51) as

$$(1 + |\lambda|) \left(\frac{1 + \cos \theta}{2} \right) = \left(\frac{1 + \cos \alpha}{2} \right)$$

$$\Rightarrow \cos \theta = \frac{\cos \alpha - |\lambda|}{1 + |\lambda|} \quad (52)$$

Thus in order for θ to be real

$$-1 < \frac{\cos \alpha - |\lambda|}{1 + |\lambda|} < 1$$

$$\Rightarrow -1 < \cos \alpha < 1 + 2|\lambda|$$

The second inequality is impossible and therefore $\cos \alpha > -1$ which gives $\alpha < 180^\circ$.

(iii) Using the expression for $\cos \theta$ in Eq.(52) and the right triangle shown in Figure 15, one can write down

$$\tan \theta = \sqrt{\left(\frac{1 + |\lambda|}{\cos \alpha - |\lambda|} \right)^2 - 1}.$$

The slope of the asymptote is $\tan \theta$. By symmetry there are two asymptotes (see Figure 16) and therefore the slopes of the lines are given by $m = \pm \tan \theta$. We have also proved that all asymptotes must pass through the center of gravity, G of charges whose coordinates are given by Eq.(46):

$$\left(\frac{1 + \mu|\lambda|}{1 + |\lambda|} \right) a.$$

Therefore, the general equation of a line $y = mx + b$ immediately yields

$$y = \pm \sqrt{\left(\frac{1 + |\lambda|}{\cos \alpha - |\lambda|} \right)^2 - 1} \left[x - \left(\frac{1 + \mu|\lambda|}{1 + |\lambda|} \right) a \right].$$

C. In this case, assume a point Q on a line of force originating from B at an angle α to \overline{AB} (see Figure 17).

$$\text{When } Q \rightarrow B: \quad \theta_1 \rightarrow \alpha \quad \text{and} \quad \theta_2 \rightarrow 0$$

For these values of θ_1 and θ_2 , Eq.(49) gives for $C = 1 + |\lambda| \cos \alpha$. Therefore, Eq.(49) gives

$$\cos \theta_1 + |\lambda| \cos \theta_2 = 1 + |\lambda| \cos \alpha$$

$$1 - 2 \sin^2 \left(\frac{1}{2} \theta_1 \right) + |\lambda| \left[1 - 2 \sin^2 \left(\frac{1}{2} \theta_2 \right) \right] = 1 + |\lambda| \left[1 - 2 \sin^2 \left(\frac{1}{2} \alpha \right) \right]$$

$$\Rightarrow \boxed{\sin^2\left(\frac{1}{2}\theta_1\right) + |\lambda| \sin^2\left(\frac{1}{2}\theta_2\right) = |\lambda| \sin^2\left(\frac{1}{2}\alpha\right)} \quad (53)$$

Now, when $Q \rightarrow \infty$, $\theta_1 = \theta_2 \equiv \theta$. With these values of θ_1 and θ_2 in Eq.(53), we get

$$\begin{aligned} \sin^2\left(\frac{1}{2}\theta\right) + |\lambda| \sin^2\left(\frac{1}{2}\theta\right) &= |\lambda| \sin^2\left(\frac{1}{2}\alpha\right) \\ \Rightarrow \theta &= 2 \sin^{-1} \left[\frac{|\lambda|}{\sqrt{1+|\lambda|}} \sin\left(\frac{1}{2}\alpha\right) \right] \end{aligned}$$

To find the range of values of α , we rewrite this last equation as

$$\begin{aligned} (1+|\lambda|) \left(\frac{1-\cos\theta}{2} \right) &= |\lambda| \left(\frac{1-\cos\alpha}{2} \right) \\ \Rightarrow \cos\theta &= \frac{1+|\lambda|\cos\alpha}{1+|\lambda|} \end{aligned}$$

Therefore, θ to be real

$$\begin{aligned} -1 &< \frac{1+|\lambda|\cos\alpha}{1+|\lambda|} < 1 \\ \Rightarrow -1 - \frac{2}{|\lambda|} &< \cos\alpha < 1. \end{aligned}$$

The first inequality is not valid and therefore, $\cos\alpha < 1$, that is, $\boxed{\alpha > 0^0}$

IX. A. Since $\lambda < 0$, we put $\lambda = -|\lambda|$ in the equation given in part **II.(B.)(i)**.

$$\cos\theta_1 - |\lambda| \cos\theta_2 = C. \quad (54)$$

Next, consider a point P on a line of force which originates from A at an angle α with respect to \overline{AB} . Thus, as P approaches A along the particular line of force we are considering (see Figure 18), $\theta_1 \rightarrow \alpha$ and $\theta_2 \rightarrow \pi$. This gives $C = \cos\alpha + |\lambda|$. Therefore Eq.(54) takes the form,

$$\begin{aligned} \cos\theta_1 - |\lambda| \cos\theta_2 &= \cos\alpha + |\lambda| \\ \Rightarrow 2 \cos^2\left(\frac{1}{2}\theta_1\right) - 1 - |\lambda| \left[2 \cos^2\left(\frac{1}{2}\theta_2\right) - 1 \right] &= 2 \cos^2\left(\frac{1}{2}\alpha\right) - 1 + |\lambda| \\ \Rightarrow \boxed{\cos^2\left(\frac{1}{2}\theta_1\right) - |\lambda| \cos^2\left(\frac{1}{2}\theta_2\right) = \cos^2\left(\frac{1}{2}\alpha\right)} & \quad (55) \end{aligned}$$

B. (i) Assume that the line of force considered in part **IX.A.** terminates at B at an angle β w.r.t. \overline{AB} (see Figure 19). Then as $P \rightarrow B$, $\theta_2 = \beta$, $\theta_1 = 0$ and Eq.(55) gives

$$1 - |\lambda| \cos^2 \left(\frac{1}{2}\beta \right) = 1 - \sin^2 \left(\frac{1}{2}\alpha \right)$$

$$\Rightarrow \beta = 2 \cos^{-1} \left[\frac{1}{\sqrt{|\lambda|}} \sin \left(\frac{1}{2}\alpha \right) \right] \quad (56)$$

(ii) Rewrite Eq.(56) as

$$|\lambda| \left(\frac{1 + \cos \beta}{2} \right) = \frac{1 - \cos \beta}{2}$$

$$\Rightarrow \cos \beta = \frac{1 - \cos \alpha - |\lambda|}{|\lambda|}$$

Thus, in order for β to exist

$$-1 < \frac{1 - \cos \alpha - |\lambda|}{|\lambda|} < 1$$

$$\Rightarrow 1 - 2|\lambda| < \cos \alpha < 1$$

$$\text{Finally, } \boxed{0 < \alpha < \cos^{-1}(1 - 2|\lambda|)}$$

Further, from this last equation,

$$-1 < 1 - 2|\lambda| < 1$$

$$\Rightarrow \boxed{-1 < \lambda < 0}$$

C. (i) When the point P on the line of force considered in part **IX.A.**, goes to infinity (see Figure 20), $AP \parallel BP$ and $\theta_1 = \theta_2 \equiv \theta$. Then, Eq.(55) implies

$$\cos^2 \left(\frac{1}{2}\theta \right) - |\lambda| \cos^2 \left(\frac{1}{2}\theta \right) = \cos^2 \left(\frac{1}{2}\alpha \right)$$

$$\theta = 2 \cos^{-1} \left[\frac{1}{\sqrt{1 - |\lambda|}} \cos \left(\frac{1}{2}\alpha \right) \right]$$

(ii) Rewriting the last equation, we get

$$(1 - |\lambda|) \left(\frac{1 + \cos \theta}{2} \right) = \frac{1 + \cos \alpha}{2}$$

$$\cos \theta = \frac{|\lambda| + \cos \alpha}{1 - |\lambda|} \quad (57)$$

Therefore,

$$-1 < \frac{|\lambda| + \cos \alpha}{1 - |\lambda|} < 1$$

$$\begin{array}{l} \frac{|\lambda| + \cos \alpha}{1 - |\lambda|} > -1 \quad \cap \quad \frac{|\lambda| + \cos \alpha}{1 - |\lambda|} < 1 \\ \frac{\cos \alpha + 1}{1 - |\lambda|} > 0 \quad \cap \quad \frac{2|\lambda| - 1 + \cos \alpha}{1 - |\lambda|} < 0 \\ \Rightarrow \alpha < 180^0 \text{ and } -1 < \lambda < 0 \quad \cap \quad \Rightarrow \left\{ \begin{array}{l} \alpha > \cos^{-1}(1 - 2|\lambda|) \text{ and } -1 < \lambda < 0 \\ \text{or} \\ \alpha < \cos^{-1}(1 - 2|\lambda|) \text{ and } \lambda < -1 \end{array} \right. \end{array}$$

Finally, $\boxed{\cos^{-1}(1 - 2|\lambda|) < \alpha < 180^0}$ and $\boxed{-1 < \lambda < 0}$

(iii) Recall that all asymptotes must through the center of gravity, G of charges which has the x -coordinates (see Eq.(46)):

$$\left(\frac{1 - \mu|\lambda|}{1 - |\lambda|} \right) a$$

The equation of the asymptote is $y = mx + b$, where $m = \pm \tan \theta$. Then from Eq.(57), we have

$$\tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{\frac{1}{\cos^2 \theta} - 1} = \sqrt{\left(\frac{1 - |\lambda|}{\cos \alpha + |\lambda|} \right)^2 - 1}$$

$$\Rightarrow \boxed{y = \pm \sqrt{\left(\frac{1 - |\lambda|}{\cos \alpha + |\lambda|} \right)^2 - 1} \left[x - \left(\frac{1 - \mu|\lambda|}{1 - |\lambda|} \right) a \right]}$$

D. (i) If the line of force originating from A, considered above, is an *extreme* line of force (see Figure 21), then as $P \rightarrow B$, $\theta_1 = 0$ and $\theta_2 = 0$. This is because the lines of force leave tangentially \overline{AB} . Eq.(55) implies

$$\cos^2 \left(\frac{1}{2} \alpha \right) = 1 - |\lambda|$$

$$\Rightarrow \frac{1 + \cos \alpha}{2} = 1 - |\lambda|$$

$$\Rightarrow \boxed{\alpha = \cos^{-1}(1 - 2|\lambda|)} \quad (58)$$

(ii) For real values of α

$$-1 < 1 - 2|\lambda| < 1 \Rightarrow \boxed{-1 < \lambda < 0}$$

E. (i) Let the angle between \overline{AB} and \overline{AC} be denoted by θ_0 (see Figure 22). Then as $P \rightarrow C$, $\theta_1 = \theta_0$ and $\theta_2 = \pi - \theta_0$. Eq.(55) implies

$$\begin{aligned} \cos^2\left(\frac{1}{2}\theta_0\right) - |\lambda| \cos^2\left(\frac{1}{2}[\pi - \theta_0]\right) &= \cos^2\left(\frac{1}{2}\alpha\right) \\ \Rightarrow 1 - \sin^2\left(\frac{1}{2}\theta_0\right) - |\lambda| \sin^2\left(\frac{1}{2}\theta_0\right) &= 1 - \sin^2\left(\frac{1}{2}\alpha\right) \\ \Rightarrow \theta_0 &= 2 \sin^{-1} \left[\frac{1}{\sqrt{1+|\lambda|}} \sin\left(\frac{1}{2}\alpha\right) \right] \end{aligned} \quad (59)$$

