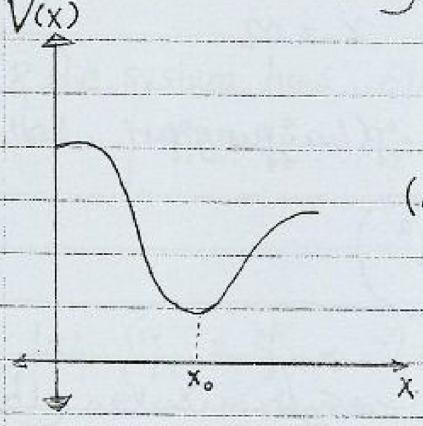
Ch 12.

General Method of Coupled Oscillation

"Theory of Small Oscillations"



كل (Small oscillations) حول (Local min.)
واللي هو موقع استقرار
"Stable equilibrium"

@ طاقة الوضع V(x) ممكن نكتبها
على شكل Taylor Series

$$V(X) = V(X_0) + (X - X_0)\frac{dV}{dX}\bigg|_{X_0} + \frac{1}{2}(X - X_0)^2\frac{d^2V}{dX^2}\bigg|_{x = X_0} + \cdots$$

أول اشي بدي آخذ $V(X_0)$ عند Zero level يعني أول حد بروح
ومن تعريف القيم القصوى ⇐ دائماً المشتقة الأولى عندها = صفر
⇐ بروح الحد الثاني

أما الحدود اللي أكبر من التربيع بنهملها لأنها صغيرة (not significant)

$$V(x) = \frac{1}{2}(X - X_0)^2\frac{d^2V}{dX^2}\bigg|_{x = X_0}$$

let $x - x_0 = \eta$ where $\eta \ll 1$

$$V(\eta) = \frac{1}{2}\frac{d^2V}{dx^2}\bigg|_{X_0}\eta^2$$

$\dot{x} = \dot{\eta}$ since X_0 const.

لوجينا نكتب طاقة الحركة بدلالة η

$$\boxed{T = \frac{1}{2} m \dot{\eta}^2}$$ since $\dot{x} = \dot{\eta}$

بصير الـ (Lagrangian)

$$\boxed{L = \frac{1}{2} m \dot{\eta}^2 - \frac{1}{2} V_{xx} \eta^2}$$

خلينا نطلع معادلات الحركة

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}}\right) - \frac{\partial L}{\partial \eta} = 0$$

$$m \ddot{\eta} + V_{xx} \eta = 0$$

$$\boxed{\ddot{\eta} + \frac{V_{xx}}{m} \eta = 0}$$ harmonic oscillator

$$\boxed{\eta(t) = \eta_0 \cos(\omega t + \varphi)}$$

where $\boxed{\omega^2 = \frac{V_{xx}}{m}}$

$V_{xx} > 0$: concave up

$m > 0$ Unless in Solid State.

$\omega^2 > 0$: (ω : real not imaginary)

⊗ Consider a system with n-degree of Freedom

We have Generalized Coordinates $\{q_1, q_2, \dots, q_n\}$

& the system has stable equilibrium at $\{q_1^\circ, q_2^\circ, q_3^\circ, \dots, q_n^\circ\} = \vec{q}^\circ$

let $q_i = q_i^\circ + \eta_i$ "Small deviation from local minimum , $\eta_i << 1$"

Taylor Series for n-dimensional function

$$V(\{q_i\}) = V(\{q_i^\circ\}) + \sum \left.\frac{\partial V}{\partial q_i}\right|_{q_i^\circ} \eta_i + \frac{1}{2}\sum \frac{\partial^2 V}{\partial q_i \partial q_j}\eta_i \eta_j$$

ومن تعريفنا لِ (local minimum) تكون المشتقة الأولى $\frac{\partial V}{\partial q_i}$ = صفر

ونختار الـ Zero level بحيث $V(q_i^\circ)$ = صفر

$$V_{ij} = \frac{1}{2}\sum \frac{\partial^2 V}{\partial q_i \partial q_j}$$

$$[V] = \begin{bmatrix} \frac{\partial^2 V}{\partial q_1^2} & \frac{\partial^2 V}{\partial q_1 \partial q_2} & \cdots & \frac{\partial^2 V}{\partial q_1 \partial q_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 V}{\partial q_n \partial q_1} & \cdots & \cdots & \frac{\partial^2 V}{\partial q_n^2} \end{bmatrix}$$

