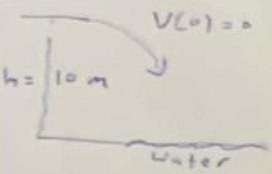
m, $h = 10\,m$, $V(0) = 0$

1 a) $mgh = \frac{1}{2} m v^2$ (conservation of energy)

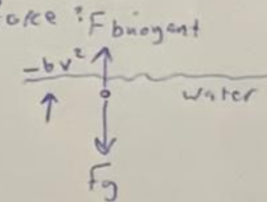
$$v = \sqrt{2gh}$$
$$= \sqrt{2 \times 9.8 \times 10}$$
$$= 14 \ m/s$$



$$V_f = \cancel{V_i} + at$$
$$t = \frac{V_f}{a} = \frac{14}{9.8} = 1.43\,s$$

b) $\Sigma F = ma$

and since the buoyant force balances the grav. force :



the eq. of motion becomes :

$$m\ddot{x} = -b\dot{x}^2$$
$$m\ddot{x} + b\dot{x}^2 = 0$$

c) Finding $V(x)$:

$$\ddot{x} = \frac{d(\dot{x})}{dt} = \frac{d(\dot{x})}{dx}\frac{dx}{dt} \qquad v = \dot{x}$$

$$\Rightarrow \ddot{x} = \frac{d\dot{x}}{dx}\dot{x} \quad \underline{\text{or}} \quad \dot{V} = \frac{dv}{dx} V$$

$$v\frac{dv}{dx} = \frac{-b}{m} v^2$$

$$\int \frac{dv}{v} = \int \frac{-b}{m} dx$$

$$\ln|v| = \frac{-b}{m} x + C \quad ; C \text{ is constant}$$

eq ① $\boxed{V = e^{-\frac{b}{m}x} \cdot V_0}$ since $V(x=0) = V_0$

Now, $\frac{-b}{m} = -0.7\ m^{-1}$, $\frac{V}{V_0} = \frac{1}{10}$

$\Rightarrow$ from eq ① : $\frac{1}{10} = e^{-0.7x}$

$$\ln 0.1 = -0.7x \quad \Rightarrow \quad x = \frac{\ln 0.1}{-0.7} = 3.29 \approx 3.3\ m$$

d) from eq ① : $V = \frac{dx}{dt} = e^{-\frac{b}{m}x} \cdot V_0$

$$e^{\frac{b}{m}x} dx = V_0\, dt$$

$$\frac{m}{b} e^{\frac{b}{m}x} = V_0 t + C \quad \text{constant}$$

$x(0) = 0 \Rightarrow \frac{m}{b} = C \Rightarrow \frac{m}{b} e^{\frac{b}{m}x} = V_0 t + \frac{m}{b} \Rightarrow e^{xb/m} = \frac{b}{m}\left(V_0 t + \frac{m}{b}\right)$

$$\Rightarrow \frac{b}{m} x = \ln \frac{b}{m}\left(V_0 t + \frac{m}{b}\right) \Rightarrow \boxed{x(t) = \frac{m}{b}\left(\ln\left[\frac{b}{m}\left(V_0 t + \frac{m}{b}\right)\right]\right)}$$

$$x(t) = \frac{m}{b}\left(\ln\left(\frac{bV_0}{m} t + 1\right)\right)$$

$\boxed{2}$ $I\ddot{\Theta} + 2b\dot{\Theta} + c\Theta = \tau\cos(2\omega_0 t) + \tau\cos(\omega_0 t)$; $\omega_0 = \sqrt{\frac{c}{I}}$, $b = \sqrt{cI}$

a) Find $\Theta_c(t)$:

$$I\ddot{\Theta} + 2b\dot{\Theta} + c\Theta = 0$$

$$Ir^2 + 2br + c = 0$$

$$r = -2b \pm \sqrt{4b^2 - \underset{\downarrow\ 4b^2}{4cI}}$$

$b^2 = cI$

$$\Rightarrow r = -2b$$

$$\Theta_c = C_1 e^{-2bt} = C_1 e^{-2\sqrt{cI}\,t}$$

b) $\Theta_p(t)$:

$$I\ddot{\Theta} + 2b\dot{\Theta} + c\Theta = \tau\cos(2\omega_0 t) + \tau\cos(\omega_0 t)$$