(ii) One can rewrite Eq.(59) as

$$\begin{aligned} \left(\frac{1 - \cos \theta_0}{2}\right) (1 + |\lambda|) &= \frac{1 - \cos \alpha}{2} \\ \Rightarrow \cos \theta_0 &= \frac{|\lambda| + \cos \alpha}{1 + |\lambda|} \end{aligned} \quad (60)$$

In order for θ_0 to be valid

$$\begin{aligned} -1 < \frac{|\lambda| + \cos \alpha}{1 + |\lambda|} < 1 \\ \Rightarrow -1 - 2|\lambda| < \cos \alpha < 1 \end{aligned}$$

First inequality is not valid, therefore $\cos \alpha < 1 \Rightarrow \boxed{\alpha > 0}$

(iii) From Figure 22, $\frac{OC}{OA} = \tan \theta_0$. Thus

$$(OC)_{\max} = OA \tan (\theta_0)_{\max} \quad (61)$$

$$\text{Eq.(60)} \Rightarrow \cos (\theta_0)_{\max} = \frac{|\lambda| + \cos (\alpha)_{\max}}{1 + |\lambda|} \quad (62)$$

$$\text{Eq.(58)} \Rightarrow \cos (\alpha)_{\max} = 1 - 2|\lambda| \quad (63)$$

Eqs.(62) and (63) gives

$$\cos (\theta_0)_{\max} = \frac{1 - |\lambda|}{1 + |\lambda|} \quad (64)$$

and that

$$\begin{aligned}
 \tan(\theta_0)_{\max} &= \sqrt{\sec^2(\theta_0)_{\max} - 1} \\
 &= \sqrt{\frac{1}{\cos^2(\theta_0)_{\max}} - 1} \\
 \text{Eqs.(64)} \Rightarrow \tan(\theta_0)_{\max} &= \frac{2\sqrt{|\lambda|}}{1 - |\lambda|} \tag{65}
 \end{aligned}$$

Using $OA = \left| \frac{a}{2}(1 + \mu) - a \right| = \left| \frac{1}{2}(1 - \mu) \right| a$, and Eq.(65) in Eq.(61), we get

$$\boxed{(OC)_{\max} = |\mu - 1| \frac{\sqrt{|\lambda|}}{1 - |\lambda|} a}$$

- F. (i) Let there be a point R on a line of force that ends at B at an angle α w.r.t. \overline{AB} (see figure 23). When $R \rightarrow B$: $\theta_1 = 0$ and $\theta_2 = \alpha$. With this set of information, Eq.(54) gives $C = 1 - |\lambda| \cos \alpha$ and for this specific value of C , Eq.(54) becomes

$$\begin{aligned}
 \cos \theta_1 - |\lambda| \cos \theta_2 &= 1 - |\lambda| \cos \alpha \\
 \Rightarrow 1 - \sin^2 \left(\frac{1}{2} \theta_1 \right) - |\lambda| \left[1 - \sin^2 \left(\frac{1}{2} \theta_2 \right) \right] &= 1 - |\lambda| \left[1 - \sin^2 \left(\frac{1}{2} \alpha \right) \right] \\
 \Rightarrow \boxed{|\lambda| \sin^2 \left(\frac{1}{2} \theta_2 \right) - \sin^2 \left(\frac{1}{2} \theta_1 \right) = |\lambda| \sin^2 \left(\frac{1}{2} \alpha \right)} &\tag{66}
 \end{aligned}$$

- (ii) If the line of force considered in part **IX.(F.)(i)**, is restricted to have been originated from A, then this implies that as $R \rightarrow A$, $\theta_1 = \beta$ (say) and $\theta_2 = \pi$ (see Figure 24). Eq.(66) then takes the form

$$\begin{aligned}
 |\lambda| - \sin^2 \left(\frac{1}{2} \beta \right) &= |\lambda| \sin^2 \left(\frac{1}{2} \alpha \right) \\
 \Rightarrow \boxed{\beta = 2 \sin^{-1} \left[\sqrt{|\lambda|} \cos \left(\frac{1}{2} \alpha \right) \right]} &\tag{67}
 \end{aligned}$$

As in previous cases, let us rewrite Eq.(67) as

$$\begin{aligned}
 \sin^2 \left(\frac{1}{2} \beta \right) &= |\lambda| \cos^2 \left(\frac{1}{2} \alpha \right) \\
 \Rightarrow \frac{1 - \cos \beta}{2} &= \left(\frac{1 + \cos \alpha}{2} \right) \\
 \Rightarrow \cos \beta &= 1 - |\lambda| - |\lambda| \cos \alpha
 \end{aligned}$$

Therefore in order for β to be real,

$$\begin{aligned} -1 < 1 - |\lambda| - |\lambda| \cos \alpha < 1 \\ \Rightarrow \frac{2}{|\lambda|} - 1 > \cos \alpha > -1 \end{aligned}$$

$$\Rightarrow \cos^{-1} \left(\frac{2}{|\lambda|} - 1 \right) < \alpha < 180^0$$

Further, in order for α to exist

$$\frac{2}{|\lambda|} - 1 < 1 \Rightarrow \boxed{\lambda < -1}$$

- (iii) In order for the line of force, that ends at B, to have been originated from infinity, $\theta_1 = \theta_2 \equiv \theta$ (see Figure 25). This is because as $R \rightarrow \infty$, AR becomes parallel to BR. Therefore Eq.(66) yields

$$\begin{aligned} [|\lambda| - 1] \sin^2 \left(\frac{1}{2} \theta \right) &= |\lambda| \sin^2 \left(\frac{1}{2} \alpha \right) \\ \Rightarrow \theta &= 2 \sin^{-1} \left[\sqrt{\frac{|\lambda|}{|\lambda| - 1}} \sin \left(\frac{1}{2} \alpha \right) \right] \end{aligned}$$

To find the restriction on the angle α , it would be convenient if we write the last equation as

$$\begin{aligned} (|\lambda| - 1) \left(\frac{1 - \cos \theta}{2} \right) &= |\lambda| \left(\frac{1 - \cos \alpha}{2} \right) \\ \Rightarrow \cos \theta &= \frac{1 - |\lambda| \cos \alpha}{1 - |\lambda|} \end{aligned}$$

$$\text{Therefore, } -1 < \frac{1 - |\lambda| \cos \alpha}{1 - |\lambda|} < 1$$

$$\begin{aligned} \frac{1 - |\lambda| \cos \alpha}{1 - |\lambda|} > -1 & \cap \frac{1 - |\lambda| \cos \alpha}{1 - |\lambda|} < 1 \\ \frac{2 - |\lambda| - |\lambda| \cos \alpha}{1 - |\lambda|} > 0 & \cap \frac{|\lambda| (1 - \cos \alpha)}{1 - |\lambda|} < 0 \\ \Rightarrow \left\{ \begin{array}{l} \cos \alpha < \frac{2}{|\lambda|} - 1 \text{ and } -1 < \lambda < 0 \\ \text{or} \\ \cos \alpha > \frac{2}{|\lambda|} - 1 \text{ and } \lambda < -1 \end{array} \right. & \cap \Rightarrow \cos \alpha < 1 \text{ and } \lambda < -1 \end{aligned}$$

Therefore, $\frac{2}{|\lambda|} - 1 < \cos \alpha < 1$ and $\lambda < -1$

$$\Rightarrow \boxed{0 < \alpha < \cos^{-1}\left(\frac{2}{|\lambda|} - 1\right)} \text{ and } \boxed{\lambda < -1}$$

(iv) If the line of force considered in **IX.(F.)(i)**, is an extreme line of force, then as $R \rightarrow A$, $\theta_1 = \theta_2 \rightarrow \pi$ (see Figure 26). Eq.(66)

$$\begin{aligned} |\lambda| \sin^2\left(\frac{1}{2}\pi\right) - \sin^2\left(\frac{1}{2}\pi\right) &= |\lambda| \sin^2\left(\frac{1}{2}\alpha\right) \\ \Rightarrow |\lambda| - 1 &= |\lambda| \left(\frac{1 - \cos \alpha}{2}\right) \end{aligned}$$

$$\boxed{\alpha = \cos^{-1}\left(\frac{2}{|\lambda|} - 1\right)}. \quad (68)$$

In order for α to exist

$$-1 < \frac{2}{|\lambda|} - 1 < 1 \Rightarrow \boxed{\lambda < -1}$$

(v) Let the angle between \overline{AB} and \overline{AC} be denoted by θ_0 (see Figure 27). As $R \rightarrow C$, $\theta_1 = \theta_0$ and $\theta_2 = \pi - \theta_0$. Eq.(66) implies

$$\begin{aligned} |\lambda| \sin^2\left(\frac{1}{2}\theta_0\right) - \cos^2\left(\frac{1}{2}\theta_0\right) &= |\lambda| \sin^2\left(\frac{1}{2}\alpha\right) \\ \Rightarrow |\lambda| \left[1 - \cos^2\left(\frac{1}{2}\theta_0\right)\right] - \cos^2\left(\frac{1}{2}\theta_0\right) &= |\lambda| \left[1 - \cos^2\left(\frac{1}{2}\alpha\right)\right] \end{aligned}$$

$$\Rightarrow \boxed{\theta_0 = 2 \cos^{-1} \left[\sqrt{\frac{|\lambda|}{1 + |\lambda|}} \cos\left(\frac{1}{2}\alpha\right) \right]} \quad (69)$$

Rewriting this last equation, we get

$$\begin{aligned} (1 + |\lambda|) \left(\frac{1 + \cos \theta_0}{2}\right) &= |\lambda| \left(\frac{1 + \cos \alpha}{2}\right) \\ \cos \theta_0 &= \frac{|\lambda| \cos \alpha - 1}{|\lambda| + 1} \end{aligned} \quad (70)$$

In the above equation, θ_0 would be real, if

$$\begin{aligned} -1 < \frac{|\lambda \cos \alpha| - 1}{|\lambda| + 1} < 1 \\ \Rightarrow -1 < \cos \alpha < 1 + \frac{2}{|\lambda|} \end{aligned}$$

The second inequality is not possible. The only solution is $\cos \alpha > -1 \Rightarrow \boxed{\alpha < 180^\circ}$.
 From Figure 27, $\frac{OC}{OA} = \tan \theta_0$. Thus

$$(OC)_{\max} = OA \tan (\theta_0)_{\max} \quad (71)$$

$$\text{Eq.(70)} \Rightarrow \cos (\theta_0)_{\max} = \frac{|\lambda| \cos (\alpha)_{\max} - 1}{|\lambda| + 1} \quad (72)$$

$$\text{Eq.(68)} \Rightarrow \cos (\alpha)_{\max} = \frac{2}{|\lambda|} - 1 \quad (73)$$

Eqs.(72) and (73) gives

$$\cos (\theta_0)_{\max} = \frac{1 - |\lambda|}{1 + |\lambda|} \quad (74)$$

and that

$$\begin{aligned} \tan (\theta_0)_{\max} &= \sqrt{\sec^2 (\theta_0)_{\max} - 1} \\ &= \sqrt{\frac{1}{\cos^2 (\theta_0)_{\max}} - 1} \end{aligned}$$

$$\text{Eqs.(74)} \Rightarrow \tan (\theta_0)_{\max} = \frac{2\sqrt{|\lambda|}}{1 - |\lambda|} \quad (75)$$