The matrix is
1. Real
2. Symmetric $V^T = V$

$$\boxed{V(\eta) = \frac{1}{2}\, \vec{\eta}^{\,t}\, [V]\, \vec{\eta}}$$

$$\vec{\eta}^{\,t} = (\eta_1, \ldots, \eta_n)$$

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}$$

هلأ بدنا نكتب طاقة الحركة T

$$T(\{q\}, \{\dot{q}\}) = \frac{1}{2} \sum T_{ij} \{q\}\, \dot{q}_i\, \dot{q}_j$$

كل الـ (expansion) حوالين (local minimum)

$$\Rightarrow q_i = q_i^{\circ} + \eta_i \quad \Rightarrow \quad \dot{q}_i = \dot{\eta}_i$$

$$T = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2} m r^2 \sin^2\theta\, \dot{\varphi}^2$$

$$\dot{\eta}_1 = \dot{r} \quad \Rightarrow \quad \dot{\eta}_1 \dot{\eta}_1 = \dot{\eta}_1^2 = \dot{r}^2$$

$$\dot{\eta}_2 = \dot{\theta} \quad \Rightarrow \quad \dot{\eta}_2 \dot{\eta}_2 = \dot{\theta}^2$$

$$\dot{\eta}_3^2 = \dot{\varphi}^2$$

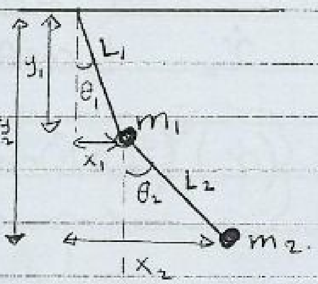
$$[T] = \frac{1}{2} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} m & 0 & 0 \\ 0 & mr^2 & 0 \\ 0 & 0 & mr^2 \sin^2\theta \end{bmatrix}$$

لذن الملخص

$$V = \frac{1}{2}\vec{\eta}^{t}[V]\vec{\eta} \quad \& \quad T = \frac{1}{2}\dot{\vec{\eta}}^{t}[T]\dot{\vec{\eta}}$$

خلينا ناخذ مثال عشان نفهم هالرموز شو تعني وكيف نستخدمها في الحل

Exp) Consider the double pendulum Problem in plane & Small Angle approximation



$$x_1 = L_1 \sin\theta_1$$
$$y_1 = -L_1 \cos\theta_1$$

$$x_2 = L_1 \sin\theta_1 + L_2 \sin\theta_2$$
$$y_2 = -L \cos\theta_1 - L_2 \cos\theta_2$$

عشان نكتب طاقة الحركة بنحتاج $\dot{x}^2$ و $\dot{y}^2$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2).$$

$$\dot{x}_1 = L_1 \cos\theta_1 \,\dot{\theta}_1 \Rightarrow (\dot{x}_1)^2 = L_1^2 \cos^2\theta_1 \,\dot{\theta}_1^2$$

$$\dot{y}_1 = L_1 \sin\theta_1 \,\dot{\theta}_1 \Rightarrow (\dot{y}_1)^2 = L_1^2 \sin^2\theta_1 \,\dot{\theta}_1^2$$

لذن لو جمعنا $\dot{x}_1^2 + \dot{y}_2^2$ بطلع معنا $(L_1\dot{\theta}_1)^2$

$$\dot{x}_1^2 + \dot{y}_1^2 = L_1^2 \cos^2\theta_1 \,\dot{\theta}_1^2 + L_1^2 \sin^2\theta_1 \,\dot{\theta}_1^2$$
$$= L_1^2 \dot{\theta}_1^2 (\cos^2\theta_1 + \sin^2\theta_1) = (L_1\dot{\theta}_1)^2$$

