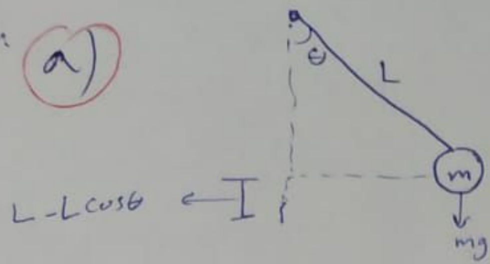
Q1 : a)



$T = \frac{1}{2} m v^2 = \frac{1}{2} m L^2 \dot{\theta}^2$; $\ddot{\theta} + \frac{g}{L} \sin\theta = 0$

$U = mg(L - L\cos\theta)$ $\qquad \dot{V}_\theta = \frac{-g}{L} \sin\theta$

$E = \frac{p_\theta^2}{2mL^2} + mg(L - L\cos\theta)$

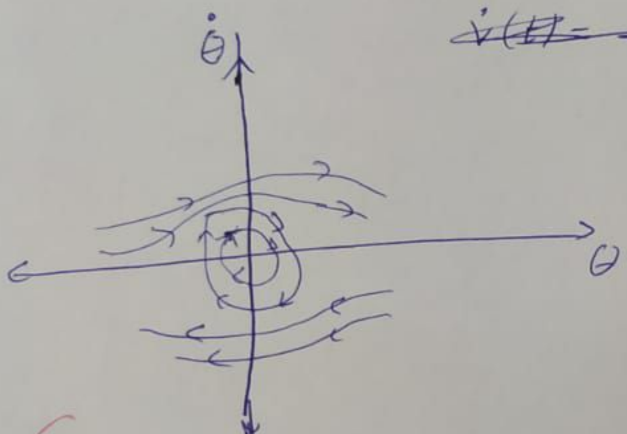
We need to plot

V_θ VS θ

$\cancel{\dot{\theta}(t) = v(t)}$

$\cancel{\dot{v}(t) = -\sin}$

$\left.\begin{aligned} \dot{\theta}(t) &= V_\theta \\ \dot{V}_\theta &= -\frac{g}{L} \sin\theta \end{aligned}\right\}$



b)

$mL^2 \ddot{\theta} + mgL \sin\theta + \beta \dot{\theta} = 0$

$\sin\theta \approx \theta$

$\Rightarrow \ddot{\theta} + \frac{g}{L} \theta + \frac{\beta}{mL^2} \dot{\theta} = 0$

~~Let β new β = β/mL~~ Let $\beta_{new} = \frac{1}{2} \frac{\beta}{mL^2}$

$\Rightarrow \ddot{\theta} + \frac{g}{L} \theta + 2\beta \dot{\theta} = 0$

c)

$\hookrightarrow \ddot{\theta} + 2\beta\dot{\theta} + w_0^2 \theta = 0$; where $w_0^2 = \frac{g}{L}$

Under damping $w_0^2 > \beta^2$

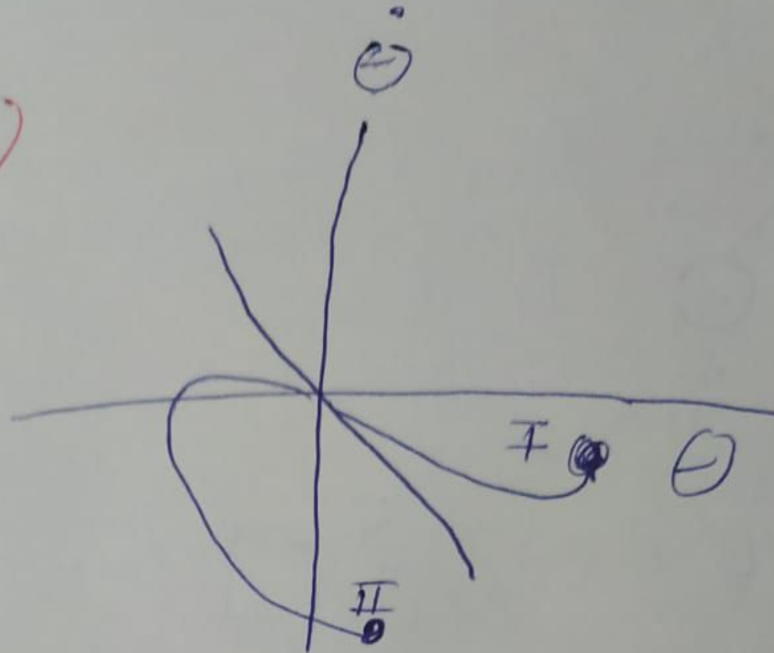
Critical damping $w_0^2 = \beta^2$

overdamping $w_0^2 < \beta^2$.

critical damping : $\boxed{\theta(t) = (A + Bt) e^{-\beta t}}$; A and B are constants (defined by IC's)