Using $OA = \left| \frac{a}{2}(1 + \mu) - a \right| = \left| \frac{1}{2}(1 - \mu) \right| a$, and Eq.(75) in Eq.(71), we get

$$\boxed{(OC)_{\max} = |\mu - 1| \frac{\sqrt{|\lambda|}}{1 - |\lambda|} a}$$

X. A. Since, $\lambda > 0$ for this part of the problem, we set $\lambda = |\lambda|$

(i) Starting from the equation for the lines of force, $\cos \theta_1 + |\lambda| \cos \theta_2 = C$, one can determine the constant C for the limiting line of force through the neutral point N by setting $\theta_1 = 0$ and $\theta_2 = \pi$ (see Figure 28). This is because for like charges the neutral point N is on the line joining the charges (\overline{AB}) and is in between the two charges (see part V.). Thus, $C = \cos 0 + |\lambda| \cos \pi \Rightarrow C = 1 - |\lambda|$. Equation for the line of force in this case, then takes the form

$$\begin{aligned} \cos \theta_1 + |\lambda| \cos \theta_2 &= 1 - |\lambda| \quad (76) \\ \Rightarrow 1 - 2 \sin^2 \left(\frac{1}{2} \theta_1 \right) + |\lambda| \left[2 \cos^2 \left(\frac{1}{2} \theta_2 \right) - 1 \right] &= 1 - |\lambda| \end{aligned}$$

$$\Rightarrow \boxed{\sin^2 \left(\frac{1}{2} \theta_1 \right) = |\lambda| \cos^2 \left(\frac{1}{2} \theta_2 \right)} \quad (77)$$

- (ii) When a point on this limiting line of force through N has receded to infinity, $\theta_1 = \theta_2 \equiv \theta_0$. We now use this information in Eq.76 to determine the slope of the asymptote through N:

$$\cos \theta_0 = \frac{1 - |\lambda|}{1 + |\lambda|} \quad (78)$$

$$\text{Further, } \tan \theta_0 = \sqrt{\frac{1}{\cos^2 \theta_0} - 1}$$

$$\text{Eq.(78)} \Rightarrow \tan \theta_0 = \frac{2\sqrt{|\lambda|}}{1 - |\lambda|}$$

And by symmetry it has two asymptotes with slopes,

$$m = \pm \tan \theta_0.$$

As before the asymptote to this limiting line of force must pass through center of gravity of charges G, whose coordinates are (see Eq.(46)):

$$\left(\left[\frac{1 - \mu|\lambda|}{1 - |\lambda|} \right] a, 0 \right)$$

Then, the y -intercept, b in the equation for an asymptote ($y = mx + b$) is given by

$$b = -m \left(\frac{1 + \mu|\lambda|}{1 + |\lambda|} \right) a$$

and finally

$$\boxed{y = \pm \frac{2\sqrt{|\lambda|}}{1 - |\lambda|} \left[x - \left(\frac{1 + \mu|\lambda|}{1 + |\lambda|} \right) a \right]}$$

B. Since $\lambda < 0$ for this part of the problem, we set $\lambda = -|\lambda|$

- (i) Without loss of generality, we assume $\mu > 1$ for this part of the question. This means that point B (charge $-|\lambda|q$) will always be to the right of point A (charge q) Further, recall from part V., that in the case of unlike charges, the neutral point N will always be either to the left of A or to the right of B and that N has x -coordinates:

$$\frac{\mu - \sqrt{|\lambda|}}{1 - \sqrt{|\lambda|}} a$$

We now determine the permissible values of λ for each of the two locations.

$$\begin{array}{ll} \frac{\mu - \sqrt{|\lambda|}}{1 - \sqrt{|\lambda|}} a > \mu a & \frac{\mu - \sqrt{|\lambda|}}{1 - \sqrt{|\lambda|}} a < \mu a \\ \frac{(\mu - 1)\sqrt{|\lambda|}}{1 - \sqrt{|\lambda|}} > 0 & \frac{(\mu - 1)}{1 - \sqrt{|\lambda|}} < 0 \\ \Rightarrow -1 < \lambda < 0 & \Rightarrow \lambda < -1 \end{array}$$

The situation is depicted in Figure 29.

- $\lambda < -1$ (see Figure 30a)

When $S_1 \rightarrow N_1$, $\theta_1 \rightarrow \pi$ and $\theta_2 \rightarrow \pi$. Then $\cos \theta_1 - |\lambda| \cos \theta_2 = C$ leads to $C = \cos \pi - |\lambda| \cos \pi = -1 + |\lambda|$. Therefore the equation for the line of force which passes through the neutral point N_1 take form

$$\begin{aligned} \cos \theta_1 - |\lambda| \cos \theta_2 &= -1 + |\lambda| \\ \Rightarrow 2 \cos^2 \left(\frac{1}{2} \theta_1 \right) - 1 - |\lambda| \left[2 \cos^2 \left(\frac{1}{2} \theta_2 \right) - 1 \right] &= -1 + |\lambda| \\ \boxed{\cos^2 \left(\frac{1}{2} \theta_1 \right) = |\lambda| \cos^2 \left(\frac{1}{2} \theta_2 \right)} & \quad (79) \end{aligned}$$

- $-1 < \lambda < 0$ (see Figure 30b)

When $S_2 \rightarrow N_2$, $\theta_1 \rightarrow 0$ and $\theta_2 \rightarrow 0$. Then $\cos \theta_1 - |\lambda| \cos \theta_2 = C$ leads to $C = \cos 0 - |\lambda| \cos 0 = 1 - |\lambda|$. Therefore the equation for the line of force which passes through the neutral point N_2 take form

$$\begin{aligned} \cos \theta_1 - |\lambda| \cos \theta_2 &= 1 - |\lambda| \\ \Rightarrow 1 - 2 \sin^2 \left(\frac{1}{2} \theta_1 \right) - |\lambda| \left[1 - 2 \sin^2 \left(\frac{1}{2} \theta_2 \right) \right] &= 1 - |\lambda| \\ \boxed{\sin^2 \left(\frac{1}{2} \theta_1 \right) = |\lambda| \sin^2 \left(\frac{1}{2} \theta_2 \right)} & \quad (80) \end{aligned}$$

- (ii) (a) When $S_2 \rightarrow A$, $\theta_1 \rightarrow \theta_{01}$ (say) and $\theta_2 \rightarrow \pi$ (see Figure 30b). Then Eq.(80) implies

$$\begin{aligned} \sin^2 \left(\frac{1}{2} \theta_{01} \right) &= |\lambda| \left[\sin^2 \left(\frac{1}{2} \pi \right) \right] \\ \Rightarrow \frac{1 - \cos \theta_{01}}{2} &= |\lambda| \\ \boxed{\theta_{01} = \cos^{-1} (1 - 2|\lambda|)} & \quad (81) \end{aligned}$$

- (b) When $S_1 \rightarrow B$, $\theta_1 \rightarrow 0$ and $\theta_2 \rightarrow \theta_{02}$ (say) (see Figure 30a). Then Eq.(79) implies

$$\begin{aligned} \cos^2 (0) &= |\lambda| \left[\cos^2 \left(\frac{1}{2} \theta_{02} \right) \right] \\ \Rightarrow 1 &= |\lambda| \left[\frac{1 + \cos \theta_{02}}{2} \right] \end{aligned}$$

$$\boxed{\theta_{02} = \cos^{-1} \left(\frac{2}{|\lambda|} - 1 \right)} \quad (82)$$

(iii) Comparing Eq.(81) with Eq.(58) and Eq.(82) with Eq.(68), we conclude that *the line of force which passes through the neutral point (N_1) N_2 separates the lines going from A to B from those (coming to B) going from A (from infinity) to infinity.* Complete situation is shown in Figures 31a and 31b.

XI. A. The statement of the problem is depicted in Figure 32. Here we are assuming that $\mu > 1$, without loss of generality. Now on using Sine Rule in $\triangle ABQ$ (see Figure 33), we get

$$\begin{aligned} \frac{\sin \left(\pi - \frac{1}{2}\theta_2 \right)}{AQ} &= \frac{\sin \left(\frac{1}{2}\theta_1 \right)}{BQ} \\ \frac{\sin^2 \left(\frac{1}{2}\theta_1 \right)}{\sin^2 \left(\frac{1}{2}\theta_2 \right)} &= \frac{BQ^2}{AQ^2} \\ \text{Eq.(80)} \Rightarrow |\lambda| &= \frac{BQ^2}{AQ^2} \end{aligned}$$

Thus,

$$\begin{aligned} \frac{(x - \mu a)^2 + y^2}{(x - a)^2 + y^2} &= |\lambda| \\ \text{expanding, } x^2 - 2a \left(\frac{\mu - |\lambda|}{1 - |\lambda|} \right) x + y^2 &= a^2 \left(\frac{|\lambda| - \mu^2}{1 - |\lambda|} \right) \\ \text{completing the square, } \left[x - \left(\frac{\mu - |\lambda|}{1 - |\lambda|} \right) a \right]^2 + y^2 &= a^2 \left(\frac{|\lambda| - \mu^2}{1 - |\lambda|} \right) + \left[\left(\frac{\mu - |\lambda|}{1 - |\lambda|} \right) a \right]^2 \\ \text{and finally, } \left[x - \left(\frac{\mu - |\lambda|}{1 - |\lambda|} \right) a \right]^2 + y^2 &= |\lambda| \left[\left(\frac{\mu - 1}{1 - |\lambda|} \right) a \right]^2 \end{aligned}$$

This circle has a (see Figure 34)

$$\text{Center : } (\mathcal{C}, 0) \quad \text{and} \quad \text{Radius, } R = \sqrt{|\lambda|} \left[\frac{\mu - 1}{1 - |\lambda|} \right] a,$$

$$\text{where } \mathcal{C} = \left[\frac{\mu - |\lambda|}{1 - |\lambda|} \right] a$$

One must now, from the above information, show that the x-coordinate of N is

$$\frac{\mu - \sqrt{|\lambda|}}{1 - \sqrt{|\lambda|}} a \quad (\text{see Eq.(28)})$$

To that end,

$$\begin{aligned}
 x\text{-coordinate of N} &= C + R \\
 &= \left[\frac{\mu - |\lambda|}{1 - |\lambda|} \right] a + \sqrt{|\lambda|} \left[\frac{\mu - 1}{1 - |\lambda|} \right] a \\
 &= \frac{a (\mu - \sqrt{|\lambda|}) (1 + \sqrt{|\lambda|})}{1 - |\lambda|} \\
 &= \frac{\mu - |\lambda|}{1 - \sqrt{|\lambda|}} a
 \end{aligned}$$

Also,

$$\begin{aligned}
 x\text{-coordinate of M} &= C - R \\
 &= \left[\frac{\mu - |\lambda|}{1 - |\lambda|} \right] a - \sqrt{|\lambda|} \left[\frac{\mu - 1}{1 - |\lambda|} \right] a \\
 &= \frac{a}{1 - |\lambda|} [\mu - |\lambda| - \mu\sqrt{|\lambda|} + \sqrt{|\lambda|}]
 \end{aligned}$$