نجي لـ $\dot{y}_2$ و $\dot{x}_2$

$$y_2 = -l_1 \cos\theta_1 - L_2 \cos\theta_2$$

$$\dot{y}_2 = l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2$$

$$(\dot{y}_2)^2 = (l_1 \sin\theta_1 \dot{\theta}_1)^2 + (l_2 \sin\theta_2 \dot{\theta}_2)^2 + 2L_1L_2 \sin\theta_1 \sin\theta_2 \dot{\theta}_1 \dot{\theta}_2$$

$$x_2 = L_1 \sin\theta_1 + L_2 \sin\theta_2$$

$$\dot{x}_2 = L_1 \cos\theta_1 \dot{\theta}_1 + L_2 \cos\theta_2 \dot{\theta}_2$$

$$(\dot{x}_2)^2 = L_1^2 \cos^2\theta_1 \dot{\theta}_1^2 + (L_2 \cos\theta_2 \dot{\theta}_2)^2 + 2L_1L_2 \cos\theta_1 \cos\theta_2 \dot{\theta}_1 \dot{\theta}_2$$

$$(\dot{x}_2)^2 + (\dot{y}_2)^2 = L_1^2 \dot{\theta}_1^2 (\sin^2\theta_1 + \cos^2\theta_1) + L_2^2 \dot{\theta}_2^2 (\cos^2\theta_2 + \sin^2\theta_1)$$
$$+ 2L_1L_2 (\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2) \dot{\theta}_1 \dot{\theta}_2$$

هون بنستخدم صيغة $\cos(\theta_1 - \theta_2)$ عشان أتخلص من الكود الكثير في آخر حد

$$(\dot{x}_2)^2 + (\dot{y}_2)^2 = (L_1 \dot{\theta}_1)^2 + (L_2 \dot{\theta}_2)^2 + 2L_1L_2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2$$

وبما أنه ماخدين Small Angles بقدر أقرب $\cos\theta$ لـ $1 - \frac{\theta^2}{2}$

$$(\dot{x}_2)^2 + (\dot{y}_2)^2 = (l_1 \dot{\theta}_1)^2 + (L_2 \dot{\theta}_2)^2 + 2L_1 l_2 \left[1 - \frac{(\theta_1 - \theta_2)^2}{2}\right] \dot{\theta}_1 \dot{\theta}_2$$

$$(\dot{x}_2)^2 + (\dot{y}_2)^2 = (L_1\dot{\theta}_1)^2 + (L_2\dot{\theta}_2)^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2 - 2L_1L_2\dot{\theta}_1\dot{\theta}_2\frac{(\theta_1-\theta_2)^2}{2}$$

نهمل الحد الأخير لأنه يصبح حاصل ضرب زوايا صغيرة وبالتالي يصغر أكثر

$$(\dot{x}_2)^2 + (\dot{y}_2)^2 \simeq (L_1\dot{\theta}_1)^2 + (L_2\dot{\theta}_2)^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2$$

بما أنه أخيراً وصلنا لما نريد، هلا نكتب طاقة الحركة

$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2}m_1(L_1\dot{\theta}_1)^2 + \frac{1}{2}m_2\left((L_1\dot{\theta}_1)^2 + (L_2\dot{\theta}_2)^2\right) + \frac{1}{2}m_2\left(2L_1L_2\dot{\theta}_1\dot{\theta}_2\right).$$

$$T = \frac{1}{2}(m_1+m_2)L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2L_2^2\dot{\theta}_2^2 + m_2L_1L_2\dot{\theta}_1\dot{\theta}_2$$

$$[T] = \begin{bmatrix} (m_1+m_2)L_1^2 & -m_2L_1L_2 \\ -m_2L_1L_2 & m_2L_2^2 \end{bmatrix}$$

هلا بدنا نكتب طاقة الوضع

$$V = m_1 g L_1 (1-\cos\theta_1) + m_2 g (l_1 - l_1\cos\theta_1 + L_2 - L_2\cos\theta_2)$$

$$= m_1 g L_1 \frac{\theta_1^2}{2} + m_2 g L_1 \left(1-1+\frac{\theta_1^2}{2}\right) + m_2 g L_2 \left(1-1+\frac{\theta_2^2}{2}\right)$$

$$= \frac{m_1 g L_1 \theta_1^2}{2} + \frac{m_2 g L_1 \theta_1^2}{2} + m_2 g L_2 \frac{\theta_2^2}{2}$$

$$= \theta_1^2 \left[m_1 \cancel{L_1} + m_2\right] \frac{g L_1}{2} + \theta_2^2 \left[\frac{m_2 g L_2}{2}\right]$$

$$[V] = \frac{1}{2} \begin{bmatrix} (m_1 + m_2) g L_1 & 0 \\ 0 & m_2 g L_2 \end{bmatrix}$$

$$\Rightarrow \frac{1}{2} (\theta_1 \quad \theta_2) [V] \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