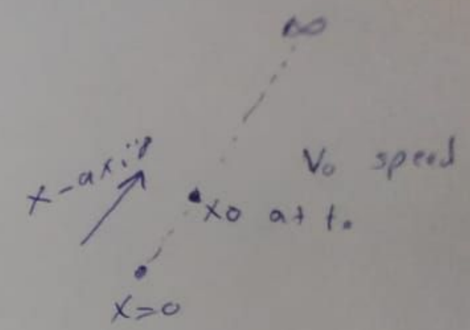
Ans

Q1) (c)



Q2)

$$U(x) = \beta e^{-2|x|/s}$$



$$F = -\frac{dU}{dx} = -\frac{2\beta}{s} e^{-2|x|/s} = m\ddot{x}$$

$$m\ddot{x} + \frac{2\beta}{s} e^{-2|x|/s} = 0$$

$x(0) = \infty$

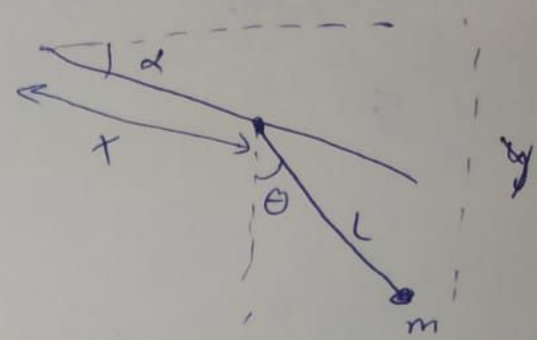
$v(0) = V_0$

The solution is:

$$x - x_0 = s \ln\left[\cosh\left(V_0(t - t_0)/s\right)\right]$$

Q3) :

(A) $X = x\cos\alpha + L\sin\theta$



$$Y = -(x\sin\alpha + L\cos\theta)$$

$$\dot{X} = \dot{x}\cos\alpha + L\cos\theta\,\dot{\theta}$$

$$\dot{Y} = -(\dot{x}\sin\alpha - L\sin\theta\,\dot{\theta})$$

□ $T = \frac{1}{2}m(\dot{X}^2 + \dot{Y}^2)$

$$= \frac{1}{2}m(\dot{x}^2 + L\dot{\theta}^2 + 2\dot{x}L\dot{\theta}\cos(\alpha+\theta))$$

□ $U = mgY = -mg(x\sin\alpha + L\cos\theta)$

$\Rightarrow L = T - U$

$$= \frac{1}{2}m(\dot{x}^2 + L^2\dot{\theta}^2 + 2\dot{x}\dot{\theta}L\cos(\alpha+\theta)) + mg(x\sin\alpha + L\cos\theta)$$

(B) $x:\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

$$\frac{d}{dt}\left(m\dot{x} + m\dot{\theta}L\cos(\alpha+\theta)\right) - mg\sin\alpha = 0$$

$$m\ddot{x} - mL\dot{\theta}^2\sin(\alpha+\theta) + mL\ddot{\theta}\cos(\alpha+\theta) - mg\sin\alpha = 0 \quad -(\mathrm{I})$$

$\dot{\theta}:\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$$\frac{d}{dt}\left[mL^2\dot{\theta} + m\dot{x}L\cos(\alpha+\theta)\right] - \left[-m\dot{x}\dot{\theta}L\sin(\alpha+\theta) - mgL\sin\theta\right] = 0$$

$$mL^2\ddot{\theta} + m\dot{x}L\sin(\alpha+\theta)\dot{\theta} + mL\ddot{x}\cos(\alpha+\theta) + m\dot{x}\dot{\theta}L\sin(\alpha+\theta)$$
$$+ mgL\sin\theta = 0 \quad -(\mathrm{II})$$

Q3)

(E) since $\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$

and $\frac{\partial L}{\partial t} = 0$

then $\frac{\partial H}{\partial t} = 0 = \frac{dH}{dt}$

So $\Rightarrow$ H is constant of motion

and U is independent of velocities,

hence, $H = E =$ constant

(D) $x:$ $P_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} m + m\dot{\theta} L \cos(\alpha + \theta)$

$\dot{x} = \frac{P_x - m\theta L\cos(\alpha+\theta)}{m}$

$\theta:$ $P_\theta = \frac{\partial L}{\partial \theta} = mL^2\dot{\theta} + m\dot{x}L\cos(\alpha+\theta)$

$\dot{\theta} = \frac{P_\theta - m\dot{x}L\cos(\alpha+\theta)}{mL^2}$

(C) $H = \sum_j P_j \dot{q}_j - L$

$= P_\theta \dot{\theta} + P_x \dot{x} - L$

where P_θ, P_x, $\dot{\theta}$ and $\dot{x}$ as above

Q4)

A) $T = \frac{1}{2} m \left(\dot{\rho}^2 + \rho^2 \dot{\varphi}^2 + \dot{z}^2 \right)$