Further,

$$\begin{aligned}
 AM &= |x\text{-coordinate of M} - x\text{-coordinate of A}| \\
 &= \left| \frac{a}{1 - \sqrt{|\lambda|}} [\mu - |\lambda| - \mu\sqrt{|\lambda|} + \sqrt{|\lambda|}] - a \right| \\
 &= \left| \frac{a(\mu - 1)(1 - \sqrt{|\lambda|})}{1 - |\lambda|} \right| \\
 &= \frac{|\mu - 1|}{1 + \sqrt{|\lambda|}} a
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 BM &= |x\text{-coordinate of M} - x\text{-coordinate of B}| \\
 &= \left| \frac{a}{1 - |\lambda|} [\mu - |\lambda| - \mu\sqrt{|\lambda|} + \sqrt{|\lambda|}] - \mu a \right| \\
 &= \left| \frac{a(\mu - 1)(\lambda - \sqrt{|\lambda|})}{1 - |\lambda|} \right| \\
 &= \sqrt{|\lambda|} \frac{|\mu - 1|}{1 + \sqrt{|\lambda|}} a
 \end{aligned}$$

Therefore,

$$\frac{AM}{BM} = \frac{a |\mu - 1|}{1 + \sqrt{|\lambda|}} \cdot \frac{1 + \sqrt{|\lambda|}}{a \sqrt{|\lambda|} |\mu - 1|}$$

$$= \boxed{\frac{1}{\sqrt{|\lambda|}}}$$

$$\begin{aligned} AN &= |x\text{-coordinate of N} - x\text{-coordinate of A}| \\ &= \left| \frac{\mu - \sqrt{|\lambda|}}{1 - \sqrt{|\lambda|}} a - a \right| \\ &= \left| \frac{\mu - 1}{1 - \sqrt{|\lambda|}} \right| a \end{aligned}$$

$$\begin{aligned} BN &= |x\text{-coordinate of N} - x\text{-coordinate of B}| \\ &= \left| \frac{\mu - \sqrt{|\lambda|}}{1 - \sqrt{|\lambda|}} a - \mu a \right| \\ &= \sqrt{|\lambda|} \left| \frac{\mu - 1}{1 - \sqrt{|\lambda|}} \right| a \end{aligned}$$

$$\begin{aligned} \frac{AN}{BN} &= \frac{a |\mu - 1|}{\left| 1 - \sqrt{|\lambda|} \right|} \cdot \frac{\left| 1 - \sqrt{|\lambda|} \right|}{a \sqrt{|\lambda|} |\mu - 1|} \\ &= \boxed{\frac{1}{\sqrt{|\lambda|}}} \end{aligned} \quad (83)$$

B. In order to look for the locus of points at which the lines of force are parallel to \overline{AB} , the component of electric field perpendicular to \overline{AB} must vanish. Thus setting E_y and E_z to zero in Eq.(1) and replacing λ by $-|\lambda|$, we get

$$\begin{aligned} E_y &= qy \left\{ \frac{1}{[(x-a)^2 + y^2 + z^2]^{\frac{3}{2}}} - \frac{|\lambda|}{[(x-\mu a)^2 + y^2 + z^2]^{\frac{3}{2}}} \right\} = 0 \\ E_z &= qz \left\{ \frac{1}{[(x-a)^2 + y^2 + z^2]^{\frac{3}{2}}} - \frac{|\lambda|}{[(x-\mu a)^2 + y^2 + z^2]^{\frac{3}{2}}} \right\} = 0 \end{aligned}$$

Either of the equations gives

$$\begin{aligned} \frac{(x - \mu a)^2 + y^2 + z^2}{(x - a)^2 + y^2 + z^2} &= |\lambda|^{\frac{2}{3}} \\ x^2 \left(1 - |\lambda|^{\frac{2}{3}} \right) + 2a \left(|\lambda|^{\frac{2}{3}} - \mu \right) x + y^2 \left(1 - |\lambda|^{\frac{2}{3}} \right) + z^2 \left(1 - |\lambda|^{\frac{2}{3}} \right) &= a^2 \left(|\lambda|^{\frac{2}{3}} - \mu^2 \right) \end{aligned}$$

Completing the square,

$$x^2 - 2a \left[\frac{\mu - |\lambda|^{\frac{2}{3}}}{1 - |\lambda|^{\frac{2}{3}}} \right] x + \left[a \left(\frac{\mu - |\lambda|^{\frac{2}{3}}}{1 - |\lambda|^{\frac{2}{3}}} \right) \right]^2 + y^2 + z^2 = \frac{a^2 (|\lambda|^{\frac{2}{3}} - \mu^2)}{1 - |\lambda|^{\frac{2}{3}}} + \left[a \left(\frac{\mu - |\lambda|^{\frac{2}{3}}}{1 - |\lambda|^{\frac{2}{3}}} \right) \right]^2$$

$$\left[x - a \left(\frac{\mu - |\lambda|^{\frac{2}{3}}}{1 - |\lambda|^{\frac{2}{3}}} \right) \right]^2 + y^2 + z^2 = \left[\frac{|\lambda|^{\frac{1}{3}} (\mu - 1) a}{1 - |\lambda|^{\frac{2}{3}}} \right]^2$$

This is the equation of a sphere with radius,

$$\boxed{\sqrt[3]{|\lambda|} \left| \frac{\mu - 1}{1 - \sqrt[3]{|\lambda|^2}} \right| a}$$

and center

$$\left(a \left[\frac{\mu - |\lambda|^{\frac{2}{3}}}{1 - |\lambda|^{\frac{2}{3}}} \right], 0, 0 \right)$$

XII. A. Refer to Figures 35a, 35b and 35c.

$$\begin{aligned} AS &= AR \cos \theta_1 & BS &= BR \cos (\pi - \theta_2) \\ AR &= \frac{AS}{\cos \theta_1} & BR &= -\frac{BS}{\cos \theta_2} \end{aligned}$$

For R close to N: $S \rightarrow N$

$$AR \approx \frac{AN}{\cos \theta_1}, \quad BR \approx -\frac{BN}{\cos \theta_2} \quad (84)$$

For the component of electric field parallel to \overline{AB} at a point R on the limiting line of force is

$$\begin{aligned} E_{\parallel AB} &= \frac{q}{AR^2} \cos \theta_1 - \frac{(\pm |\lambda| q)}{BR^2} \cos (\pi - \theta_2) \\ &= q \left[\frac{\cos \theta_1}{AR^2} \pm |\lambda| \frac{\cos \theta_2}{BR^2} \right] \end{aligned} \quad (85)$$

Similarly, the component of electric field perpendicular to \overline{AB} at a point R on the limiting line of force is

$$\begin{aligned} E_{\perp AB} &= \frac{q}{AR^2} \sin \theta_1 + \frac{(\pm |\lambda| q)}{BR^2} \sin (\pi - \theta_2) \\ &= q \left[\frac{\sin \theta_1}{AR^2} \pm |\lambda| \frac{\sin \theta_2}{BR^2} \right] \end{aligned} \quad (86)$$

Eqs.(85) and (86) implies

$$\begin{aligned}
\frac{E_{\perp} \text{ AB}}{E_{\parallel} \text{ AB}} &= \frac{\frac{\sin \theta_1}{AR^2} \pm |\lambda| \frac{\sin \theta_2}{BR^2}}{\frac{\cos \theta_1}{AR^2} \pm |\lambda| \frac{\cos \theta_2}{BR^2}} \\
\text{Eq.(84)} \Rightarrow &\approx \frac{\frac{\sin \theta_1 \cos^2 \theta_1}{AN^2} \pm |\lambda| \frac{\sin \theta_2 \cos^2 \theta_2}{BN^2}}{\frac{\cos^3 \theta_1}{AN^2} \pm |\lambda| \frac{\cos^3 \theta_2}{BN^2}} \\
\text{Eq.(83)} \Rightarrow &= \frac{\frac{\sin \theta_1 \cos^2 \theta_1}{AN^2} \pm |\lambda| \frac{\sin \theta_2 \cos^2 \theta_2}{(\sqrt{|\lambda|AN})^2}}{\frac{\cos^3 \theta_1}{AN^2} \pm |\lambda| \frac{\cos^3 \theta_2}{(\sqrt{|\lambda|AN})^2}}
\end{aligned}$$

Finally,

$$\frac{E_{\perp} \text{ AB}}{E_{\parallel} \text{ AB}} \approx \frac{\sin \theta_1 \cos^2 \theta_1 \pm \sin \theta_2 \cos^2 \theta_2}{\cos^3 \theta_1 \pm \cos^3 \theta_2} \quad \text{for } \begin{cases} \lambda > 0 \\ \lambda < 0 \end{cases} \quad (87)$$

We need to evaluate Eq.(87) when R os close to N, i.e., $S \rightarrow N$, so that for

$$\lambda > 0 : \begin{cases} \theta_1 \rightarrow 0 \\ \theta_2 \rightarrow \pi \end{cases}, \quad -1 < \lambda < 0 : \begin{cases} \theta_1 \rightarrow 0 \\ \theta_2 \rightarrow 0 \end{cases}, \quad \lambda < -1 : \begin{cases} \theta_1 \rightarrow \pi \\ \theta_2 \rightarrow \pi \end{cases}$$

Taylor series expansion of a function, $f(\theta)$ about the point $\theta = \theta_0$:

$$f(\theta) = f(\theta_0) + (\theta - \theta_0) f'(\theta)|_{\theta=\theta_0} + \frac{(\theta - \theta_0)^2}{2!} f''(\theta)|_{\theta=\theta_0} + \dots$$

$$\begin{aligned}
f(\theta) &= \cos \theta, & f'(\theta) &= -\sin \theta, & f''(\theta) &= -\cos \theta \\
f(\theta)|_{\theta=0} &= 1, & f'(\theta)|_{\theta=0} &= 0, & f''(\theta)|_{\theta=0} &= -1 \\
f(\theta)|_{\theta=\pi} &= -1, & f'(\theta)|_{\theta=\pi} &= 0, & f''(\theta)|_{\theta=\pi} &= 1
\end{aligned}$$

$$\begin{aligned}
f(\theta) &= \sin \theta, & f'(\theta) &= \cos \theta, & f''(\theta) &= -\sin \theta \\
f(\theta)|_{\theta=0} &= 0, & f'(\theta)|_{\theta=0} &= 1, & f''(\theta)|_{\theta=0} &= -1 \\
f(\theta)|_{\theta=\pi} &= 0, & f'(\theta)|_{\theta=\pi} &= -1, & f''(\theta)|_{\theta=\pi} &= 0
\end{aligned}$$

$$\Rightarrow \cos \theta \approx \begin{cases} 1 - \frac{\theta^2}{2} & \text{about } \theta = 0 \\ -1 + \frac{(\theta - \pi)^2}{2} & \text{about } \theta = \pi \end{cases} \quad \text{and} \quad \sin \theta \approx \begin{cases} \theta & \text{about } \theta = 0 \\ -(\theta - \pi) & \text{about } \theta = \pi \end{cases}$$

$$\Rightarrow \cos^3 \theta \approx \begin{cases} \left(1 - \frac{\theta^2}{2}\right)^3 \approx 1 - \frac{3\theta^2}{2} & \text{about } \theta = 0 \\ \left[-1 + \frac{(\theta - \pi)^2}{2}\right]^3 \approx -1 + \frac{3(\theta - \pi)^2}{2} & \text{about } \theta = \pi \end{cases} \quad (88)$$

$$\sin \theta \cos^2 \theta \approx \begin{cases} \theta \left(1 - \frac{\theta^2}{2}\right)^2 \approx \theta & \text{about } \theta = 0 \\ -(\theta - \pi) \left[-1 + \frac{(\theta - \pi)^2}{2}\right]^2 \approx -(\theta - \pi) & \text{about } \theta = \pi \end{cases} \quad (89)$$

Using Eqs.(88) and (89), we evaluate Eq.(87) for $\lambda > 0$, $-1 < \lambda < 0$ and $\lambda < -1$:

- $\lambda > 0$ (about $\theta_1 = 0$ and $\theta_2 = \pi$)

$$\begin{aligned} \frac{E_{\perp \text{ AB}}}{E_{\parallel \text{ AB}}} &\approx \frac{\theta_1 + [-(\theta_2 - \pi)]}{\left[1 - \frac{3\theta_1^2}{2}\right] + \left[-1 + \frac{3(\theta_2 - \pi)^2}{2}\right]} \\ &= \frac{\theta_1 - (\theta_2 - \pi)}{\frac{3}{2} [\theta_1^2 - (\theta_2 - \pi)^2]} \\ &= \frac{\theta_1 - (\theta_2 - \pi)}{\frac{3}{2} [\theta_1 - (\theta_2 - \pi)] [\theta_1 + (\theta_2 - \pi)]} \\ &= \boxed{-\frac{2}{3} \left(\frac{1}{\pi - \theta_1 - \theta_2} \right)} \end{aligned} \quad (90)$$

- $-1 < \lambda < 0$ (about $\theta_1 = 0$ and $\theta_2 = 0$)

$$\begin{aligned} \frac{E_{\perp \text{ AB}}}{E_{\parallel \text{ AB}}} &\approx \frac{\theta_1 - \theta_2}{\left[1 - \frac{3\theta_1^2}{2}\right] - \left[1 - \frac{3\theta_2^2}{2}\right]} \\ &= \frac{-(\theta_2 - \theta_1)}{\frac{3}{2} [\theta_2^2 - \theta_1^2]} \\ &= \frac{-(\theta_2 - \theta_1)}{\frac{3}{2} (\theta_2 - \theta_1) (\theta_2 + \theta_1)} \\ &= \boxed{-\frac{2}{3} \left(\frac{1}{\theta_1 + \theta_2} \right)} \end{aligned} \quad (91)$$

- $\lambda < -1$ (about $\theta_1 = \pi$ and $\theta_2 = \pi$)

$$\begin{aligned}
\frac{E_{\perp AB}}{E_{\parallel AB}} &\approx \frac{[-(\theta_1 - \pi)] - [-(\theta_2 - \pi)]}{\left[-1 + \frac{3(\theta_1 - \pi)^2}{2}\right] - \left[-1 + \frac{3(\theta_2 - \pi)^2}{2}\right]} \\
&= \frac{(\theta_2 - \theta_1)}{\frac{3}{2} [(\theta_1 - \pi)^2 - (\theta_2 - \pi)^2]} \\
&= \frac{-(\theta_1 - \theta_2)}{\frac{3}{2} (\theta_1 - \theta_2) (\theta_1 + \theta_2 - 2\pi)} \\
&= \boxed{-\frac{2}{3} \left(\frac{1}{\theta_1 + \theta_2 - 2\pi} \right)} \tag{92}
\end{aligned}$$

B. Setting $\theta_1 = 0$, $\theta_2 = \pi$ in Eq.(90), $\theta_1 = 0$, $\theta_2 = 0$ in Eq.(91) and $\theta_1 = \pi$, $\theta_2 = \pi$ in Eq.(92) gives

$$\frac{E_{\perp AB}}{E_{\parallel AB}} \rightarrow \infty \Rightarrow \tan^{-1} \left(\frac{E_{\perp AB}}{E_{\parallel AB}} \right) = \tan^{-1}(\infty) = \boxed{\frac{\pi}{2}}$$

Therefore, the limiting line of force through N intersects \overline{AB} at right angles.

XIII. A. Here refer to Figures 36a, 36b and 36c. Recall that the coordinates of A, B and N are

$$A : (a, 0, 0), \quad B : (\mu a, 0, 0), \quad N : \left(\left[\frac{\mu \pm \sqrt{|\lambda|}}{1 \pm \sqrt{|\lambda|}} \right] a, 0, 0 \right) \equiv (x_N, 0, 0) \quad \text{for } \begin{cases} \lambda > 0 \\ \lambda < 0 \end{cases}$$

Let $S(x,y,z)$ be any point on an equipotential surface. Then

$$r^2 = (x - x_N)^2 + y^2 + z^2 \tag{93}$$

$$r \cos \theta = x - x_N \tag{94}$$

The potentials, Φ at the points S and N are given by

$$\begin{aligned}
\Phi_S &= \frac{q}{AS} + \frac{\pm(|\lambda|q)}{BS}, & \Phi_N &= \frac{q}{AN} + \frac{\pm(|\lambda|q)}{BN} \\
\Rightarrow \Phi_S - \Phi_N &= \frac{q}{AS} \pm \frac{|\lambda|q}{BS} - \left(\frac{q}{AN} \pm \frac{|\lambda|q}{BN} \right) \\
\Rightarrow \frac{\Phi_S - \Phi_N}{q} &= \left(\frac{1}{AS} - \frac{1}{AN} \right) \pm |\lambda| \left(\frac{1}{BS} - \frac{1}{BN} \right) \tag{95}
\end{aligned}$$

We next compute AS and BS. To that end,

$$\begin{aligned}
AS^2 &= AN^2 + r^2 - 2rAN \left[\frac{\cos(\pi - \theta)}{\cos \theta} \right] & \text{for } \begin{cases} \lambda > 0, & -1 < \lambda < 0 \\ \lambda < -1 \end{cases} \\
&= AN^2 \left[1 + \left(\frac{r}{AN} \right)^2 \pm 2 \left(\frac{r}{AN} \right) \cos \theta \right] & \text{for } \begin{cases} \lambda > 0, & -1 < \lambda < 0 \\ \lambda < -1 \end{cases} \\
\Rightarrow \frac{1}{AS} &= \frac{1}{AN} \left[1 + \left(\frac{r}{AN} \right)^2 \pm 2 \left(\frac{r}{AN} \right) \cos \theta \right]^{-\frac{1}{2}} & \text{for } \begin{cases} \lambda > 0, & -1 < \lambda < 0 \\ \lambda < -1 \end{cases}
\end{aligned}$$

We now expand the expression for $\frac{1}{AS}$ in the limit $r \ll AN$ (i.e. S close to N) using $(1 + \epsilon)^n = 1 + n\epsilon + \frac{n(n-1)}{2}\epsilon^2 + \dots$. Setting, $\epsilon = \pm 2 \left(\frac{r}{AN} \right) \cos \theta + \left(\frac{r}{AN} \right)^2$, we get

$$\begin{aligned}
\frac{1}{AS} &= \frac{1}{AN} \left\{ 1 + \left(-\frac{1}{2} \right) \left[\pm 2 \frac{r}{AN} \cos \theta + \left(\frac{r}{AN} \right)^2 \right] \right. \\
&\quad \left. + \frac{\left(-\frac{1}{2} \right) \left(-\frac{3}{2} \right)}{2} \left[\pm 2 \frac{r}{AN} \cos \theta + \left(\frac{r}{AN} \right)^2 \right]^2 + \dots \right\} & \text{for } \begin{cases} \lambda > 0, & -1 < \lambda < 0 \\ \lambda < -1 \end{cases} \\
&= \frac{1}{AN} \left\{ 1 \mp \frac{r}{AN} \cos \theta - \frac{1}{2} \left(\frac{r}{AN} \right)^2 \right. \\
&\quad \left. + \frac{3}{8} \left[4 \left(\frac{r}{AN} \right)^2 \cos^2 \theta + \left(\frac{r}{AN} \right)^4 \pm 4 \left(\frac{r}{AN} \right)^3 \cos \theta \right] + \dots \right\} & \text{for } \begin{cases} \lambda > 0, & -1 < \lambda < 0 \\ \lambda < -1 \end{cases}
\end{aligned}$$

Neglecting terms $(r/AN)^3$ and higher powers of (r/AN) , we get

$$\frac{1}{AS} \approx \frac{1}{AN} \left[1 \mp \frac{r}{AN} \cos \theta + \frac{1}{2} \left(\frac{r}{AN} \right)^2 (3 \cos^2 \theta - 1) \right] \text{ for } \begin{cases} \lambda > 0, & -1 < \lambda < 0 \\ \lambda < -1 \end{cases} \quad (96)$$

Similarly,

$$\begin{aligned}
BS^2 &= BN^2 + r^2 - 2rAN \left[\frac{\cos \theta}{\cos(\pi - \theta)} \right] & \text{for } \begin{cases} \lambda > 0, & \lambda < -1 \\ -1 < \lambda < 0 \end{cases} \\
&= BN^2 \left[1 + \left(\frac{r}{AN} \right)^2 \mp 2 \left(\frac{r}{BN} \right) \cos \theta \right] & \text{for } \begin{cases} \lambda > 0, & \lambda < -1 \\ -1 < \lambda < 0 \end{cases}
\end{aligned}$$

This expression for BS^2 is the same as AS^2 given on the previous page, provided we make the following transformation

$$AN \rightarrow BN, \quad AS \rightarrow BS, \quad \mp \cos \theta \rightarrow \pm \cos \theta.$$

Hence this transformation in Eq.(96) gives

$$\frac{1}{BS} \approx \frac{1}{BN} \left[1 \pm \frac{r}{BN} \cos \theta + \frac{1}{2} \left(\frac{r}{BN} \right)^2 (3 \cos^2 \theta - 1) \right] \text{ for } \begin{cases} \lambda > 0, & \lambda < -1 \\ -1 < \lambda < 0 \end{cases} \quad (97)$$

We now evaluate Eq.(95) using Eqs.(96) and (97) for each of cases where $\lambda > 0$, $-1 < \lambda < 0$ and $\lambda < -1$:

$$\frac{\Phi_S - \Phi_N}{q} \approx \begin{cases} \frac{1}{AN} \left[1 - \frac{r}{AN} \cos \theta + \frac{1}{2} \left(\frac{r}{AN} \right)^2 (3 \cos^2 \theta - 1) \right] - \frac{1}{AN} & (\lambda > 0) \\ + \frac{|\lambda|}{BN} \left[1 + \frac{r}{BN} \cos \theta + \frac{1}{2} \left(\frac{r}{BN} \right)^2 (3 \cos^2 \theta - 1) \right] - \frac{|\lambda|}{BN} & \\ \frac{1}{AN} \left[1 - \frac{r}{AN} \cos \theta + \frac{1}{2} \left(\frac{r}{AN} \right)^2 (3 \cos^2 \theta - 1) \right] - \frac{1}{AN} & (-1 < \lambda < 0) \\ - \frac{|\lambda|}{BN} \left[1 - \frac{r}{BN} \cos \theta + \frac{1}{2} \left(\frac{r}{BN} \right)^2 (3 \cos^2 \theta - 1) \right] + \frac{|\lambda|}{BN} & \\ \frac{1}{AN} \left[1 + \frac{r}{AN} \cos \theta + \frac{1}{2} \left(\frac{r}{AN} \right)^2 (3 \cos^2 \theta - 1) \right] - \frac{1}{AN} & (\lambda < -1) \\ - \frac{|\lambda|}{BN} \left[1 + \frac{r}{BN} \cos \theta + \frac{1}{2} \left(\frac{r}{BN} \right)^2 (3 \cos^2 \theta - 1) \right] + \frac{|\lambda|}{BN} & \end{cases}$$

$$\frac{\Phi_S - \Phi_N}{q} \approx \begin{cases} \frac{1}{AN} \left[-\frac{r}{AN} \cos \theta + \frac{1}{2} \left(\frac{r}{AN} \right)^2 (3 \cos^2 \theta - 1) \right] & (\lambda > 0) \\ + \frac{|\lambda|}{BN} \left[\frac{r}{BN} \cos \theta + \frac{1}{2} \left(\frac{r}{BN} \right)^2 (3 \cos^2 \theta - 1) \right] & \\ \frac{1}{AN} \left[-\frac{r}{AN} \cos \theta + \frac{1}{2} \left(\frac{r}{AN} \right)^2 (3 \cos^2 \theta - 1) \right] & (-1 < \lambda < 0) \\ - \frac{|\lambda|}{BN} \left[-\frac{r}{BN} \cos \theta + \frac{1}{2} \left(\frac{r}{BN} \right)^2 (3 \cos^2 \theta - 1) \right] & \\ \frac{1}{AN} \left[\frac{r}{AN} \cos \theta + \frac{1}{2} \left(\frac{r}{AN} \right)^2 (3 \cos^2 \theta - 1) \right] & (\lambda < -1) \\ - \frac{|\lambda|}{BN} \left[\frac{r}{BN} \cos \theta + \frac{1}{2} \left(\frac{r}{BN} \right)^2 (3 \cos^2 \theta - 1) \right] & \end{cases}$$