خلينا هلا نستخدم الـ (Lagrangian) في إيجاد معادلات الحركة

$$L = T - V = \frac{1}{2}\left[T_{ij}\,\dot{\eta}_i\,\dot{\eta}_j - V_{ij}\,\eta_i\,\eta_j \right]$$ Assume Sum Rule

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_k}\right) - \frac{\partial L}{\partial \eta_k} = 0$$

$$T_{1k}\,\ddot{\eta}_1 + V_{1k}\,\eta_1 = 0$$

$$\vdots$$

$$T_{ik}\,\ddot{\eta}_k + V_{ik}\,\eta_i = 0$$

$$\vdots$$

$$T_{nk}\,\ddot{\eta}_n + V_{nk}\,\eta_n = 0$$

راح يطلع عندي n من المعادلات على شكل 2nd order linear eq.

لازم نحلهم كلهم مع بعض (simultaneously)

منحول معاملات $\ddot{\eta}_k$ لل (Matrix وكذلك η_k

$$[T]\,\ddot{\vec{\eta}} + [V]\,\vec{\eta} = 0$$

let $\vec{\eta} = C\,\vec{a}\,e^{i\omega t}$

$$\ddot{\vec{\eta}} = -\omega^2 C\,\vec{a}\,e^{i\omega t} = -\omega^2\vec{\eta}$$

$$[T]\ddot{\vec{\eta}} + [V]\vec{\eta} = 0$$

$$-\omega^2 [T]\, c\vec{a}\, e^{i\omega t} + [V]\, c\vec{a}\, e^{i\omega t} = 0$$

$$\left(-\omega^2 [T] + [V]\right) c\vec{a}\, e^{i\omega t} = 0$$

وإذا كنا بندور على non trivial Solution لازم تكون محددة ما بين الأقواس تساوي صفر

$$\boxed{\left| [V] - \omega^2 [T] \right| = 0}$$ Secular Equation

Recall) $V_{ij} = V_{ji}$ & $T_{ij} = T_{ji}$ since Both are Real & Symmetric

This is an eigenvalue problem & $\omega_1^2 \dots \omega_n^2$ are eigenvalues.

هلا بدنا نثبت انه ω^2 حقيقية وأكبر من صفر

أول بنثبت انها Real بعدها انها أكبر من صفر :

let $\lambda_L = \omega_L^2$

A) $\sum \left(V_{ij}\, a_{jL} - \lambda_L T_{ij}\, a_{jL}\right) = 0$

B) $\sum \left(V_{ij}\, a^*_{jL} - \lambda^*_L T_{ij}\, a_{jL}\right) = 0$

أخذنا الـ (Complex conjugate) للأولى عشان نطلع B

هلا بدنا نضرب A بـ (a^*_{iL}) و B بـ (a_{iL})

A') $\sum V_{ij}\, a^*_{iL}\, a_{jL} = \sum \lambda_L T_{ij}\, a^*_{iL}\, a_{jL}$

B') $\sum V_{ij}\, a_{iL}\, a^*_{jL} = \sum \lambda^*_L T_{ij}\, a_{iL}\, a^*_{jL}$

But we can Interchange the label since [T] & [V] symmetric. $\Rightarrow a_{iL}, a^*_{iL}$

$$\lambda_L = \frac{\sum V_{ij}\, a^*_{iL}\, a_{jL}}{\sum T_{ij}\, a^*_{iL}\, a_{jL}} = \lambda^*_L$$

& We can put the zero level of at local minim.
& Any Variation will be positive & T > 0 always

So $\lambda_L = \lambda^*_L > 0$

$\Rightarrow \omega_L^2 > 0$

بالرجوع لمثال (double pendulum)

let $m_1 = m_2$ & $l_1 = l_2 = L$

$$[T] = \frac{1}{2}\begin{bmatrix} 2mL^2 & -mL^2 \\ -mL^2 & mL^2 \end{bmatrix}$$

$$[V] = \frac{1}{2}\begin{bmatrix} 2mgL & 0 \\ 0 & mgL \end{bmatrix}$$

$$|[V] - \omega^2[T]| = 0$$

$$\begin{vmatrix} 2(mgL - \omega^2 mL^2) & +\omega^2 mL^2 \\ \omega^2 mL^2 & mgL - \omega^2 mL^2 \end{vmatrix} = 0$$