$L = T - U$

$$H = \frac{P_\rho^2}{2m} + \frac{P_\varphi^2}{2m\rho^2} + \frac{P_z^2}{2m} + V_0 \left(\cosh \frac{\rho}{b} + \cos \rho - \frac{z}{a} \right)$$

B) Hamilton equations:

ρ: $\quad \dot{\rho} = \frac{\partial H}{\partial P_\rho} = \frac{P_\rho}{m}$

$\quad -\dot{P}_\rho = \frac{\partial H}{\partial \rho} = \frac{V_0}{b} \cosh \frac{\rho}{b} - V_0 \sin \rho$

φ: $\quad \dot{\varphi} = \frac{\partial H}{\partial P_\varphi} = \frac{P_\varphi}{m\rho^2}$

$\quad -\dot{P}_\varphi = \frac{\partial H}{\partial \varphi} = 0$

z: $\quad \dot{z} = \frac{\partial H}{\partial P_z} = \frac{P_z}{m}$

$\quad -\dot{P}_z = \frac{\partial H}{\partial z} = -\frac{V_0}{a}$

c) Constants of motion are:

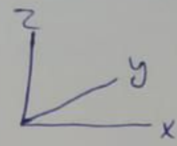
P_φ

H is constant iff $\frac{\partial H}{\partial t} = 0$ ✓

$\Rightarrow$ H and P_φ

Q5) a) $T = \frac{1}{2} m v^2$

$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$



$U = mgz$

$$H = \frac{P_x^2 + P_y^2 + P_z^2}{2m} + mgz$$

b) $\dot{x} = \frac{\partial H}{\partial P_x} = \frac{P_x}{m}$ | $\dot{y} = \frac{\partial H}{\partial P_y} = \frac{P_y}{m}$ | $\dot{z} = \frac{\partial H}{\partial P_z} = \frac{P_z}{m}$

$-\dot{P}_x = \frac{\partial H}{\partial x} = 0$ | $-\dot{P}_y = \frac{\partial H}{\partial y} = 0$ | $-\dot{P}_z = \frac{\partial H}{\partial z} = mg$

We have : P_x and P_y are constants of motion.

$P_x = m\dot{x}$, $P_y = m\dot{y}$, $P_z = m\dot{z}$

$-\dot{P}_z = -m\ddot{z} = mg$

$m\ddot{z} + mg = 0$

$\ddot{z} = -g$

$\dot{z} = -gt + C_1$

let $\dot{z}(t=0) = V_z^0$

$\Rightarrow \boxed{\dot{z} = -gt + V_z^0}$

$z = -\frac{1}{2} g t^2 + V_z^0 t + C_2$

let $z(t=0) = 0 \Rightarrow C_2 = 0$

$\Rightarrow \boxed{z = -\frac{1}{2} g t^2 + V_z^0 t}$

$m\dot{x} = P_x = \text{constant}$

$\dot{x} = \frac{P_x}{m}$

$x = \frac{P_x}{m} t + C_1$

let $x(0) = 0 \Rightarrow C_1 = 0$

$\boxed{x = \frac{P_x}{m} t}$

and hence, $\boxed{y = \frac{P_y}{m} t}$

$\Rightarrow x = \frac{\cancel{m} V_x}{\cancel{m}} t$

$\boxed{x = V_x t}$ and $\boxed{y = V_y t}$] V_x, V_y constants

- which are the same equations for projectile motion
- $H = E$ if 1) $KE \propto v^2$

2) $\frac{\partial U}{\partial \dot{x}_i} = 0$ (Potential energy is independent of velocities)

Hence, $H = E = T + U$

c) The equations of motion are:

$m\ddot{z} + mg + kmV_z = 0$ — (1)

$\dot{V}_z + kV_z + g = 0$

$\Rightarrow V_z = \frac{-g}{k} + C_1 e^{-kt}$

$V_z(t=0) = 0 \Rightarrow C_1 = \frac{g}{k}$

$\Rightarrow \boxed{V_z = \frac{-g}{k} + \frac{g}{k} e^{-kt}}$

$m\ddot{x} + mg + F_0 = 0$ — (2)

$m\dot{V}_x + mg + F_0 = 0$

$V_x = -gt - \frac{F_0}{m} t + C_2$

$V_x(t=0) = V_x^0 = \text{constant}$

$\Rightarrow \boxed{V_x = -gt - \frac{F_0}{m} t + V_x^0}$

تكملة

Q5) c) $\vec{V}(t) = \left(-gt - \frac{F_o}{m} t + V_x^\circ\right)\hat{i} + \left(V_y^\circ\right)\hat{j} + \left(\frac{-g}{k} + \frac{g}{k} e^{-kt}\right)\hat{k}$

terminal velocity:

$$\vec{V}_t = V_y^\circ \hat{j} - \frac{g}{k}\hat{k}$$