On using Eq.(83): $BN = AN\sqrt{|\lambda|}$ above, we get

$$\left(\frac{\Phi_S - \Phi_N}{q}\right)_{AN} \approx \begin{cases} -\frac{r}{AN} \cos \theta + \frac{1}{2} \left(\frac{r}{AN}\right)^2 (3 \cos^2 \theta - 1) & (\lambda > 0) \\ +\frac{r}{AN} \cos \theta + \frac{1}{2} \frac{1}{\sqrt{|\lambda|}} \left(\frac{r}{AN}\right)^2 (3 \cos^2 \theta - 1) & \\ -\frac{r}{AN} \cos \theta + \frac{1}{2} \left(\frac{r}{AN}\right)^2 (3 \cos^2 \theta - 1) & (-1 < \lambda < 0) \\ +\frac{r}{AN} \cos \theta - \frac{1}{2} \frac{1}{\sqrt{|\lambda|}} \left(\frac{r}{AN}\right)^2 (3 \cos^2 \theta - 1) & \\ \frac{r}{AN} \cos \theta + \frac{1}{2} \left(\frac{r}{AN}\right)^2 (3 \cos^2 \theta - 1) & (\lambda < -1) \\ -\frac{r}{AN} \cos \theta - \frac{1}{2} \frac{1}{\sqrt{|\lambda|}} \left(\frac{r}{AN}\right)^2 (3 \cos^2 \theta - 1) & \end{cases}$$

$$\left(\frac{\Phi_S - \Phi_N}{q}\right)_{AN} \approx \begin{cases} \frac{1}{2} \left(\frac{r}{AN}\right)^2 (3 \cos^2 \theta - 1) \left[1 + \frac{1}{\sqrt{|\lambda|}}\right] & (\lambda > 0) \\ \frac{1}{2} \left(\frac{r}{AN}\right)^2 (3 \cos^2 \theta - 1) \left[1 - \frac{1}{\sqrt{|\lambda|}}\right] & (\lambda < 0) \end{cases}$$

One can now write this last expression as

$$3r^2 \cos^2 \theta - r^2 = \frac{2 AN^3}{q} \left(\frac{\sqrt{|\lambda|}}{\sqrt{|\lambda| \pm 1}}\right) (\Phi_S - \Phi_N) \quad \begin{cases} \lambda > 0 \\ \lambda < 0 \end{cases}$$

The expression for AN can be calculated from Eq.(28):

$$\begin{aligned} AN &= \left| x\text{-coordinate of N } (x_N) - x\text{-coordinate of A} \right| \\ &= \left| \frac{\mu \pm \sqrt{|\lambda|}}{1 \pm \sqrt{|\lambda|}} a - a \right| \\ &= \left| \frac{\mu - 1}{1 \pm \sqrt{|\lambda|}} \right| a \quad \begin{cases} \lambda > 0 \\ \lambda < 0 \end{cases} \end{aligned} \quad (98)$$

On using Eqs.(93), (94) and (98),

$$3(x - x_N)^2 - \left[(x - x_N)^2 + y^2 + z^2 \right] = \frac{2a^3}{q} \left| \frac{\mu - 1}{1 \pm \sqrt{|\lambda|}} \right|^3 \left(\frac{\sqrt{|\lambda|}}{\sqrt{|\lambda| \pm 1}}\right) (\Phi_S - \Phi_N) \quad \begin{cases} \lambda > 0 \\ \lambda < 0 \end{cases}$$

or

$$2(x - x_{\mathbf{N}})^2 - y^2 - z^2 = \kappa \quad (99)$$

$$\text{where } \kappa \equiv \frac{2a^3}{q} \left| \frac{\mu - 1}{1 \pm \sqrt{|\lambda|}} \right|^3 \left(\frac{\sqrt{|\lambda|}}{\sqrt{|\lambda| \pm 1}} \right) (\Phi_{\mathbf{S}} - \Phi_{\mathbf{N}}) \quad \begin{cases} \lambda > 0 \\ \lambda < 0 \end{cases}$$

$$\text{and } x_{\mathbf{N}} \equiv \left[\frac{\mu \pm \sqrt{|\lambda|}}{1 \pm \sqrt{|\lambda|}} \right] a \quad \begin{cases} \lambda > 0 \\ \lambda < 0 \end{cases}$$

- CASE 1. $\kappa = 0$

Eq.(99) gives

$$2(x - x_{\mathbf{N}})^2 - y^2 - z^2 = 0$$

$$\boxed{2(x - x_{\mathbf{N}})^2 = y^2 + z^2} \quad (100)$$

This is an equation for a *right circular cone* with its vertex at $x = x_{\mathbf{N}}$.

- CASE 2. $\kappa > 0$

Replacing κ by $|\kappa|$ in Eq.(99) we get

$$2(x - x_{\mathbf{N}})^2 - y^2 - z^2 = |\kappa|$$

$$\boxed{\frac{(x - x_{\mathbf{N}})^2}{\frac{|\kappa|}{2}} - \frac{y^2}{|\kappa|} - \frac{z^2}{|\kappa|} = 1} \quad (101)$$

This is an equation for a *hyperboloid of revolution of two sheets* about the line of charges, \overline{AB} .

- CASE 3. $\kappa < 0$

Replacing κ by $-|\kappa|$ in Eq.(99) we get

$$2(x - x_{\mathbf{N}})^2 - y^2 - z^2 = -|\kappa|$$

$$\boxed{\frac{y^2}{|\kappa|} + \frac{z^2}{|\kappa|} - \frac{(x - x_{\mathbf{N}})^2}{\frac{|\kappa|}{2}} = 1} \quad (102)$$

This is an equation for a *hyperboloid of revolution of one sheet* about the line of charges, \overline{AB} .

All the three cases are demonstrated in Figure 37.

B. In the xy -plane ($z = 0$), Eq.(100) implies

$$2(x - x_{\mathbf{N}})^2 - y^2 = 0 \Rightarrow y = \pm\sqrt{2}x \mp \sqrt{2}x_{\mathbf{N}}$$

Identifying this last equation with that of straight line: $y = (\pm \tan \alpha)x + b$ gives

$$\tan \alpha = \sqrt{2} \Rightarrow \boxed{\alpha = \tan^{-1}(\sqrt{2})}$$

See Figure 37.

C. In the xy -plane ($z = 0$), Eq.(99) implies

$$2(x - x_{\mathbf{N}})^2 - y^2 = |\kappa|$$

Taking the derivative of this last equation w.r.t. x ,

$$4(x - x_{\mathbf{N}}) - 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{2(x - x_{\mathbf{N}})}{y}$$

This is the slope of a tangent line to an equipotential curve *near* N at the point (x, y) . Therefore, the slope of a tangent line to a line of force at the same point will be

$$\frac{dy}{dx} = \frac{-1}{\frac{2(x - x_{\mathbf{N}})}{y}}$$

This is because line of force is perpendicular to its corresponding equipotential curve. Thus,

$$\begin{aligned} \int \frac{dy}{y} &= -\frac{1}{2} \int \frac{dx}{x - x_{\mathbf{N}}} \\ \Rightarrow \ln y &= -\frac{1}{2} \ln(x - x_{\mathbf{N}}) + \ln(\text{Constant}) \\ y &= (x - x_{\mathbf{N}})^{-\frac{1}{2}} \cdot \text{Constant} \end{aligned}$$

Using the expression for x_N , we obtain

$$y^2 \left[x - \left(\frac{\mu \pm \sqrt{|\lambda|}}{1 \pm \sqrt{|\lambda|}} \right) a \right] = \text{Constant}$$

PROBLEM 2

- I. A. (i) Refer to Figure 38. The equations of motion for the charges ($m_1 = m$, $q_1 = q$) and ($m_2 = \sigma m$, $q_2 = \lambda q$) are

$$m_2 \frac{d^2 x_2}{dt^2} = \frac{q_1 q_2}{(x_2 - x_1)^2}$$

$$m_1 \frac{d^2 x_1}{dt^2} = -\frac{q_1 q_2}{(x_2 - x_1)^2}$$

Multiplying the first equation by m_1 and second by m_2 and subtracting we get

$$m_1 m_2 \frac{d^2}{dt^2} (x_2 - x_1) = (m_1 + m_2) \frac{q_1 q_2}{(x_2 - x_1)^2}$$

Defining $x \equiv x_2 - x_1$ and noting that $\frac{m_1 + m_2}{m_1 m_2} = \frac{1 + \sigma}{m\sigma}$, $q_1 q_2 = \lambda q^2$,

the last equation can be written as

$$\frac{d^2 x}{dt^2} = \frac{\omega}{x^2}, \quad \omega \equiv \frac{\lambda q^2}{m} \frac{\lambda(\sigma + 1)}{\sigma} \quad (103)$$

Now,

$$\frac{d^2 x}{dt^2} = \frac{d}{dt} \left(\frac{dx}{dt} \right) = \frac{dv}{dt} = \frac{dx}{dt} \frac{dv}{dx} = v \frac{dv}{dx} = \frac{d}{dx} \left(\frac{1}{2} v^2 \right)$$

With this expression for $\frac{d^2 x}{dt^2}$ in Eq.(103), we have

$$\int d \left(\frac{1}{2} v^2 \right) = \omega \int \frac{dx}{x^2}$$

$$\Rightarrow \frac{1}{2} v^2 = -\frac{\omega}{x} + C_1$$

The constant of integration C_1 is determined from the given initial conditions: $v = 0$ at $x = x_0 \equiv x_{02} - x_{01} = a(\mu - 1)$. This gives $C_1 = \omega/x_0$. Therefore,

$$\frac{1}{2} v^2 = \frac{1}{2} \left(\frac{dx}{dt} \right)^2 = \left(\frac{1}{x_0} - \frac{1}{x} \right)$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = 2\omega \left(\frac{x-x_0}{xx_0}\right)$$

- CASE 1: $\lambda < 0$ ($\lambda = -|\lambda|$)

$$\omega = -\frac{|\lambda|q^2 \lambda(\sigma+1)}{m \sigma} = -|\omega|$$

$$\frac{dx}{dt} = \pm \sqrt{2|\omega| \left(\frac{x_0-x}{xx_0}\right)}$$

Choosing the negative sign, we get

$$\int \frac{\sqrt{x}}{\sqrt{x_0-x}} dx = -\sqrt{\frac{2|\omega|}{x_0}} \int dt$$