$$2(mgL - \omega^2 mL^2)^2 - (\omega^2 mL^2)^2 = 0$$

$$\sqrt{2}\,(\cancel{m}gL - \omega^2 \cancel{m}L^2) = \omega^2 \cancel{m}L^2$$

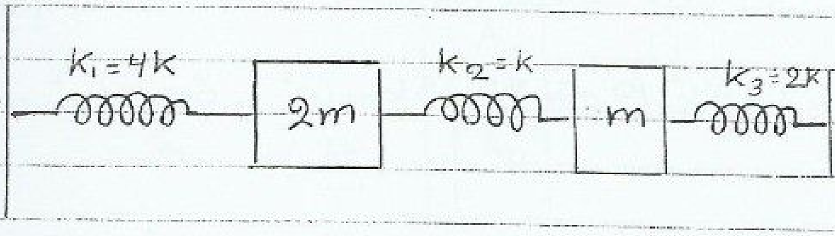
$$\sqrt{2}g - \omega^2 L = \mp \omega^2 L$$

$$\sqrt{2}\frac{g}{L} - \omega^2 = +\omega^2 \Rightarrow \frac{\sqrt{2}}{2}\frac{g}{L} = \omega^2$$

$$\omega = \mp\sqrt{\frac{g}{\sqrt{2}L}}$$

خلينا ناخد أمثلة عالموضوع عشان نتدرب عالحل

Exp)



$$L = T - V = \frac{1}{2} m_1 \dot{X}_1^2 + \frac{1}{2} m_2 \dot{X}_2^2 - \frac{1}{2} k_1 X_1^2$$

$$- \frac{1}{2} k_2 (X_2 - X_1)^2 - \frac{1}{2} k_3 X_2^2$$

$$[T] = \frac{\partial^2 T}{\partial \dot{x}_i \partial \dot{x}_j} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = \begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix}$$

$$[V] = \frac{\partial^2 V}{\partial x_i \partial x_j} = \begin{pmatrix} 5k & -k \\ -k & 3k \end{pmatrix}$$

كيف اجينا بهدول ؟ بنشوف معامل $(X_1)^2$ وبكون $5K$

ومعامل $X_1 X_2 = -k$

ومعامل $X_2 X_1 = -K$

ومعامل $(X_2)^2 = 3K$

يعني لو عدنا ترتيب الـ (Lagrngian) هيك بطلع معنا المعاملات

$$\left| [V] - \omega^2 [T] \right| = 0$$

$$\begin{vmatrix} 5k - 2\omega^2 m & -k \\ -k & 3k - m\omega^2 \end{vmatrix} = 0$$

$$(5k-2m\omega^2)(3k-m\omega^2)-k^2=0$$

$$15k^2+(2m^2\omega^4)-5km\omega^2-6mk\omega^2-k^2=0$$

$$2m^2\omega^4-11mk\omega^2+14k^2=0$$

let $x=\omega^2$

$$2mx^2-11mkx+14k^2=0$$

$$x=\frac{-b\mp\sqrt{b^2-4ac}}{2a}=\frac{11mk\mp\sqrt{121m^2k^2-4(2m^2)(1\ldots}}{2(2m^2)}$$

$$=\frac{11mk\mp\sqrt{121m^2k^2-112m^2k^2}}{4m^2}$$

$$=\frac{11k}{4m}\mp\sqrt{\frac{9m^2k^2}{16m^4}}$$

$$=\frac{11}{4}\frac{k}{m}\mp\frac{3k}{4m}$$

$$\omega_1^2=\left(\frac{11+3}{4}\right)\frac{k}{m}=\frac{7k}{3m}$$

$$\omega_2^2=\left(\frac{11-3}{4}\right)\frac{k}{m}=\frac{8}{4}\frac{k}{m}=\frac{2k}{m}$$

$$\begin{bmatrix} 5k - 2m\left(\frac{7k}{2m}\right) & -k \\ -k & 3k - m\left(\frac{7k}{2m}\right) \end{bmatrix} \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} = \vec{O}$$

$$(5k - 7k)\, A_{11} - k A_{21} = 0$$

$$\boxed{-2A_{11} = A_{21}} \Rightarrow \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