Let $\Theta = A\cos(2\omega_0 t) + B\sin(2\omega_0 t) + C\cos(\omega_0 t) + D\sin(\omega_0 t)$

$$\dot{\Theta} = -2\omega_0 A\sin(2\omega_0 t) + 2\omega_0 B\cos(2\omega_0 t) - \omega_0 C\sin(\omega_0 t) + \omega_0 D\cos(\omega_0 t)$$

$$\ddot{\Theta} = -4\omega_0^2 A\cos(2\omega_0 t) - 4\omega_0^2 B\sin(2\omega_0 t) - \omega_0^2 C\cos(\omega_0 t) - \omega_0^2 D\sin(\omega_0 t)$$

$$-4\omega_0^2 AI - 4\omega_0^2 BI - 4b\omega_0 A + 4b\omega_0 B$$

$\vdots$

or Using Superposition principle :

$$L\left(\sum_{n=1}^{N}\alpha_n x_n(t)\right) = \sum_{n=1}^{N}\alpha_n F_n(t) = \sum_{n=1}^{N}\alpha_n\cos(\omega_n t)$$

Comparing: $\alpha_1 = \tau = \alpha_2$, $\omega_1 = 2\omega_0$, $\omega_2 = \omega_0$, $\beta = b$

$$\omega_0 = \sqrt{\frac{c}{I}}$$

the steady-state solution :

$$\Theta_p(t) = \frac{1}{I}\sum_{n=1}^{2}\frac{\alpha_n}{(\omega_0^2 - \omega_n^2)^2 + 4\omega_n^2 b^2}\cos(\omega_n t - \delta_n)$$

$$= \frac{1}{I}\left[\frac{\tau}{(\omega_0^2 - 4\omega_0^2)^2 + 16\omega_0^2 b^2}\cos(2\omega_0 t - \delta_1) + \frac{\tau}{(\omega_0^2 - \omega_0^2)^2 + 4\omega_0^2 b^2}\cos(\omega_0 t - \delta_2)\right]$$

$$\delta_1 = \tan^{-1}\left(\frac{4\omega_0 b}{\omega_0^2 - 4\omega_0^2}\right)$$

$$\delta_2 = \tan^{-1}\left(\frac{\cdots}{0}\right) = \frac{\pi}{2}$$

$\boxed{2}$ تكملة :

$$\theta_p(t) = \frac{1}{I}\left[\frac{\tau}{\cancel{3}\,9 w_0^4 + 16 w_0^2 b^2} \cos(2w_0 t - \delta_1) \right.$$

$$\left. + \frac{\tau}{4 w_0^2 b^2} \underbrace{\cos\left(w_0 t - \frac{\pi}{2}\right)}_{\downarrow \; \sin(w_0 t)} \right]$$

where $\delta_1 = \tan^{-1}\left(\dfrac{4 w_0 b}{-3 w_0^2}\right)$

$$b = \sqrt{cI}$$

$$w_0 = \sqrt{\frac{c}{I}}$$

d)

$\boxed{3}$ $V(x) = -V_0 x^2 e^{-x^2/a}$, $V_0 > 0$

a) $\frac{dV}{dx} = -2x V_0 e^{-\frac{x^2}{a}} + \left(-V_0 x^2 \left(\frac{-2x}{a}\right) e^{-\frac{x^2}{a}}\right)$

$= -2V_0 x e^{-\frac{x^2}{a}} + \frac{2V_0}{a} x^3 e^{-\frac{x^2}{a}} = 0$

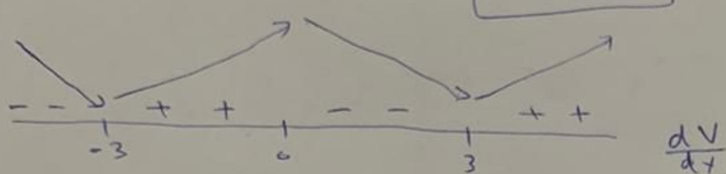
$\Rightarrow -2x e^{-\frac{x^3}{a}} + \frac{2}{a} x^3 e^{-\frac{x^2}{a}} = 0$

$e^{-\frac{x^2}{a}} \left(\frac{2}{a} x^3 - 2x\right) = 0$

$\boxed{x = 0}$ or $\Rightarrow \frac{2}{a} x^2 - 2 = 0$

$x^2 = a$

$\boxed{x = \pm 3}$



at $x = -3 \Rightarrow V(-3) = -V_0 a e^{-1} = -a e^{-1} V_0 \rightarrow$ stable

at $x = 3 \Rightarrow V(3) = V(-3) = -a e^{-1} V_0 \rightarrow$ stable } min

at $x = 0 \Rightarrow V(0) = 0$ unstable (max)

b) $E = -V_0 a e^{-1}$ (min ~~max~~ energy)

