Q1): Using cylindrical coordinates:

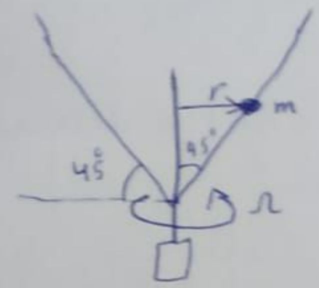
r, θ, z

velocity in cylindrical coordinates:

$$v^2 = (\dot r)^2 + (r\dot\theta)^2 + (\dot z)^2$$

$$T = \frac{1}{2} m v^2$$

$$U = mgz = mgr$$



$\tan\frac{\pi}{4} = 1 = \frac{z}{r} \Rightarrow z = r$

$\Omega = \dot\theta$

$$L = T - U = \frac{1}{2} m(\dot r^2 + r^2\dot\theta^2 + \dot z^2) - mgr$$

$$\dot r = \dot z$$

(a) $\Rightarrow L = \frac{1}{2} m(2\dot r^2 + r^2\dot\theta^2) - mgr$

the generalized coordinat are: r, Ω

r: $\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot r} = 0$

$$\frac{\partial L}{\partial r} = mr\dot\theta^2 - mg$$

$$\frac{\partial L}{\partial \dot r} = 2m\dot r$$

$$\Rightarrow mr\dot\theta^2 - mg - 2m\ddot r = 0 \quad \text{— (1)}$$

$$\Rightarrow mr\Omega^2 - mg - 2m\ddot r = 0$$

θ: $\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = 0$

$$\frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot\theta} = mr^2\dot\theta = mr^2\Omega$$

$$\rightarrow \frac{d}{dt}(mr^2\dot\theta) = 0 \quad \text{— (2)}$$

(c)

$$mr^2\dot\theta = \text{constant}$$

P_θ = constant (constant of motion)

(b) The two constants of motion are:

P_θ, $\theta = 45° \Rightarrow \tan 45° = \frac{z}{r} \Rightarrow z = r$

Q2) $P = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$

$$-\dot{P} = \frac{\partial H}{\partial y} = -m\dot{y}$$

$$\boxed{H = -my\dot{y}}$$

(a) — $\boxed{\begin{aligned} H &= T+V \\ &= \frac{1}{2}m\dot{y}^2 + mgye^{\gamma t} \end{aligned}}$

$$\boxed{= \frac{p^2}{2m} + mgye^{\gamma t}}$$

Hamilton's eqs:

(b)

□ $-\dot{p} = \frac{\partial H}{\partial y} = mge^{\gamma t}$

$\Rightarrow mge^{\gamma t} + \dot{p} = 0$

$mge^{\gamma t} + m\dot{y} = 0$ — (1) which is first order diff. eq.

□ $\dot{y} = \frac{\partial H}{\partial p} = \frac{p}{m}$

$p = m\dot{y}$ — (2) which is as calculated above

(c) $\frac{dH}{dt} = \frac{\partial H}{\partial t} = \gamma mgye^{\gamma t}$ which is dependent of time t

Hence, H is not a constant of motion

Q 3):

$$n(x,y) = n_o(1+ky)$$

$$V = \frac{c}{n} = \frac{c}{n_o(1+ky}$$

$$V = \frac{ds}{dt}$$

$$dt = \frac{ds}{V}$$

$$t = \int_{(x_o,y_o)}^{(x_1,y_1)} \frac{ds}{V} = \int \frac{(dx^2+dy^2)^{1/2}(n_o(1+ky))}{c}$$

$$= \int \frac{n_o}{c}(1+ky)(1+y'^2)^{1/2}\,dx$$

Let $f = \frac{n_o}{c}(1+ky)(1+y'^2)^{1/2}$

which is the function that describes the path of light in this medium.

Using the second form of the Euler-Lagrange equation:

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$$

Note that $\frac{\partial f}{\partial x} = 0 \Rightarrow f - y'\frac{\partial f}{\partial y'} = \text{constant} = A$

$$\frac{\partial f}{\partial y'} = y'(1+y'^2)^{-3/2}$$

$$\Rightarrow \frac{n_o}{c}(1+ky)(1+y'^2)^{1/2} - y'^2(1+y'^2)^{-3/2} = A$$

$$\left.\begin{array}{l} x(0)=0 \\ y(0)=0 \end{array}\right\} \text{start from origin}$$

Q1 (d) :

d) : First, we need to determine the effective potential: that is,

$$H = T + U$$
$$= m\dot{r}^2 + \frac{P_\theta^2}{2mr^2} + mgr$$
$$= \frac{1}{2}\gamma\dot{r}^2 + \underbrace{\frac{P_\theta^2}{\gamma r^2} + \frac{\gamma g r}{2}}_{U_{eff}}$$

where $\boxed{\gamma = 2m}$

$$\frac{\partial U_{eff}}{\partial r} = -\frac{2P_\theta^2}{\gamma r^3} + \frac{\gamma g}{2} = 0$$

$$\Rightarrow r^3 = \frac{4P_\theta^2}{\gamma^2 g} = \frac{r^4\dot{\theta}^2}{g} = \frac{r^4}{g}\Omega^2$$

$$\Rightarrow \boxed{r = \frac{g}{\Omega^2}}$$