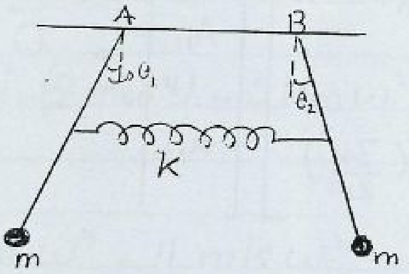
for $\omega_2^2 = \frac{2k}{m}$

$$\begin{bmatrix} 5k - 2m\left(\frac{2k}{m}\right) & -k \\ -k & 3k - m\left(\frac{2k}{m}\right) \end{bmatrix} \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = \vec{O}$$

$$(5k - 4k)\, A_{12} - k A_{22} = 0$$

$$A_{12} = +A_{22} \Rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Exp)



for simple pendulum $L = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(\underbrace{1-\cos\theta}_{1-(1-\frac{\theta^2}{2})}) \approx \frac{1}{2} m l^2 \dot{\theta}^2 - \frac{1}{2} mgl(\theta^2)$

for two uncoupled pendulum

$$L = \frac{1}{2} m l^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2) - \frac{1}{2} mgl(\theta_1^2 + \theta_2^2)$$

لما يصيروا Coupled بنضيف عاللي فوق حد جديد من طاقة الوضع

$$L = \frac{1}{2} m l^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2) + \frac{-1}{2} mgl(\theta_1^2 + \theta_2^2) - \frac{1}{2} k \left(\frac{l}{2}\right)^2 (\theta_2 - \theta_1)^2$$

$$L = \frac{1}{2} m l^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2) - \frac{1}{2} mgl(\theta_1^2 + \theta_2^2) - \frac{1}{8} k l^2 (\theta_2 - \theta_1)^2$$

$$= \frac{1}{2} m l^2 \dot{\theta}_1^2 + \frac{1}{2} m l^2 \dot{\theta}_2^2 - \frac{1}{2} mgl\theta_1^2 - \frac{1}{2} mgl\theta_2^2 - \frac{1}{8} k l^2 \theta_2^2 - \frac{1}{8} k l^2 \theta_1^2 + \frac{1}{4} k l^2 \theta_2 \theta_1$$

$$[T] = \begin{bmatrix} ml^2 & 0 \\ 0 & ml^2 \end{bmatrix}, \quad [V] = \begin{bmatrix} mgl + \frac{kl^2}{4} & -\frac{kl^2}{4} \\ -\frac{kl^2}{4} & mgl + \frac{kl^2}{4} \end{bmatrix}$$

$$\left| [V] - \omega^2 [T] \right| = 0$$

$$\begin{vmatrix} mgL + \frac{kL^2}{4} - \omega^2 m l^2 & -\frac{kl^2}{4} \\ -\frac{kl^2}{4} & mgl + \frac{kl^2}{4} - \omega^2 m l^2 \end{vmatrix} = 0$$

$$\left(mgL + \frac{kL^2}{4} - \omega^2 m l^2\right)\left(mgl + \frac{kl^2}{4} - \omega^2 m l^2\right) - \left(\frac{kl^2}{4}\right)^2 = 0$$

$$mgl + \frac{kl^2}{4} - \omega^2 m l^2 = \mp \frac{kL^2}{4}$$

$\boxed{1}$ $mgl - \omega^2 m l^2 = 0 \quad \Rightarrow \quad \omega_1^2 = \frac{mgl}{mL^2} = \frac{g}{L}$

$\boxed{2}$ $mgl - \omega^2 m l^2 = -\frac{kL^2}{2} \quad \Rightarrow \quad \omega_2^2 = \frac{-kL^2}{-2mL^2} - \frac{mgL}{-mL^2}$

$$\omega_2^2 = +\frac{k}{2m} + \frac{g}{L}$$

To find eigenstates

for $\omega_1^2 = \frac{g}{L}$

$$\begin{bmatrix} mgL + \frac{kl^2}{4} - \frac{g}{L} ml & -\frac{kl^2}{4} \\ -\frac{kL^2}{4} & \frac{kl^2}{4} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \vec{0}$$