$E = 0$ (max energy) } otherwise, the motion is unbounded

c) $F = -kx = \frac{dV}{dx} = -2x V_0 e^{-\frac{x^2}{a}} + \frac{2V_0}{a} x^3 e^{-\frac{x^2}{a}}$

~~$\frac{dF}{dx}$~~ $k = 2V_0 e^{-\frac{x^2}{a}} - \frac{2V_0}{a} x^2 e^{-\frac{x^2}{a}}$

$k(-3) = k(3) = 2V_0 e^{-1} - 2V_0$

$E = V_0 a e^{-1} = \frac{1}{2} k A^2 = \frac{1}{2} A^2 w_0^2 m$

$w_0^2 = \frac{2V_0 a e^{-1}}{A^2 m}$

or $\frac{dF}{dx} = -k = \frac{d^2V(x)}{dx^2}$

then find $k(-3) = k(3)$

then, from $k \Rightarrow w = \sqrt{\frac{k}{m}} \Rightarrow \boxed{T = \frac{2\pi}{w}}$

3 d) $E = KE + PE$

$$= \frac{p^2}{2m} - V_0 x^2 e^{-x^2/9}$$

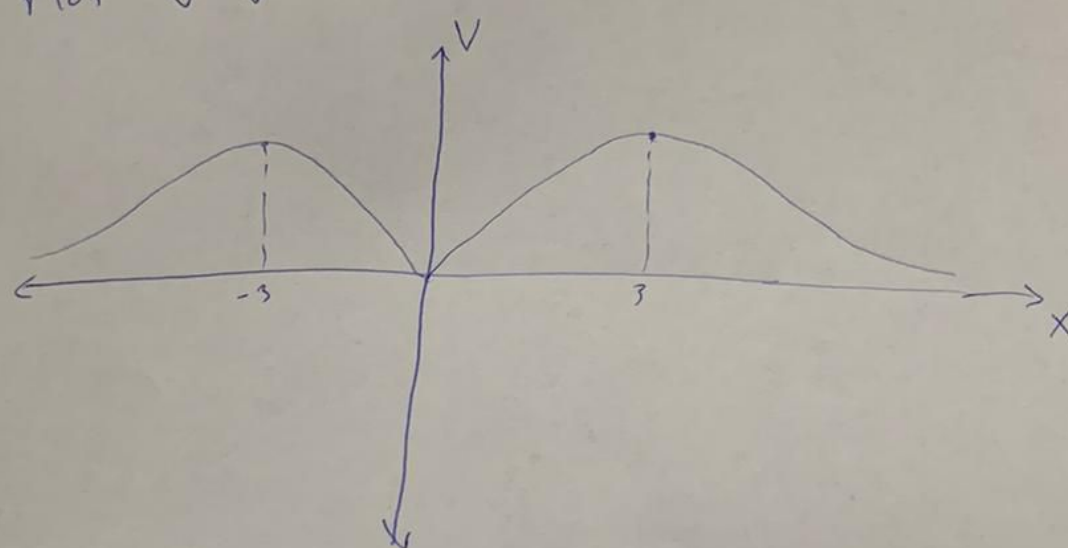
$$p^2 = 2m\left(E + V_0 x^2 e^{-x^2/9}\right)$$

let $E = 0$

$$\Rightarrow p^2 = 2 m V_0 x^2 e^{-x^2/9}$$

$$v^2 = \frac{2V_0}{m} x^2 e^{-x^2/9}$$

Plot v vs x:



3 c)

$$k = -\frac{d^2V}{dx^2}$$

$$= 2e^{-\frac{x^2}{9}} V_0 - \frac{10}{9} e^{-\frac{x^2}{9}} V_0 x^2 + \frac{4}{81} e^{-\frac{x^2}{9}} V_0 x^4$$

$$k(3) = \frac{2V_0}{e} - \frac{10V_0}{e} + \frac{4V_0}{e}$$

$$= -\frac{4V_0}{e}$$

$$w_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{-4V_0e^{-1}}{m}} = \frac{2\pi}{T}$$

$$T = \frac{2\pi}{w_0}$$ ✓