The result for the left hand side integral is proven in the appendix (see Eq.(.)). Performing the integration in the last equation above, we obtain,

$$x_0 \sin^{-1} \left(\sqrt{\frac{x}{x_0}} \right) + \sqrt{x(x_0-x)} = -\sqrt{\frac{2|\omega|}{x_0}} t + C_2.$$

The constant of integration C_2 can easily be found from the information that $t = 0$, $x = x_0$. This gives $C_2 = x_0\pi/2$. Thus

$$\sqrt{\frac{2|\omega|}{x_0}} t = \frac{x_0\pi}{2} - x_0 \sin^{-1} \left(\sqrt{\frac{x}{x_0}} \right) - \sqrt{x(x_0-x)}$$

Inserting expressions for $x_0 = a(\mu-1)$ and $|\omega| = \frac{|\lambda|q^2 \lambda(\sigma+1)}{m \sigma}$, we obtain

$$t = \sqrt{\frac{ma \sigma(\mu-1)}{2q^2 |\lambda|(\sigma+1)}} \left\{ \frac{a}{2}(\mu-1) \left[\pi - \sin^{-1} \left(\sqrt{\frac{2x}{a(\mu-1)}} \right) \right] - \sqrt{x[a(\mu-1)-x]} \right\} \quad (104)$$

- CASE 2: $\lambda > 0$ ($\lambda = |\lambda|$)

$$\omega = \frac{|\lambda|q^2 \lambda(\sigma+1)}{m \sigma} = |\omega|$$

$$\frac{dx}{dt} = \pm \sqrt{2|\omega| \left(\frac{x-x_0}{xx_0}\right)}$$

Choosing the positive sign, we get

$$\int \frac{\sqrt{x}}{\sqrt{x-x_0}} dx = \sqrt{\frac{2|\omega|}{x_0}} \int dt$$

The result for the left hand side integral is proven in the appendix (see Eq.(.)). Performing the integration in the last equation above, we obtain,

$$\frac{1}{2}x_0 \cosh^{-1} \left[\frac{2x}{x_0} - 1 \right] + \sqrt{x[x-x_0]} = \sqrt{\frac{2|\omega|}{x_0}}t + C_3.$$

Again the constant of integration, C_3 is obtained using the initial condition: at $t = 0$, $x = x_0$. This gives $C_3 = 0$. As before, inserting the expression for x_0 and $|\omega|$, we finally obtain,

$$t = \sqrt{\frac{ma}{2q^2} \frac{\sigma(\mu-1)}{|\lambda|(\sigma+1)}} \left\{ \frac{1}{2}a(\mu-1) \cosh^{-1} \left[\frac{2x}{a(\mu-1)} - 1 \right] + \sqrt{x[x-a(\mu-1)]} \right\}$$

- (ii) On collision, $x = 0$. With this value for x , the time for collision is easily obtained from Eq.(104):

$$t_c = \frac{a(\mu-1)}{2} \pi \sqrt{\frac{ma}{2q^2} \frac{\sigma(\mu-1)}{|\lambda|(\sigma+1)}}$$

- B. (i) With the information provided in the statement of the problem, one can write the equations of motion for the charges as

$$\beta_2 \frac{dx_2}{dt} = \frac{q_1 q_2}{(x_2 - x_1)^2}$$

$$\beta_1 \frac{dx_1}{dt} = -\frac{q_1 q_2}{(x_2 - x_1)^2}$$

Here $\beta_2 = \beta$ and $\beta_1 = 1$. Multiplying the first equation by β_1 and second by β_2 and subtracting we get

$$\begin{aligned} \frac{d}{dt} (x_2 - x_1) &= \left(\frac{\beta_1 + \beta_2}{\beta_1 \beta_2} \right) \frac{q_1 q_2}{(x_2 - x_1)^2} \\ \Rightarrow \frac{dx}{dt} &= q^2 \frac{\lambda(1+\beta)}{\beta} \frac{1}{x^2} \end{aligned}$$

- CASE 1: $\lambda < 0$ ($\lambda = -|\lambda|$)

$$\int x^2 dx = -q^2 \frac{|\lambda|(1+\beta)}{\beta} \int dt$$

$$\Rightarrow \frac{x^3}{3} = -q^2 \frac{|\lambda|(1+\beta)}{\beta} t + C_4.$$

Using the initial condition: $t = 0$, $x = x_0$, gives $C_4 = x_0^3/3$. With this value for C_4 , we get

$$\boxed{t = \frac{1}{3q^2} \frac{\beta}{|\lambda|(1+\beta)} \left[\frac{a^3}{8} (\mu-1)^3 - x^3 \right]} \quad (105)$$

- CASE 2: $\lambda > 0$ ($\lambda = |\lambda|$)

$$\int x^2 dx = q^2 \frac{|\lambda|(1+\beta)}{\beta} \int dt$$

$$\Rightarrow \frac{x^3}{3} = q^2 \frac{|\lambda|(1+\beta)}{\beta} t + C_5.$$

Using the initial conditions gives for $C_5 = x_0^3/3$. Thus

$$\boxed{t = \frac{1}{3q^2} \frac{\beta}{|\lambda|(1+\beta)} \left[x^3 - \frac{a^3}{8} (\mu-1)^3 \right]}$$

- (ii) On collision, $x = 0$. With this value for x , the time for collision is easily obtained from Eq.(105):

$$\boxed{t_c = \frac{a^3}{24q^2} \frac{\beta (\mu-1)^3}{|\lambda| (\beta+1)}}$$

- II. A. For this part of the problem we set: $\lambda = |\lambda|$. The situation is sketched in Figure 39a. The free-body diagram of the charge $|\lambda|q$ is shown in Figure 39b. Application of Newton's second law immediately gives

$$T - (F + \sigma mg \cos \theta) = m \frac{v^2}{l}.$$

Here v is the tangential velocity of the charge at the instant the string makes an angle θ with the downward vertical. l is the length of the string given by $a(\mu-1)$. F is the coulomb force between the charges and it is given by $\frac{|\lambda|q^2}{l^2}$. At the highest point B_h where $\theta = \pi$, the tension, T_{B_h} and the velocity v_{B_h} are related by

$$T_{B_h} = \frac{mv_{B_h}^2}{l} - \sigma mg + \frac{|\lambda|q^2}{l^2}.$$

In order to find the minimum speed of the charge $|\lambda|q$ at the point B_h to make one complete revolution, we set $T_{B_h}|_{\min} = 0$ in the last equation to get

$$0 = \frac{m v_{B_h}^2|_{\min}}{l} - \sigma mg + \frac{|\lambda|q^2}{l^2}$$

$$\Rightarrow v_{B_h}^2|_{\min} = \sigma gl - \frac{|\lambda|q^2}{ml} \quad (106)$$

Conservation of mechanical energy between the points B_0 and B_h (see Figures 40a and 40b) gives

$$\frac{1}{2}\sigma m v_{B_0}^2|_{\min} + \sigma mgl(1 - \cos \alpha) = \frac{1}{2}\sigma m v_{B_h}^2|_{\min} + 2\sigma mgl$$

$$\Rightarrow v_{B_0}|_{\min} = \sqrt{v_{B_h}^2|_{\min} + 2gl(1 + \cos \alpha)}$$

On using Eq.(106) we get,

$$v_{B_0}|_{\min} = \sqrt{\sigma gl - \frac{|\lambda|q^2}{ml} + 2gl(1 + \cos \alpha)}$$

Finally using $l = a(\mu - 1)$ we get

$$v_{B_0}|_{\min} = \sqrt{ga(\mu - 1)(\sigma + 2 + 2 \cos \alpha) - \frac{|\lambda|q^2}{ma(\mu - 1)}}.$$

B. The situation is depicted in Figure 41a. Using the free-body (Figure 41b) for the system one can write down the Newton's second law:

$$F_{\text{system}}^{\text{net}} = qE - |\lambda|qE = (m + \sigma m)a$$

$$\Rightarrow a = \frac{qE(1 - |\lambda|)}{m(1 + \sigma)} \quad (107)$$

Here, a represents the common acceleration of the two charges. Next consider the net force on anyone of the charges (say q). From Figure 41c we have

$$F_m^{\text{net}} = qE - \frac{|\lambda|q^2}{l^2} = ma$$

$$\Rightarrow a = \frac{qE}{m} - \frac{|\lambda|q^2}{ml^2} \quad (108)$$

Eqs.(107) and (108) gives

$$\frac{qE(1 - |\lambda|)}{m(1 + \sigma)} = \frac{qE}{m} - \frac{|\lambda|q^2}{ml^2}$$

simplifying,

$$\frac{|\lambda|q}{l^2} = E \left(\frac{\sigma + |\lambda|}{1 + \sigma} \right)$$

On using $l = a(\mu - 1)$, we get

$$\boxed{\mu = 1 + \frac{1}{a} \sqrt{\frac{q}{E_0} \frac{|\lambda|(1 + \sigma)}{\sigma + |\lambda|}}}$$

III. See Figure 42. Newton's second law in angular form gives

$$\sum \tau = I(-\alpha)$$

Here, the left hand side represents sum of the torques about the pivot O. I is the moment of inertia of the system about O and perpendicular to the length of the rod and α represents the angular acceleration of the system. The negative sign signifies that the angular displacement is opposite to the angular acceleration. Thus,

$$|\lambda|qE \left(\frac{l}{2} \sin \theta \right) + qE \left(\frac{l}{2} \sin \theta \right) = -I \frac{d^2\theta}{dt^2}$$

The moment of inertia is calculated to be

$$I = m \left(\frac{l}{2} \right)^2 + \sigma m \left(\frac{l}{2} \right)^2 = \frac{ml^2}{4}(1 + \sigma)$$

Thus,

$$\frac{d^2\theta}{dt^2} + \left[\frac{2qE(1 + |\lambda|)}{ml(1 + \sigma)} \right] \sin \theta = 0$$

For small θ , the angular frequency, ω is

$$\omega^2 = \frac{2qE(1 + |\lambda|)}{ml(1 + \sigma)}$$

and the period of small oscillations, $T(\equiv \frac{2\pi}{\omega})$ is

$$\boxed{T = 2\pi \sqrt{\frac{ma}{2qE_0} \frac{(\mu - 1)(\sigma + 1)}{|\lambda + 1|}}}$$

Appendix

$$\boxed{\int \frac{d\alpha}{(\alpha^2 + \beta^2)^{\frac{3}{2}}} = \frac{1}{\beta^2} \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}} \quad (109)$$

Proof:

$$\begin{aligned} \int \frac{d\alpha}{(\alpha^2 + \beta^2)^{\frac{3}{2}}} &= \int \frac{\beta \sec^2 \gamma}{(\beta^2 \tan^2 \gamma + \beta^2)^{\frac{3}{2}}} d\gamma && (\alpha \equiv \beta \tan \gamma \Rightarrow d\alpha = \beta \sec^2 \gamma d\gamma) \\ &= \int \frac{\beta \sec^2 \gamma}{\beta^3 \sec^3 \gamma} d\gamma \\ &= \frac{1}{\beta^2} \int \cos \gamma d\gamma \\ &= \frac{1}{\beta^2} \sin \gamma \\ &= \frac{1}{\beta^2} \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} && \text{(see Figure 43)} \end{aligned}$$

$$\boxed{\int \frac{\sqrt{\alpha^2 - \gamma^2}}{\beta^2 + \gamma^2} d\gamma = -\sin^{-1} \left(\frac{\gamma}{\alpha} \right) - \frac{\sqrt{\alpha^2 + \beta^2}}{\beta} \sin^{-1} \left(\frac{\beta}{\alpha} \sqrt{\frac{\alpha^2 - \gamma^2}{\beta^2 + \gamma^2}} \right)} \quad (110)$$