$$\frac{kL^2}{4} a_{11} - \frac{kL^2}{4} a_{21} = 0$$

$$a_{11} = a_{21}$$

for $\omega_2^2 = \frac{k}{2m} + \frac{g}{L}$

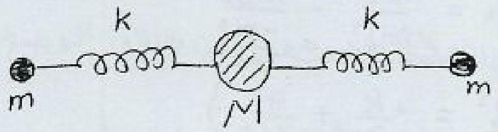
$$\begin{bmatrix} mgL + \frac{kL^2}{4} - \left(\frac{k}{2m}+\frac{g}{L}\right) mL^2 & -\frac{kL^2}{4} \\ -\frac{kL^2}{4} & mgL + \frac{kL^2}{4} - \cancel{\omega^2} mL^2\left(\frac{k}{2m}\ \frac{g}{L}\right) \end{bmatrix}\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \vec{0}$$

$$\left(\cancel{mgL} + \frac{kL^2}{4} - \frac{k}{2}L^2 - \cancel{mgL}\right) a_{12} - \frac{kL^2}{4} a_{22} = 0$$

Anti-Symmetric

$$a_{11}\left(-\frac{kL^2}{4}\right) = \frac{kL^2}{4} a_{22} \quad \Rightarrow \quad a_{22} = -a_{11}$$

Exp)



$$L = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_3^2\right) + \frac{1}{2} M \dot{x}_2^2 - \frac{k}{2}\left[(x_2 - x_1)^2 + (x_3 - x_1)^2\right]$$

$$[T] = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}, \quad [V] = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}$$

$$\left| [V] - \omega^2 [T] \right| = 0$$

$$\begin{vmatrix} k-\omega^2 m & -k & 0 \\ -k & 2k-\omega^2 M & -k \\ 0 & -k & k-\omega^2 \end{vmatrix} = 0$$

$$(k-\omega^2 m)\left[(2k-\omega^2 M)(k-\omega^2) - k^2\right] + k^2(k-\omega^2) = 0$$

$$(k-\omega^2 m)\left[(2k-\omega^2 M)(k-\omega^2) - \cancel{k^2} + \cancel{k^2}\right] = 0$$

$k-\omega^2 m = 0$ or $2k^2 - \omega^2(2k+kM) + M\omega^4 = 0$ let $x=\omega^2$

$\omega_1^2 = \dfrac{k}{m}$ or $x = \omega_2^2 = \dfrac{+(2k+kM) \mp \sqrt{(2k+kM)^2 - 4(0)}}{2M}$

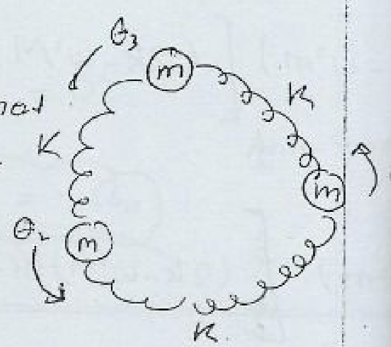
$$= \frac{(2k+kM) \mp (2k+kM)}{2M}$$

$$\omega_2^2 = \frac{2k+kM-(2k+kM)}{2M} = 0$$

$$\omega_3^2 = \frac{(2k+kM)+(2k+kM)}{2M}$$

$$= \frac{4k+2kM}{2M} = \frac{2k}{M} + k$$

Exp) Three masses m each, initially located equidistant from one another on a horizontal circle of Radius "R". They are connected in pairs by three springs of force constant k each & of unstretched length $\frac{2\pi R}{3}$. The spring threads the circuloar tract so that the mass is constrained to move on the circle. Find the normal modes.



let θ_1, θ_2, θ_3 be Angular displacement of three masses from their equilibrim

$$L = T - V$$

$$T = \frac{1}{2} m R^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2)$$

$$V = \frac{1}{2} k R^2 [(\theta_1 - \theta_2)^2 + (\theta_2 - \theta_3)^2 + (\theta_3 - \theta_1)^2]$$

$$[V] = \cancel{kR^2} \begin{bmatrix} 2kR^2 & -kR^2 & -kR^2 \\ -kR^2 & 2kR^2 & -kR^2 \\ -kR^2 & -kR^2 & 2kR^2 \end{bmatrix}$$

$$[T] = \begin{pmatrix} mR^2 & 0 & 0 \\ 0 & mR^2 & 0 \\ 0 & 0 & mR^2 \end{pmatrix}$$

$$| [V] - \omega^2 [T] | = 0$$

$$\omega_1 = 0 \quad , \quad \omega_2 = \omega_3 = \sqrt{\frac{3K}{m}}$$