Proof:

$$\begin{aligned} \int \frac{\sqrt{\alpha^2 - \gamma^2}}{\beta^2 + \gamma^2} d\gamma &= \int \frac{\sqrt{\alpha^2 - \gamma^2}}{\beta^2 + \gamma^2} \cdot \frac{\sqrt{\alpha^2 - \gamma^2}}{\sqrt{\alpha^2 - \gamma^2}} d\gamma \\ &= \int \frac{1}{\sqrt{\alpha^2 - \gamma^2}} \left[\frac{\alpha^2 - \gamma^2}{\beta^2 + \gamma^2} \right] d\gamma \\ &= \int \frac{1}{\sqrt{\alpha^2 - \gamma^2}} \left[-1 + \frac{\alpha^2 + \beta^2}{\gamma^2 + \beta^2} \right] d\gamma \\ &= -\int \frac{1}{\sqrt{\alpha^2 - \gamma^2}} d\gamma + (\alpha^2 + \beta^2) \int \frac{1}{(\beta^2 + \gamma^2) \sqrt{\alpha^2 - \gamma^2}} d\gamma \\ &= I_1 + I_2 \end{aligned}$$

where,

$$I_1 = -\int \frac{1}{\sqrt{\alpha^2 - \gamma^2}} d\gamma \quad \text{and} \quad I_2 = (\alpha^2 + \beta^2) \int \frac{1}{(\beta^2 + \gamma^2) \sqrt{\alpha^2 - \gamma^2}} d\gamma$$

Therefore,

$$I_1 = -\int \frac{1}{\sqrt{\alpha^2 - \gamma^2}} d\gamma$$

$$\begin{aligned}
&= - \int \frac{\alpha \cos \theta}{\alpha \cos \theta} d\theta && (\gamma \equiv \alpha \sin \theta \Rightarrow d\gamma = \alpha \cos \theta d\theta) \\
&= -\theta \\
&= -\sin^{-1}\left(\frac{\gamma}{\alpha}\right)
\end{aligned}$$

To evaluate I_2 , we use the substitution

$$\eta = \sqrt{\frac{\alpha^2 - \gamma^2}{\beta^2 + \gamma^2}}$$

$$\begin{aligned}
\Rightarrow \frac{d\eta}{d\gamma} &= \frac{\sqrt{\beta^2 + \gamma^2} \left(\frac{1}{2} \cdot \frac{-2\gamma}{\sqrt{\alpha^2 - \gamma^2}} \right) - \sqrt{\alpha^2 - \gamma^2} \left(\frac{1}{2} \cdot \frac{2\gamma}{\sqrt{\beta^2 + \gamma^2}} \right)}{\beta^2 + \gamma^2} \\
&= -(\alpha^2 + \beta^2) \cdot \frac{\gamma}{\sqrt{\beta^2 + \gamma^2}} \cdot \frac{1}{(\beta^2 + \gamma^2) \sqrt{\alpha^2 - \gamma^2}} \\
\frac{d\gamma}{(\beta^2 + \gamma^2) \sqrt{\alpha^2 - \gamma^2}} &= -\frac{1}{\alpha^2 + \beta^2} \left(\frac{\sqrt{\beta^2 + \gamma^2}}{\gamma} \right) d\eta
\end{aligned}$$

Further,

$$\begin{aligned}
\eta^2 &= \frac{\alpha^2 - \gamma^2}{\beta^2 + \gamma^2} \\
\Rightarrow \gamma^2 &= \frac{\alpha^2 - \beta^2 \eta^2}{1 + \eta^2} \\
\Rightarrow \beta^2 + \gamma^2 &= \frac{\alpha^2 + \beta^2}{1 + \eta^2} \\
\Rightarrow \frac{\beta^2 + \gamma^2}{\gamma^2} &= \frac{\alpha^2 + \beta^2}{1 + \eta^2} \cdot \frac{1 + \eta^2}{\alpha^2 - \beta^2 \eta^2} \\
\Rightarrow \frac{\sqrt{\beta^2 + \gamma^2}}{\gamma} &= \sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2 - \beta^2 \eta^2}}
\end{aligned}$$

Thus,

$$\frac{d\gamma}{(\beta^2 + \gamma^2) \sqrt{\alpha^2 - \gamma^2}} = -\frac{1}{\alpha^2 + \beta^2} \sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2 - \beta^2 \eta^2}} d\eta$$

$$\begin{aligned}
I_2 &= (\alpha^2 + \beta^2) \int \frac{1}{(\beta^2 + \gamma^2) \sqrt{\alpha^2 - \gamma^2}} d\gamma \\
&= (\alpha^2 + \beta^2) \int -\frac{1}{\alpha^2 + \beta^2} \sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2 - \beta^2 \eta^2}} d\eta
\end{aligned}$$

$$\begin{aligned}
&= -\frac{\sqrt{\alpha^2 + \beta^2}}{\beta} \int \frac{1}{\sqrt{(\alpha/\beta)^2 - \eta^2}} d\eta \\
&= -\frac{\sqrt{\alpha^2 + \beta^2}}{\beta} \int \frac{\alpha/\beta \cos \theta'}{\alpha/\beta \cos \theta'} d\theta' \quad \left(\eta = \frac{\alpha}{\beta} \sin \theta' \Rightarrow d\eta = \frac{\alpha}{\beta} \cos \theta' d\theta' \right) \\
&= -\frac{\sqrt{\alpha^2 + \beta^2}}{\beta} \theta' \\
&= -\frac{\sqrt{\alpha^2 + \beta^2}}{\beta} \sin^{-1} \left(\frac{\beta}{\alpha} \eta \right) \\
&= -\frac{\sqrt{\alpha^2 + \beta^2}}{\beta} \sin^{-1} \left(\frac{\beta}{\alpha} \sqrt{\frac{\alpha^2 - \gamma^2}{\beta^2 + \gamma^2}} \right)
\end{aligned}$$

$$\boxed{\int \frac{\sqrt{\alpha}}{\sqrt{\beta - \alpha}} d\alpha = \beta \sin^{-1} \left(\sqrt{\frac{\alpha}{\beta}} \right) + \sqrt{\alpha(\beta - \alpha)}} \quad (111)$$

Proof:

$$\begin{aligned}
\int \frac{\sqrt{\alpha}}{\sqrt{\beta - \alpha}} d\alpha &= \int \frac{\sqrt{\beta} \sin \theta (2\beta \sin \theta \cos \theta)}{\sqrt{\beta} \cos \theta} d\theta \quad (\alpha \equiv \beta \sin^2 \theta \Rightarrow d\alpha = 2\beta \sin \theta \cos \theta d\theta) \\
&= 2\beta \int \sin^2 \theta d\theta \\
&= 2\beta \int \left(\frac{1 - \cos 2\theta}{2} \right) d\theta \\
&= \beta \left(\theta + \frac{\sin 2\theta}{2} \right) \\
&= \beta (\theta + \sin \theta \cos \theta) \\
&= \beta \sin^{-1} \left(\sqrt{\frac{\alpha}{\beta}} \right) + \beta \sqrt{\frac{\alpha}{\beta}} \cdot \frac{\sqrt{\beta - \alpha}}{\sqrt{\beta}} \quad (\text{see Figure 44}) \\
&= \beta \sin^{-1} \left(\sqrt{\frac{\alpha}{\beta}} \right) + \sqrt{\alpha(\beta - \alpha)}
\end{aligned}$$

$$\boxed{\int \frac{\sqrt{\alpha}}{\sqrt{\alpha - \beta}} d\alpha = \frac{1}{2} \beta \cosh^{-1} \left(\frac{2\alpha}{\beta} - 1 \right) + \sqrt{\alpha(\alpha - \beta)}} \quad (112)$$

Proof:

$$\begin{aligned}
\int \frac{\sqrt{\alpha}}{\sqrt{\alpha - \beta}} d\alpha &= \int \frac{\sqrt{\alpha}}{\sqrt{\alpha - \beta}} \frac{\sqrt{\alpha}}{\alpha} d\alpha \\
&= \int \frac{\alpha}{\sqrt{\alpha^2 - \beta\alpha}}
\end{aligned}$$

Note that

$$\frac{d}{d\alpha} (\alpha^2 - \beta\alpha) = 2\alpha - \beta$$

Need to write:

$$\alpha \equiv \lambda(2\alpha - \beta) + \mu$$

Comparing coefficients we get: $\lambda = 1/2$ and $\mu = 1/2\beta$. Thus,

$$\begin{aligned} \int \frac{\sqrt{\alpha}}{\sqrt{\alpha - \beta}} d\alpha &= \int \frac{1/2(2\alpha - \beta) + 1/2\beta}{\sqrt{\alpha^2 - \beta\alpha}} \\ &= \frac{1}{2}\beta \int \frac{1}{\sqrt{\alpha^2 - \beta\alpha}} d\alpha + \frac{1}{2} \int \frac{2\alpha - \beta}{\sqrt{\alpha^2 - \beta\alpha}} d\alpha \\ &= I_1 + I_2 \end{aligned}$$

where,

$$I_1 = \frac{1}{2}\beta \int \frac{1}{\sqrt{\alpha^2 - \beta\alpha}} d\alpha \quad \text{and} \quad I_2 = \frac{1}{2} \int \frac{2\alpha - \beta}{\sqrt{\alpha^2 - \beta\alpha}} d\alpha$$

Therefore,

$$\begin{aligned} I_1 &= \frac{1}{2}\beta \int \frac{1}{\sqrt{\alpha^2 - \beta\alpha}} d\alpha \\ &= \int \frac{1}{\sqrt{\left(\alpha - \frac{1}{2}\beta\right)^2 - \left(\frac{1}{2}\beta\right)^2}} d\alpha \\ &= \frac{1}{2}\beta \int \frac{1}{\sqrt{v^2 - \left(\frac{1}{2}\beta\right)^2}} dv \quad \left(v \equiv \alpha - \frac{1}{2}\beta \Rightarrow dv = d\alpha\right) \\ &= \frac{1}{2}\beta \int \frac{1/2\beta \sinh \theta}{1/2\beta \sinh \theta} d\theta \quad \left(v \equiv \frac{1}{2}\beta \cosh \theta \Rightarrow dv = \frac{1}{2}\beta \sinh \theta d\theta\right) \\ &= \frac{1}{2}\beta \theta \\ &= \frac{1}{2}\beta \cosh^{-1} \left(\frac{2v}{\beta}\right) \\ &= \frac{1}{2}\beta \cosh^{-1} \left(\frac{2\alpha}{\beta} - 1\right) \end{aligned}$$

and

$$\begin{aligned} I_2 &= \frac{1}{2} \int \frac{2\alpha - \beta}{\sqrt{\alpha^2 - \beta\alpha}} d\alpha \\ &= \frac{1}{2} \int \frac{1}{\sqrt{u}} du \quad (u \equiv \alpha^2 - \beta\alpha \Rightarrow du = (2\alpha - \beta)) \end{aligned}$$

$$\begin{aligned} &= \sqrt{u} \\ &= \sqrt{\alpha(\alpha - \beta)} \end{aligned}$$